13. Let \( \mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\} \) where
\[
A_1 = \{1, 2\}, \quad A_2 = \{2, 3\}, \quad A_3 = \{3, 4\}, \\
A_4 = \{4, 5\}, \quad A_5 = \{5, 6\}, \quad A_6 = \{6, 1\}
\]
Determine the number of different SDRs that \( \mathcal{A} \) has. Generalize to \( n \) sets.

**Solution.** Suppose that \( e_1, e_2, e_3, e_4, e_5, e_6 \) is an SDR such that \( e_i \) is in \( A_i \). \( e_1 \) is either 1 or 2. If \( e_1 = 1 \), the distinctness forces \( e_2 = 2 \), \( e_3 = 3 \), \( e_4 = 4 \), \( e_5 = 5 \), and \( e_6 = 6 \). If \( e_1 = 2 \), the distinctness forces \( e_2 = 3 \), \( e_3 = 4 \), \( e_4 = 4 \), \( e_5 = 5 \), and \( e_6 = 6 \).

Thus there are exactly two SDRs.

For \( n \)-sets, in which \( \mathcal{A} = \{A_1, A_2, A_3, A_4, \ldots, A_n\} \) with
\[
A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 4\}, \ldots, A_{n-1} = \{n-1, n\}, A_n = \{n, 1\},
\]
there are exactly two SDRs by the same argument. \( e_1 = 1 \) will force \( e_i = i \) for all \( i \) and \( e_1 = 2 \) will force \( e_i = i + 1 \) for \( i = 2, 3, \ldots, n-1 \) and \( e_n = 1 \).

15. Suppose \( \mathcal{A} = \{A_1, A_2, A_3, A_4, \ldots, A_n\} \) is a family of sets which “more than satisfies” the Marriage Condition. More precisely, suppose,
\[
|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| \geq k + 1
\]
for each \( k = 1, 2, \ldots, n \) and each choice of \( k \)-distinct indices \( i_1, i_2, \ldots, i_k \). Let \( x \) be an element of \( A_1 \). Prove that \( \mathcal{A} \) has an SDR in which \( x \) represents \( A_1 \).

**Proof.** Consider the new family of sets \( \mathcal{B} = \{B_1, B_2, \ldots, B_n\} \) in which \( B_i = A_i - \{x\} \), i.e., \( B_i = A_i \) if \( x \) is not in \( A_i \) and \( B_i = A_i \) with \( x \) removed if \( x \) is in \( A_i \). Then the new family \( \mathcal{B} \) satisfies the Marriage Condition since, for each \( k \) and each choice of indices \( i_1, i_2, \ldots, i_k \), the set \( A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} \) has at most one more element (which is \( x \)) than the set \( B_{i_1} \cup B_{i_2} \cup \cdots \cup B_{i_k} \).

Thus we have
\[
|B_{i_1} \cup B_{i_2} \cup \cdots \cup B_{i_k}| \geq |A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| - 1 \geq k.
\]
Thus the family \( \mathcal{B} \) has an SDR, say \( e_1, e_2, \ldots, e_n \) with \( e_i \) in \( B_i \). Replacing \( e_1 \) by \( x \), we get that \( x, e_2, \ldots, e_n \) is an SDR for the family \( \mathcal{A} \) and \( x \) is in \( A_1 \).

20. Consider a preferential ranking matrix in which the woman \( A \) ranks the man \( a \) first and the man \( a \) ranks the woman \( A \) first. Show that, in every stable marriage, \( A \) is paired with \( a \).

