3. Consider the sum of the binomial coefficients along the diagonals of Pascal’s triangle running upward from the left. The first few are: 1, 1, 1 + 1 = 2, 1 + 2 = 3, 1 + 3 + 1 = 5, 1 + 4 + 3 = 8. Computer several more of these diagonal sums and determine how these sums are related.

**Solution.** Let us set \(a_0 = 1\), \(a_1 = 1\), and \(a_2 = 2\), ... with \(a_n\) representing the sum starting with \(\binom{n}{0}\). By the examples given, one should be able to see the relation \(a_n = a_{n-1} + a_{n-2}\). (Do a few more list if you can not see the pattern). How the observation from the first few numbers does not always guarantee that the relation always holds. A proof must be given. Note that diagonal sums in the question is \(a_{n} = \sum_{r=0}^{n} \binom{n-r}{r}\). One should note that \(\binom{k}{r}\) = 0 if \(r > k \geq 0\) or \(k < 0\) are all integers. By Pascal’s formula, we have

\[
a_{n} = \sum_{r=0}^{n} \left( \binom{n-r-1}{r} + \binom{n-r-1}{r-1} \right)
\]

\[
= \sum_{r=0}^{n} \left( \binom{n-1-r-1}{r} + \sum_{k=0}^{n-1} \binom{n-2-k}{k} \right)
\]

\[
= a_{n-1} + a_{n-2}.
\]

Here in the second sum, we have changed the index by \(k = r - 1\).

5. Expand \((2x - y)^7\) using the binomial coefficient theorem.

**Solution.**

\[
(2x - y)^7 = \binom{7}{0}(2x)^7 + \binom{7}{1}(2x)^6(-y) + \binom{7}{2}(2x)^5(-y)^2 + \binom{7}{3}(2x)^4(-y)^3
\]

\[
+ \binom{7}{4}(2x)^3(-y)^4 + \binom{7}{5}(2x)^2(-y)^5 + \binom{7}{6}(2x)^1(-y)^6 + \binom{7}{7}(-y)^7
\]

\[
= \binom{7}{0}2^7x^7 - \binom{7}{1}2^6x^6y + \binom{7}{2}2^5x^5y^2 - \binom{7}{3}2^4x^4y^3 + \binom{7}{4}2^3x^3y^4
\]

\[
- \binom{7}{5}2^2x^2y^5 + \binom{7}{6}2x^1y^6 - \binom{7}{7}y^7.
\]

8. Use the binomial theorem to prove that

\[2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}.\]

**Proof.** In the binomial theorem, we set \(x = 3\) and \(y = -1\). Then the we have

\[2^n = (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

\[= \sum_{k=0}^{n} \binom{n}{k} 3^{n-k}(-1)^k.
\]
12. Let $n$ be a positive integer. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \binom{n}{m} 2^m & \text{if } n = 2m. \end{cases}$$

**Proof.** Let us consider identity of polynomials $(1 + x)^n(1 - x)^n = (1 - x^2)^n$ and then use the binomial theorem to expand both sides.

$$(1 - x^2)^n = \sum_{k=0}^{n} \binom{n}{k} (-x^2)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2$$

$$(1 - x)^n(1 + x)^n = \left( \sum_{i=0}^{n} \binom{n}{i} (-x)^i \right) \left( \sum_{j=0}^{n} \binom{n}{j} x^j \right)$$

$$= \sum_{k=0}^{2n} \left( \sum_{i=0}^{k} \binom{n}{i} (-1)^i \binom{n}{k-i} \right) x^k.$$ 

We now compare the coefficient of $x^n$ in both expansions. The coefficient in the second expansion is

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \binom{n}{n-i} = \sum_{i=0}^{n} (-1)^i \binom{n}{i}^2,$$

which is exactly the left hand side of the identity to be proved. If $n$ is odd, then the coefficient of $x^n$ in the first expansion is 0. If $n = 2m$ is even, then the coefficient of $x^n$ in the first expansion is $(-1)^m \binom{n}{m}$ by $2k = n = 2m$. This proves the given identity.

16. By integrating the binomial expansion, prove that for a positive integer $n$,

$$1 + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \cdots + \frac{1}{n+1} \binom{n}{n} = \frac{2^{n+1} - 1}{n+1}.$$ 

**Proof.** Consider the polynomial function

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Take the definite integral on the interval $[0, 1]$ we have

$$\int_{0}^{1} (1 + x)^n \, dx = \frac{1}{n+1} (1 + x)^{n+1}|_{0}^{1} = \frac{1}{n+1} ((1 + 1)^{n+1} - (1 + 0)^{n+1}) = \frac{2^{n+1} - 1}{n+1}.$$ 

This is exactly the right hand side of the identity. The integration on $[0, 1]$ to the expansion is

$$\sum_{k=0}^{n} \binom{n}{k} \int_{0}^{1} x^k \, dx = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}.$$
which is just the left hand side of the identity. Thus the both sides of the identity holds for all positive integer \( n \).

22. Prove that for all real numbers \( r \) and all integers \( k \) and \( m \),

\[
\left( \begin{array}{c} r \\ m \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) = \left( \begin{array}{c} r \\ k \end{array} \right) \left( \begin{array}{c} r - k \\ m - k \end{array} \right).
\]

Proof. First by convention, we have \( \left( \begin{array}{c} m \\ k \end{array} \right) = 0 \) if \( k > m \geq 0 \) are integers or \( k < 0 \) is an integer.

If \( k < 0 \) then both sides are zero. If \( k \geq 0 \) and \( m < k \) then both sides are zero. Thus we only need to consider cases when \( m \geq k \geq 0 \) are integers. In this case

\[
\left( \begin{array}{c} r \\ m \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) = \frac{r(r - 1) \cdots (r - m + 1) \frac{m!}{k!(m - k)!}}{m!} = \frac{r(r - 1) \cdots (r - m + 1)}{k!(m - k)!} = \frac{r(r - 1) \cdots (r - k + 1) (r - k)(r - k - 1) \cdots (r - m + 1)}{k!(m - k)!} = \frac{r(r - 1) \cdots (r - k + 1) (r - k)(r - k - 1) \cdots (r - k - (m - k) + 1)}{k!(m - k)!} = \left( \begin{array}{c} r \\ k \end{array} \right) \left( \begin{array}{c} r - k \\ m - k \end{array} \right).
\]