16. \( \binom{n}{3} \) Prove that in a group of \( n > 1 \) people there are two who have the same number of acquaintances in the group. (it is assumed that no is acquainted with himself or herself.)

**Proof.** Let \( a_1, a_2, \ldots, a_n \) be the number of people acquainted with the first, second, \ldots, and the last person respectively. Then \( 0 \leq a_i \leq n - 1 \) for all \( n \). If these \( n \) numbers are all distinct, then they have to take each and all the values \( 0, 1, 2, \ldots, n - 1 \) by Pigeonhole principle. Thus \( a_i = 0 \) and \( a_j = n - 1 \) for some \( i \) and \( j \). Since \( n > 1 \), then \( i \neq j \). But this is impossible since the \( j \)-th person is acquainted with all other \( n - 1 \) people in the group (everyone except himself), in particular the \( i \)-th person. But the \( i \)-th person is acquainted with no one, in particular the \( j \)-th person. Since we assumed that acquaintance is a mutual acquaintance. This is a contradiction and is impossible to both numbers 0 and \( n - 1 \) be taken as the values of \( a_1, \ldots, a_n \).

18. Prove that of any five points chosen within a square of side length 2, there are two whose distance apart is at most \( \sqrt{2} \).

**Proof.** We divide the square with side 2 into four squares with side 1 by bisecting sides. Each of the five points has to be in one of the four small squares. By Pigeonhole principle, there must be a small square containing at least two of the five points. The diameter of the square (i.e., the largest possible distance between two points in the square) is the length diagonal, which is \( \sqrt{2} \). Thus the distance between those two points which lies in the same small square is at most \( \sqrt{2} \).

23. The line segments joining 10 points are arbitrarily colored red or blue. Prove that there must exist 3 points such that the three line segments joining them are all red or there exist 4 points such that all 6 lines joining them are blue.

**Proof.** Pick one the ten points, say, the point 1. There nine line segments joining 1. Either there are four the nine line segments are red or there are six of the nine line segments are blue.

If there are four line segments are red, let us say the four points are \( a, b, c, d \) joining 1 with red line segments. If the there is red line segment joining \( a, b, c, d \). Then that line segments two end points together with the point 1 form the three points such that all three line segments joining them are red. Otherwise, there is no red line segments joining \( a, b, c, d \). Then all line segments joining \( a, b, c, d \) are blue.

In the second situation when there are six blue line segments joining 1, say, \( a, b, c, d, e, f \) are joining 1 with blue line segments. Since each line segment joining these six points are either red of blue and we know that \( K_6 \rightarrow K_3, K_3 \). Thus there are either three points among \( a, b, c, d, e, f \) such that all line segments joining them are red or there are three points among \( a, b, c, d, e, f \) such that all line segments joining them are blue. In the former situation we have three points all joined by red line segments and the latter situation, those three point together with the point 1 form the four points such that all line segments joining them are blue.

In summary, on matter what there is either an red \( K_3 \) or blue \( K_4 \).
Section 3.6

2. How many orderings are there for a deck of cards if all the cards of the same suit are together?

**Solution.** A deck of cards consists of 13 cards for each of the four suits, Spades, Diamonds, Hearts, and Clubs. The ordering for each suit is 13!. But there are 4! ways to order the four suits and with order of the suits, there are $13! \times 13! \times 13! \times 13!$ ways to order the cards. Thus the total number of ways ordering a deck of cards with each suits together is $4! \times 13!^4 = 3.608548172 \times 10^{40}$.

4. How many distinct positive divisors do each of the following numbers have?
   (a). $3^4 \times 5^2 \times 7^6 \times 11$.

   **Solution.** There are $(4 + 1)(2 + 1)(6 + 1)(1 + 1) = 210$ positive divisors.

   (b). 620.

   **Solution.** $620 = 31 \times 20 = 2^2 \times 5 \times 31$. Then there are $(2 + 1)(1 + 1)(1 + 1) = 12$ positive divisors for 620.

