Structure of finite $W$-algebras

in type A

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1. \( W \)-algebras

\( g \) finite dimensional reductive Lie algebra/\( \mathbb{C} \).

\( \langle \cdot, \cdot \rangle \) non-degenerate symmetric invariant bilinear form on \( g \).

\( e \in g \) nilpotent.

**Definition** (Kac-Roan-Wakimoto). A \( \mathbb{Z} \)-grading

\[
g = \bigoplus_{j \in \mathbb{Z}} g_j
\]

is good for \( e \) if \( e \in g_2 \) and \( \text{ad} e : g_j \to g_{j+2} \) is injective for \( j \leq -1 \) and surjective for \( j \geq -1 \).

**Example 1.** Jacobson-Morozov grading

\[
g_j := \{ x \in g \mid [h, x] = jx \}
\]

where \( (e, h, f) \) is an \( \mathfrak{sl}_2 \)-triple.

Elashvili and Kac classify the good gradings of semisimple \( g \) up to conjugacy.
Example. \[ \mathfrak{g} = \mathfrak{gl}_{12}(\mathbb{C}) \].

\[ \deg e_{ij} := \text{col}(j) - \text{col}(i) \]

\[ e = e_{5,9} + e_{1,3} + e_{3,3} + e_{3,10} + e_{2,4} + e_{6,8} + e_{8,n} + e_{n,12} \]
Example 2. $g = gl_N(\mathbb{C})$. Here good gradings are classified by the pyramids with $N$ boxes.

Following Kawanaka, Moeglin, Premet, we consider the symplectic form $\langle \cdot , \cdot \rangle$ on $g_{-1}$:

$$\langle x , y \rangle := ([x , y] , e) \quad (x , y \in g_{-1}).$$

$\ell$ a Lagrangian subspace of $g_{-1}$ wrt $\langle \cdot , \cdot \rangle$.

$m := (\bigoplus_{j \leq -2} g_j) \oplus \ell$. It is clear that $m$ is a nilpotent subalgebra of $g$.

$\chi : m \to \mathbb{C} , \ x \mapsto (x , e)$ is a representation of $m$.

$\mathbb{C}_\chi$ the corresponding $U(m)$-module.

$W := \text{End}_g(U(g) \otimes_{U(m)} \mathbb{C}_\chi)^{\text{op}}$, finite $W$-algebra.

$W$ has a very interesting structure and representation theory. Important for physics (de-Boer, Tjin, ...), non-commutative geometry
(Premet, Gan-Ginzburg), representations of quivers (Crawley-Boevey and Holland), finite groups (Kawanaka), modular Lie algebras (Premet), Yangians (Ragoucy and Sorba), etc.

(Skryabin’02, Lynch’79 for even gradings) The category of $W$-modules is equivalent to the category of $\mathfrak{g}$-modules on which $x - \chi(x)$ act locally nilpotently for all $x \in \mathfrak{m}$ (generalized Whittaker modules).

(Premet’02, Gan and Ginzburg’02, Lynch’79 for even gradings) $W$ has a filtration with

$$\text{gr}W \cong \mathbb{C}[e + \mathfrak{z}_\mathfrak{g}(f)]$$

as Poisson algebras.

(Kostant’78) If $e$ is regular, $W \cong Z(U(\mathfrak{g}))$, a key result in the classical theory of Whittaker modules.

Plan: to describe the structure of $W$ and its finite dimensional simple modules for $\mathfrak{g} = \mathfrak{gl}_N$. 
Dirty type A trick: We can shift ‘odd’ rows of the pyramid one box to the left to get an even grading, but \( \chi \) and \( m \) will not change. For this new grading, we have

\[
m = \bigoplus_{j < 0} g_j \text{ and } g = m \oplus p
\]

where

\[
p = \bigoplus_{j \geq 0} g_j
\]

is a parabolic with Levi subalgebra \( l = g_0 \). It follows from PBW-theorem that

\[
U(g) = U(p) \oplus U(g)(x - \chi(x) \mid x \in m).
\]

Let

\[
pr : U(g) \to U(p)
\]

be the projection along this direct sum decomposition. Consider the twisted \( m \)-action:

\[
x \cdot y = pr([x,y]) \quad (x \in m, \ y \in U(p)).
\]

