21. Prove that $D_n$ is an even number if and only if $n$ is an odd number.

Proof. Note that $D_n = nD_{n-1} + (-1)^n$. We will use induction on $n$. When $n = 1$, then $D_1 = 0$ is even and when $h = 2$, $D_2 = 1$ is odd. Assume that for any $m < n$, $D_m$ is even if and only if $m$ is odd. Now for $m = n$, there are two situations: $n$ is even or $n$ is odd.

If $n$ is even, then $nD_{n-1}$ is even and $D_n = nD_{n-1} + (-1)^n$ (adding/subtracting 1 to an even number will result an odd number) is odd. When $n$ is odd, then $n - 1$ is even, thus $D_{n-1}$ is odd by the induction assumption. Hence, $nD_{n-1}$ is also odd (as product of two odd number is still odd). Thus adding/subtracting 1 will result an even number. Thus $D_n$ is even.

24. What is the number of ways to place six non-attacking rooks on the $6 \times 6$ board with forbidden positions as shown.

Solution. (c). We use the inclusion exclusion principle and the formula the number of ways to placing six non-attacking rooks is

$$6! - r_1 \cdot 5! + r_2 \cdot 4! - r_3 \cdot 3! + r_4 \cdot 2! - r_5 \cdot r_6,$$

where $r_i$ is the number of ways of placing $i$-non-attacking rooks on the the forbidden positions. $r_1 = 8$. To compute $r_2$, two rooks have to be in two different rows. To compute $r_3$ three rooks have to be in three different rows, etc.

<table>
<thead>
<tr>
<th>For $r_2$, rows</th>
<th>1, 2</th>
<th>1, 3</th>
<th>1, 4</th>
<th>1, 5</th>
<th>2, 3</th>
<th>2, 4</th>
<th>2, 5</th>
<th>3, 4</th>
<th>3, 5</th>
<th>4, 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>ways</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>For $r_3$, rows</th>
<th>1, 2, 3</th>
<th>1, 2, 4</th>
<th>1, 2, 5</th>
<th>1, 3, 4</th>
<th>1, 3, 5</th>
<th>1, 4, 5</th>
<th>2, 3, 4</th>
<th>2, 3, 5</th>
<th>2, 4, 5</th>
<th>3, 4, 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>ways</td>
<td>1</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>For $r_4$, rows</th>
<th>1, 2, 3, 4</th>
<th>1, 2, 3, 5</th>
<th>1, 2, 4, 5</th>
<th>1, 3, 4, 5</th>
<th>2, 3, 4, 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>ways</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus $r_2 = 22$, $r_3 = 24$, $r_4 = 9$. Note that $r_5 = 1$ an do $r_6 = 0$. Thus the number of ways of placing six non-attacking rooks is

$$6! - 8 \cdot 5! + 22 \cdot 4! - 24 \cdot 3! + 9 \cdot 2! - 1 + 0 = 161.$$

25. Count the permutations $i_1i_2i_3i_4i_5i_6$ of $\{1, 2, 3, 4, 5, 6\}$ where $i_1 \neq 1, 5$, $i_3 \neq 2, 3, 5$, $r_4 \neq 4$, and $i_6 \neq 5, 6$.

Solution. We change the problem into a question of chess board with forbidden positions.

```
X  X
X  X  X
X  X
X  X
```
Then \( r_1 = 8 \). For \( r_2, r_3, r_4 \) we have the following tables

<table>
<thead>
<tr>
<th>rows</th>
<th>1,3</th>
<th>1,4</th>
<th>1,6</th>
<th>3,4</th>
<th>3,6</th>
<th>4,6</th>
</tr>
</thead>
<tbody>
<tr>
<td>ways</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Thus \( r_2 = 20 \), and \( r_3 = 20 \). \( r_4 = 7 \). Thus the number of the permutations with forbidden positions is

\[
6! - 8 \cdot 5! + 20 \cdot 4! - 20 \cdot 3! + 7 \cdot 2! = 134.
\]

28. A carousel has eight seats each representing a different animal. Eight boys are seated around a carousel but facing inward so that each boy faces another (each boy looks at another boy’s front). In how many ways can they change seats so that each faces a different boy? How does the problem change if all the seats are identical?