   (c). $10^{10}$.

   **Solution.** $10^{10} = 2^{10} \times 5^{10}$ with both 2 and 5 prime. Then $10^{10}$ has $(10 + 1)(10 + 1) = 121$ positive divisors.

6. How many integers greater than 5400 have the properties: (a) the digits are distinct, and (b) 2 and 7 do not occur.

**Solution.** Since 2 and 7 do not occur, and all digits are distinct, the numbers should have at most eight digits and at least 4. Let $S_4$ be the set of those integers satisfying the required conditions and have exactly four digits. Similarly, $S_5, S_6, S_7,$ and $S_8$ are defined. The numbers in $S_5$ can further be divided into two groups with either the leading number is larger than 5 (thus only 6, 8, 9 are possible choice) or equal to 5 (the second place can be 4, 6, 8, 9). Thus

$$|S_4| = 3 \times 7 \times 6 \times 5 + 4 \times 6 \times 5 = 750.$$ \[|S_5|, |S_6|, |S_7|, |S_8|\] are easier to compute with 7 choices for the leading digit (0 is not leading digit).

$$|S_5| = 7 \times 7 \times 6 \times 5 \times 4 = 5880, \quad |S_6| = 7 \times 7 \times 6 \times 5 \times 4 \times 3 = 17640,$$

$$|S_7| = |S_8| = 7 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 35280.$$ 

Thus by the addition principle, the total number of such numbers is

$$|S_4| + |S_5| + |S_6| + |S_7| + |S_8| = 750 + 5880 + 17640 + 2 \times 35280 = 94830.$$ 

7. Determine the number of poker hands of the following types:

(a) Full houses (3 cards of one rank and 2 of a different rank).

(b) Straights (5 consecutive ranks).

(c) Flushes (5 cards of the same suit).

(d) Straight flushes (5 consecutive cards of the same suit).
(e) Exactly two pairs (2 cards of the one rank, 2 cards of another rank and 1 card of a third rank).

(f) Exactly pair (2 cards of one rank and 3 cards of three other different ranks).

**Solution.**

(a) To make a full house, one choose one rank of the three cards and another rank of 2 cards. There are \( \binom{13}{1} \binom{4}{2} \) ways to choose three cards of the same rank. Then there are \( \binom{12}{1} \binom{4}{2} \) ways to make the remaining to cards of the same rank. Thus the multiplication principle implies that the total number of ways to make a full house is
\[
13 \cdot 12 \cdot \binom{4}{2} = 13 \cdot 12 \cdot 4 \cdot 6.
\]

(b) To make a straight, only need to choose the beginning rank of the five consecutive ranks (the possible choices are \( \{1, 2, \ldots, 9\} \)) and then choose one card for each of the five chosen ranks (there are \( 4^5 \) ways). Thus the total number of the straights is \( 9 \cdot 4^5 \).

(c) There four suits to choose and then there five cards to choose out of 13 cards to the chosen suit. Hence there are totally \( 4 \cdot \binom{13}{5} \) flushes.

(d) Step 1: choose a suit, (there are 4 ways). Step 2, choose the straight from the chosen suit (there are 9 ways). Thus the total number of straight flushes is \( 4 \cdot 9 = 36 \) different straight flushes.

(e) First choose 2 ranks to get the two pairs. There are \( \binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \). The choose the third rank and get the last card of the different rank. There are \( \binom{11}{1} \cdot \binom{4}{1} = 44 \) ways to choose the last card. Hence the total number of “exactly two pairs” is \( \binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 \).

(f) Choose one rank and get the pair. There are \( \binom{13}{1} \cdot \binom{4}{2} \) outcomes. Now choose three different ranks to form three cards of three different other ranks. There are \( \binom{12}{3} \cdot \binom{4}{1}^3 = \binom{12}{3} \cdot 4^3 \). Thus total number of “exactly one pair” is \( 13 \cdot 6 \cdot \binom{12}{3} \cdot 4^3 \).