Math 811 Additional Problems

1 Let $P$ and $J$ be any two $R$-modules. Prove that for any short exact sequence
$$0 \to K \overset{\alpha}{\to} N \overset{\beta}{\to} M \to 0$$
of $R$-modules, the following sequences of abelian groups are exact:
$$0 \to \text{Hom}_R(P, K) \overset{\alpha^*}{\to} \text{Hom}_R(P, N) \overset{\beta^*}{\to} \text{Hom}_R(P, M) \to 0$$
$$0 \to \text{Hom}_R(M, J) \overset{\beta^*}{\to} \text{Hom}_R(N, J) \overset{\alpha^*}{\to} \text{Hom}_R(K, J).$$
Here for any homomorphism $\phi : A \to B$ of $R$-modules, $\phi^*_j : \text{Hom}_R(B, J) \to \text{Hom}_R(A, J)$ is defined by $\phi^*_j(f) = f \circ \phi$ for all $f \in \text{Hom}_R(B, J)$ and $\phi^*_p : \text{Hom}_R(A, B) \to \text{Hom}_R(P, B)$ is defined by $\phi^*_p(f) = \phi \circ f$ for all $f \in \text{Hom}_R(P, A)$.

2 Let $P$ be an $R$-module. Prove that $P$ is projective if and only if for any short exact sequence
$$0 \to K \overset{\alpha}{\to} N \overset{\beta}{\to} M \to 0$$
of $R$-modules, the following sequence of abelian groups is exact:
$$0 \to \text{Hom}_R(P, K) \overset{\alpha^*}{\to} \text{Hom}_R(P, N) \overset{\beta^*}{\to} \text{Hom}_R(P, M) \to 0.$$

3 Let $J$ be an $R$-module. Prove that $J$ is injective if and only if for any short exact sequence
$$0 \to K \overset{\alpha}{\to} N \overset{\beta}{\to} M \to 0$$
of $R$-modules, the following sequence of abelian groups is exact:
$$0 \to \text{Hom}_R(M, J) \overset{\beta^*}{\to} \text{Hom}_R(N, J) \overset{\alpha^*}{\to} \text{Hom}_R(K, J) \to 0.$$

4 For a collection $\{J_i \mid i \in I\}$ of $R$-modules, $\prod_{i \in I} J_i$ is an injective $R$-module if and only if each $J_i$ is an injective $R$-module.