We claim that

\[
W \cong U(p)^m \subset U(p).
\]
This follows from Frobenius reciprocity:

\[ W = \text{Hom}_g(U(g) \otimes_{U(m)} \mathbb{C}_\chi, U(g) \otimes_{U(m)} \mathbb{C}_\chi) \]
\[ = \text{Hom}_m(\mathbb{C}_\chi, U(g) \otimes_{U(m)} \mathbb{C}_\chi) \]
\[ = \text{Hom}_m(\mathbb{C}_\chi, U(p)) \]
\[ = U(p)^m. \]

Remark. Since from now on our grading is even, it makes sense to divide by 2 to get a shape of a ‘real’ pyramid:
2. Yangians

**Definition.** The Yangian of $\mathfrak{gl}_n$, denoted $Y_n$, is the associative unital algebra given by generators

$$\{T_{i,j}^{(r)} \mid 1 \leq i,j \leq n, \ r \in \mathbb{Z}_{>0}\},$$

with $T_{i,j}^{(0)}$ interpreted as $\delta_{i,j}1$, and relations

$$[T_{i,j}^{(r)}, T_{h,k}^{(s)}] = \min(r,s)-1 \sum_{t=0}^{\min(r,s)-1} \left( T_{i,k}^{(r+s-1-t)}T_{h,j}^{(t)} - T_{i,k}^{(t)}T_{h,j}^{(r+s-1-t)} \right)$$

for every $1 \leq h, i, j, k \leq n$ and $r, s \geq 0$.

Let $u$ be an indeterminate, and define

$$T_{i,j}(u) := \sum_{r \geq 0} T_{i,j}^{(r)} u^{-r} \in Y_n[[u^{-1}]].$$

$Y_n$ is actually a Hopf algebra. The coproduct is given by

$$\Delta : T_{i,j}(u) \mapsto \sum_{k=1}^{n} T_{i,k}(u) \otimes T_{k,j}(u).$$
There is another important set of generators of \( Y_n \), called \textit{Drinfeld generators}. These are defined from Gauss factorization:

\[
T(u) = F(u)D(u)E(u),
\]

where

\[
D(u) = \begin{pmatrix}
D_1(u) & 0 & \cdots & 0 \\
0 & D_2(u) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_n(u)
\end{pmatrix},
\]

\[
E(u) = \begin{pmatrix}
1 & E_{1,2}(u) & \cdots & E_{1,n}(u) \\
0 & 1 & \cdots & E_{2,n}(u) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix},
\]

\[
F(u) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
F_{1,2}(u) & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
F_{1,n}(u) & F_{2,n}(u) & \cdots & 1
\end{pmatrix}.
\]

This defines power series

\[
D_i(u) = \sum_{r \geq 0} D_i^{(r)} u^{-r},
\]
\[ E_{i,j}(u) = \sum_{r \geq 1} E_{i,j}^{(r)} u^{-r}, \]
\[ F_{i,j}(u) = \sum_{r \geq 1} F_{i,j}^{(r)} u^{-r}. \]

Let
\[ E_i(u) = \sum_{r \geq 1} E_i^{(r)} u^{-r} := E_{i,i+1}(u), \]
\[ F_i(u) = \sum_{r \geq 1} F_i^{(r)} u^{-r} := F_{i,i+1}(u) \]
for short. Also let
\[ \widetilde{D}_i(u) = \sum_{r \geq 0} \widetilde{D}_i^{(r)} u^{-r} := -D_i(u)^{-1}. \]

**Theorem** (Drinfeld, sort of). The algebra $Y_n$ is generated by the elements
\[ \{D_i^{(r)}\}_{1 \leq i \leq n, r \geq 0}, \{E_i^{(r)}, F_i^{(r)}\}_{1 \leq i < n, r \geq 1} \]
subject only to the following relations:
Drinfeld relations:

\[[D_i^{(r)}, D_j^{(s)}] = 0\]

\[[E_i^{(r)}, F_j^{(s)}] = \delta_{ij} \sum_{t=0}^{r+s-1} \tilde{D}_i^{(t)} D_j^{(r+s-1-t)}\]

\[[D_i^{(r)}, E_j^{(s)}] = (\delta_{ij} - \delta_{ij+1}) \sum_{t=0}^{r+s-1} D_i^{(t)} E_j^{(r+s-1-t)}\]