**Solution.** Since each seat represents a different animal, the question is not a circular permutation question rather a linear permutations. We will name the boys by the numbers as they sit first time so that 1 faces 5, 2 faces 6, 3 faces 7 and 4 faces 8. We use inclusion-exclusion principle. Let \( S \) be the set of all permutations of \( \{1, 2, 3, 4, 5, 6, 7, 8\} \) and \( A_1 \) be the set of permutations in \( S \) such that 1 faces 5; \( A_2 \) be the set of permutations in \( S \) such that 2 faces 6; \( A_3 \) be the set of permutations in \( S \) such that 3 faces 7; \( A_4 \) be the set of permutations in \( S \) such that 4 faces 8.

Note that \( |S| = 8! \). To compute \( |A_1| \), seat 1, there are 8 ways (to theeight animals). Then sit 5 to the opposite of 1 (only one way). Then seat the remaining six boys. Then \( |A_1| = 8 \cdot 6! \). Similarly, \( |A_2| = |A_3| = |A_4| = 8 \cdot 6! \). To compute \( |A_1 \cap A_2| \), seat 1 and 5 first (8 ways), then seat 2 and 6 by just seating 2 ( 6 has to face 2). There are 6 choices and 4! ways to seat the remaining 4 boys. Thus \( |A_1 \cap A_2| = 8 \cdot 6 \cdot 4! \). Similarly \( |A_i \cap A_j| = 8 \cdot 6 \cdot 4! \) for all \( i \neq j \). In \( A_1 \cap A_2 \cap A_3 \) first seat 1 and 5 (8 ways) then seat 2 and 6, (6 ways) followed by seating 3 and 7 (4 ways) then remaining 2 boys (2 ways). Thus \( A_1 \cap A_2 \cap A_3 \) and any other intersections of three sets have \( 8 \cdot 6 \cdot 4 \cdot 2! \) elements.

Similarly, \( |A_1 \cap A_2 \cap A_3 \cap A_4| = 6 \cdot 4 \cdot 2 \). Thus by the inclusion-exclusion principle, the number of ways to change the seating so no boy faces the same person is

\[
8! - 8 \cdot 6! + \binom{4}{2} \cdot 8 \cdot 6 \cdot 4! - \binom{4}{3} \cdot 8 \cdot 6 \cdot 4 \cdot 2! + \binom{4}{4} \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 4608 \cdot 82880 \cdot 8
\]

If all seats are identical, the question becomes a circular permutation question. Now we let \( S \) be the set of all circular permutations of \( \{1, 2, 3, 4, 5, 6, 7, 8\} \) and \( A_1 \) be the set of permutations in \( S \) such that 1 faces 5; \( A_2 \) be the set of permutations in \( S \) such that 2 faces 6; \( A_3 \) be the set of permutations in \( S \) such that 3 faces 7; \( A_4 \) be the set of permutations in \( S \) such that 4 faces 8.

Note that \( |S| = \frac{8!}{8} = 7! \). To compute \( |A_1| \), seat 2, and 5 first (only one way), then seat the remaining six boys. Then \( |A_1| = 6! \). Similarly, \( |A_2| = |A_3| = |A_4| = 6! \). To compute \( |A_1 \cap A_2| \), seat 1, and 5 first, then seat 2 and 6 by just seating 2 (then 6 has to face 2), there are 6 choices and 4! ways to seat the remaining 4 boys. Thus \( |A_1 \cap A_2| = 6 \cdot 4! \). Similarly \( |A_i \cap A_j| = 6 \cdot 4! \) for all \( i \neq j \). In \( A_1 \cap A_2 \cap A_3 \) first seat 1 and 5 (one way) then seat 2 and 6, (6 ways) followed seating 3 and 7 (4 ways) then remaining 2 boys (two ways). Thus \( A_1 \cap A_2 \cap A_3 \) and any other intersections of three sets have \( 6 \cdot 4 \cdot 2! \) elements. Similarly, \( |A_1 \cap A_2 \cap A_3 \cap A_4| = 6 \cdot 4 \cdot 2 \). Thus
by the inclusion-exclusion principle, the number of ways to change the seating so no boy faces the same person is