**Proof.** If there is a marriage in which the woman \( A \) married a man \( m \) and the man \( a \) married a woman \( W \). If \( m \) is not \( a \) then \( W \) is not \( A \). Thus \( A \) ranked \( a \) better than \( A \) ranked \( m \) and \( a \) ranked \( A \) better than \( a \) ranked \( W \). Thus the marriage is an unstable marriage. Thus for a marriage to be a stable marriage, it is necessary that \( A \) and \( a \) are paired together.
22. Use the deferred acceptance algorithm to obtain both the women-optimal and men-optimal stable complete marriages for the preferential matrix

\[
\begin{bmatrix}
A & 1,3 & 2,3 & 3,2 & 4,3 \\
B & 1,4 & 4,1 & 3,3 & 2,2 \\
C & 2,2 & 1,4 & 3,4 & 4,1 \\
D & 4,1 & 2,2 & 3,1 & 1,4 \\
\end{bmatrix}
\]

Conclude that for the given preferential ranking matrix there is only one stable marriage.

**Solution.** First determine the women-optimal marriage:

\[
\begin{align*}
A & \rightarrow a^\text{hold} & A & \rightarrow a^\text{hold} & A & \rightarrow a^\text{reject} & A & \rightarrow b^\text{reject} & A & \rightarrow c^\text{hold} \\
B & \rightarrow a^\text{reject} & B & \rightarrow d^\text{hold} & B & \rightarrow d^\text{hold} & B & \rightarrow d^\text{hold} & B & \rightarrow d^\text{hold} \\
C & \rightarrow b^\text{hold} & C & \rightarrow b^\text{hold} & C & \rightarrow a^\text{hold} & C & \rightarrow a^\text{hold} & C & \rightarrow a^\text{hold} \\
D & \rightarrow d^\text{hold} & C & \rightarrow d^\text{reject} & D & \rightarrow b^\text{hold} & D & \rightarrow b^\text{hold} & D & \rightarrow b^\text{hold} \\
\end{align*}
\]

Then the marriage is \(A \rightarrow c\), \(B \rightarrow d\), \(C \rightarrow a\) and \(D \rightarrow b\).

The same algorithm by starting with men proposing to the their best women who have not rejected them will result in a men-optimal marriage.

\[
\begin{align*}
a & \rightarrow D \overset{r}{\rightarrow} a \rightarrow C \\
b & \rightarrow B \rightarrow b \rightarrow B \\
c & \rightarrow D \rightarrow c \rightarrow D \\
d & \rightarrow C \rightarrow d \rightarrow C \overset{r}{\rightarrow} d \rightarrow B \\
\end{align*}
\]

The result is the same as the women proposing one presented above. Thus in every Women optimal is also men optimal, both men and women got all their best possible spouses among all stable marriages and their least possible spouses. Thus it is be only stable marriage.

26. Apply the deferred acceptance algorithm to obtain a stable complete marriage for the preferential ranking matrix

\[
\begin{bmatrix}
A & 1,3 & 2,3 & 3,2 & 4,3 \\
B & 1,4 & 4,1 & 3,3 & 2,2 \\
C & 2,2 & 1,4 & 3,4 & 4,1 \\
D & 4,1 & 2,2 & 3,1 & 1,4 \\
\end{bmatrix}
\]

**Solution.** We use \(A, B, C, D\) propose algorithm. In the following, “\(h\)” means hold and “\(r\)” mean reject

\[
\begin{align*}
A & \rightarrow a^h & A & \rightarrow a^h & A & \rightarrow a^r & A & \rightarrow b^h & A & \rightarrow b^h & A & \rightarrow c^h & A & \rightarrow c^h \\
B & \rightarrow a^r & B & \rightarrow b^h & B & \rightarrow b^h & B & \rightarrow b^h & B & \rightarrow b^h & B & \rightarrow c^h & B & \rightarrow c^h \\
C & \rightarrow b^h & C & \rightarrow b^r & C & \rightarrow c^h & C & \rightarrow c^h & C & \rightarrow c^h & C & \rightarrow a^h & C & \rightarrow a^h \\
D & \rightarrow c^h & D & \rightarrow c^h & D & \rightarrow c^h & D & \rightarrow a^h & D & \rightarrow a^h & D & \rightarrow a^r & D & \rightarrow b^h \\
\end{align*}
\]