\[[D_i^{(r)}, F_j^{(s)}] = (\delta_{ij} - \delta_{ij+1}) \sum_{t=0}^{r+s-1} F_j^{(t)} D_i^{(r+s-1-t)}\]

\[[E_i^{(r)}, E_i^{(s)}] = \sum_{t=0}^{r+s-1} E_i^{(t)} E_i^{(r+s-1-t)} - \sum_{t=0}^{r+s-1} E_i^{(t)} E_i^{(r+s-1-t)}\]

\[[F_i^{(r)}, F_i^{(s)}] = \sum_{t=0}^{r+s-1} F_i^{(t)} F_i^{(r+s-1-t)} - \sum_{t=0}^{r+s-1} F_i^{(t)} F_i^{(r+s-1-t)}\]

\[[E_i^{(r)}, E_{i+1}^{(s)}] - [E_i^{(r)}, E_{i+1}^{(s)}] = -E_i^{(r)} E_{i+1}^{(s)}\]

\[[F_i^{(r)}, F_{i+1}^{(s)}] - [F_i^{(r)}, F_{i+1}^{(s)}] = -F_{i+1}^{(s)} F_i^{(r)}\]

\[[E_i^{(r)}, E_j^{(s)}] = 0 = [F_i^{(r)}, F_j^{(s)}]\] if \( 1 \cdot i - j \cdot 1 > 1 \)

\[[E_i^{(r)}, [E_i^{(s)}, E_j^{(t)}]] + [E_i^{(s)}, [E_i^{(r)}, E_j^{(t)}]] = 0\] if \( 1 \cdot i - j \cdot 1 = 1 \)

\[[F_i^{(r)}, [F_i^{(s)}, F_j^{(t)}]] + [F_i^{(s)}, [F_i^{(r)}, F_j^{(t)}]] = 0\] if \( 1 \cdot i - j \cdot 1 = 1 \)

Here \( \tilde{D}_i^{(r)} \) is defined from \( \tilde{D}_i(u) = -D_i(u)^{-1} \)

(Note that \( D_i^{(0)} = 1 \)).
Let $\pi$ be a pyramid with $n$ rows of lengths $p_1 \leq p_2 \leq \ldots \leq p_n$. For $1 \leq i \leq j \leq n$ denote by $s_{ij}$ (resp. $s_{ji}$) the difference between the right end of the $j$th row and the right end of the $i$th row (resp. the difference between the left end of the $j$th row and the left end of the $i$th row).

**Definition** The truncated shifted Yangian $Y(\pi)$ is the subalgebra of $Y_n$ generated by all

$$ \{D_{i}^{(r)}\}_{1 \leq i \leq n, r > 0}, $$
$$ \{E_{i}^{(r)}\}_{1 < i < n, r > s_{i,i+1}}, \quad (*) $$
$$ \{F_{i}^{(r)}\}_{1 < i < n, r > s_{i+1,i}}. $$

factored out by the ideal generated by all

$$ \{D_{1}^{(r)} \mid r > p_1\}. $$

We use the same symbols for elements of $Y(\pi)$ as for $Y_n$, e.g. $E_{i}^{(r)}$ denotes the coset of $E_{i}^{(r)}$ in the quotient $Y(\pi)$. 
Theorem. $Y(\pi)$ is generated by (the cosets of) the generators (*) subject only to the Drinfeld relations which make sense for them, and relations $D_{1}^{(r)} = 0$ for $r > p_1$.

Corollary. The isomorphism type of $Y(\pi)$ is determined by the row lengths $p_1, p_2, \ldots, p_n$, the isomorphism given by the corresponding shift in the upper indices.

Theorem. The monomials in the elements

\[
\{D_{i}^{(r)}\} 1\leq i \leq n, 1 \leq r \leq p_i,
\]

\[
\{E_{i,j}^{(r)}\} 1 \leq i < j \leq n, s_{i,j} < r \leq s_{i,j} + p_i,
\]

\[
\{F_{i,j}^{(r)}\} 1 \leq i < j \leq n, s_{j,i} < r \leq s_{j,i} + p_i
\]

taken in some fixed order form a basis for $Y(\pi)$.