\[ 7! - \binom{4}{1} \cdot 6! + \binom{4}{2} \cdot 6 \cdot 4! - \binom{4}{3} \cdot 6 \cdot 4 \cdot 2 + 6 \cdot 4 \cdot 2 \cdot 1 = 2880. \]

Note that each of the numbers we computed for identical seats are just the numbers of those with different animals divided by 8. This reflects that the first case (seat with different animals) the seating are linear permutations, while the identical cases are circular permutations and the total numbers differed by a factor of 8.

29. How many circular permutations are there of the multiset \( \{3 \cdot a, 4 \cdot b, 2 \cdot c, 1\cdot\} \), where for each type of letters, all letters of that type do not appear consecutively.

(This is an example of unclear statement: “all letters ... do not ...”. It should be interpreted as not all letter of each type appear consecutively, it is possible some letter of same type (as long as not all) can appear consecutively.)

Solution. Note the general problem if circular permutations of a multiset is much more complicated. However, in this problem, there one type with repetition 1. One can simply place \( d \) on a circle (with just one way to do so). Then the total number of circular permutations of the original multiset is the same as the number of linear permutations of the multiset \( \{3 \cdot a, 4 \cdot b, 2 \cdot c\} \), which can be computed as follows:

Let \( S \) be the set of all linear permutations of the multiset \( \{3 \cdot a, 4 \cdot b, 2 \cdot c\} \). Then \( |S| = \frac{9!}{3! \cdot 4! \cdot 2!} \) (by Theorem 3.4.2).

Let \( A_1 \) be the set of all permutations of \( \{3 \cdot a, 4 \cdot b, 2 \cdot c\} \) with all three \( a \)'s appearing consecutively, \( A_2 \) be the set of all permutations of \( \{3 \cdot a, 4 \cdot b, 2 \cdot c\} \) with all 4 \( b \)'s appearing consecutively, and \( A_3 \) the set of all permutations with all 2 \( c \)'s appearing consecutively.

Then \( |A_1| = \) the number of permutations of the multiset \( \{1 \cdot a, 4 \cdot b, 2 \cdot c\} \) i.e., \( |A_1| = \frac{7!}{1! \cdot 4! \cdot 2!} \).

Similarly, \( |A_2| = \frac{6!}{4! \cdot 1! \cdot 2!} \), \( |A_3| = \frac{8!}{3! \cdot 4! \cdot 1!} \).

\[ |A_1 \cap A_2| = \frac{4!}{4! \cdot 1! \cdot 1!} \cdot \frac{6!}{4! \cdot 1! \cdot 2!} \cdot \frac{6!}{4! \cdot 1! \cdot 2!} \cdot \frac{5!}{3! \cdot 1! \cdot 1!} \cdot \frac{5!}{3! \cdot 1! \cdot 1!} \cdot \frac{3!}{2! \cdot 1! \cdot 1!} = 1260 \]

\[ |A_1 \cap A_2 \cap A_3| = 3 \cdot 3 \cdot 2 = 18 \]

\[ |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = \frac{9!}{3! \cdot 4! \cdot 2!} - \frac{7!}{1! \cdot 4! \cdot 2!} - \frac{6!}{3! \cdot 1! \cdot 2!} - \frac{8!}{3! \cdot 4! \cdot 1!} + \frac{4!}{1! \cdot 1! \cdot 2!} + \frac{1! \cdot 4! \cdot 1!}{1! \cdot 1! \cdot 1!} = 781 \]