Let $\pi = \pi' \sqcup \pi''$ be a decomposition of the pyramid $\pi$ into the union of two pyramids obtained by cutting $\pi$ with a vertical line.
Theorem. The coproduct $\Delta$ factors through to give a homomorphism

$$\Delta : Y(\pi) \to Y(\pi') \otimes Y(\pi'').$$

Moreover, $\Delta$ is injective. In particular, by iterating, we can embed $Y(\pi)$ into

$$U(gl_{q_1}) \otimes \ldots \otimes U(gl_{q_l})$$

where $q_1, \ldots, q_l$ are the column lengths of the pyramid $\pi$.

Main Theorem. $W(\pi)$ is isomorphic to $Y(\pi)$.

Remark. It follows from the Main Theorem and a corollary above that the isomorphism type of $W(\pi)$ depends only on the row lengths of $\pi$, i.e. only on the conjugacy class of $e$.

Remark. In the case of $e$ having all blocks of the same size this was observed by Ragoucy and Sorba in 1999.
Remark. Our isomorphism is explicit: we construct the invariants in \( U(\mathfrak{p})^m = W(\pi) \) which correspond to the generators \( E_i^{(r)}, F_i^{(r)}, D_i^{(r)} \). These are in spirit of classical invariant theory.

Remark. A conceptual way to see the isomorphism between \( Y(\pi) \) and \( W(\pi) \) is to embed them both into \( U(\mathfrak{gl}_{q_1}) \otimes \ldots \otimes U(\mathfrak{gl}_{q_l}) \), and identify the images. For \( Y(\pi) \) this is done using coproduct \( \Delta \), as explained above. For \( W(\pi) \) embedding is more transparent. It comes from the restriction to \( W(\pi) = U(\mathfrak{p})^m \) of the natural projection

\[
U(\mathfrak{p}) \to U(\mathfrak{l}) = U(\mathfrak{gl}_{q_1}) \otimes \ldots \otimes U(\mathfrak{gl}_{q_l}),
\]

followed by a shift by \( \rho \). It is non-trivial that the restriction is injective though.
Remark We do not know an invariant description of the image of $W(\pi)$ in $U(\mathfrak{gl}_{q_1}) \otimes \ldots \otimes U(\mathfrak{gl}_{q_l})$. In the special case where $e$ is regular, the image is $\mathbb{C}[\mathfrak{h}]^W$, in agreement with Kostant and Chevalley.

Remark. The coproduct $\Delta$ has a very natural interpretation in terms of $W$-algebras.
3. Representation Theory

**Theorem** Every irreducible finite dimensional $W(\pi)$-module $V$ has a unique up to scalar high weight vector, i.e. a vector $v_+$ killed by all $E_i^{(r)}$ and on which all $D_i^{(r)}$ act diagonally. The isomorphic type of $V$ is determined by the eigenvalues of the $D_i^{(r)}$ (the ‘highest weight’). Moreover, for each $i$ we have $D_i^{(r)}v_+ = 0$ for $r > p_i$.

Now, let us analyze the ‘highest weight’ in more detail. In view of the theorem we can write

\[
\begin{align*}
u^{p_1} D_1(u)v_+ &= P_1(u)v_+, \\
(u - 1)^{p_2} D_2(u - 1)v_+ &= P_2(u)v_+, \\
&\vdots \\
(u - n + 1)^{p_n} D_n(u - n + 1)v_+ &= P_n(u)v_+.
\end{align*}
\]

for monic polynomials $P_1(u), \ldots, P_n(u)$ of degrees $p_1, \ldots, p_n$, respectively.
We denote our irreducible module
\[ V(P_1(u), \ldots, P_n(u)). \]

Next, decompose each polynomial into linear factors:
\[ P_i(u) = (u + a_{i1}) \ldots (u + a_{i,p_i}) \quad (1 \leq i \leq n). \]

The answer is in terms of the array of the roots \( a_{ij} \). For pedagogical reasons let us make two non-essential assumptions: a) that \( \pi \) has a shape of a Young diagram (in French notation), b) that all roots are integers. Now, order the roots so that
\[ a_{i1} \leq a_{i2} \leq \ldots \leq a_{i,p_i} \quad (1 \leq i \leq n). \]

and allocate the number \( a_{ij} \) into the \((i, j)\) box of \( \pi \) to get a \( \pi \)-tableau.

**Theorem.** The irreducible module
\[ V(P_1(u), \ldots, P_n(u)) \]
is finite dimensional if and only the corresponding \( \pi \)-tableau is standard, i.e. entries increase from bottom to top along the columns.