THE SECOND MOMENT OF $GL(4) \times GL(2)$ $L$-FUNCTIONS AT SPECIAL POINTS

VORRAPAN CHANDEE AND XIANNAN LI

Abstract. In this paper, we obtain upper bounds for the second moment of $L(u_j \times \phi, \frac{1}{2} + it_j)$, where $\phi$ is a Hecke Maass form for $SL(4, \mathbb{Z})$, and $u_j$ is taken from an orthonormal basis of Hecke-Maass forms on $SL(2, \mathbb{Z})$ with eigenvalue $1/4 + t_j^2$. The bounds are consistent with the Lindelöf hypothesis. Previously these types of upper bounds are available for only $GL(n) \times GL(2)$, where $n \leq 3$.

1. Introduction

Statistical distribution of the values of $L$-functions are fundamentally related to interesting arithmetic objects. Here, we are interested in understanding a $GL(4) \times GL(2)$ Rankin-Selberg $L$-functions at special points. To be precise, we shall study the second moment of $L(u_j \times \phi, \frac{1}{2} + it_j)$, where $\phi$ is a Hecke Maass form for $SL(4, \mathbb{Z})$ of type $(\nu_1, \nu_2, \nu_3)$, and $u_j$ is taken from an orthonormal basis of Hecke-Maass forms on $SL(2, \mathbb{Z})$ with eigenvalue $1/4 + t_j^2$.

The point $\frac{1}{2} + it_j$ is a zero of the Selberg zeta function, and the question of how certain $L$-functions behave at these points appears in the work of Phillips and Sarnak [23] on deformation of cusp forms. These points are also distinguished in the analytic theory of $L$-functions for leading to conductor dropping. The latter phenomenon has made subconvex results at these points quite difficult to achieve through moments computation.

In particular, the conductor dropping necessitates understanding higher moments when deriving subconvexity results while making high moments apparently more achievable. The latter statement should be viewed critically - in particular, the study of such families introduces unfamiliar and delicate problems. It is of great interest to understand how the difficulty scales. In this direction, Luo [18] showed that

$$\sum_{t_j \leq T} \left| L(u_j, \frac{1}{2} + it_j) \right|^6 \ll T^{9/4+\epsilon},$$

and

$$\sum_{t_j \leq T} \left| L(u_j, \frac{1}{2} + it_j) \right|^8 \ll T^{5/2+\epsilon}.$$
This should be compared to the conjectured optimal bound of
\[
\sum_{t_j \leq T} \left| L(u_j, \frac{1}{2} + it_j) \right|^{2k} \ll T^{2+\epsilon},
\]
for all \( k \geq 0 \). Achieving the above bound for all \( k \) implies the deep Lindelöf hypothesis, while subconvexity follows if the bound is achieved for any \( k > 4 \). Luo’s work was based on a large sieve type inequality he derived for this family and the power loss in Luo’s result arises from suboptimal large sieve bound. This is a superficial indication of deeper complexities in this \( GL(2) \) family which are still not well understood. Later, Young showed that
\[
\sum_{t_j \leq T} \left| L(u_j, \frac{1}{2} + it_j) \right|^{6} \ll T^{2+\epsilon},
\]
thereby achieving essentially the optimal bound for the sixth moment. This resulted not from general improvements to the large sieve bound, but rather the use of Fourier analytic techniques which use specific information about the coefficients involved. It should be noted that the Young’s sixth moment result above translates without change to give the same quality bound for the second moment of a \( GL(3) \times GL(2) \) family. Specifically, Young [25] proved that
\[
\sum_{t_j \leq T} \left| L(F \times u_j, \frac{1}{2} + it_j) \right|^{2} \ll T^{2+\epsilon},
\]
for \( F \) a fixed \( GL(3) \) Hecke Maass form.

Here, our focus shall be on the second moment of \( GL(4) \times GL(2) \) \( L \)-functions. To be precise, the main object of this paper is to prove the following theorem.

**Theorem 1.1.** With notations as above, we have
\[
\sum_{t_j \leq T} \left| L(u_j, \frac{1}{2} + it_j) \right|^{2} \ll T^{2+\epsilon}.
\]

The techniques underlying the proof of our Theorem 1.1 can be modified to give the following bound on the eighth moment of the \( GL(2) \) \( L \)-functions:
\[
\sum_{t_j \leq T} \left| L(u_j, \frac{1}{2} + it_j) \right|^{8} \ll T^{2+\epsilon}.
\]

As mentioned before, these bounds are consistent with the Lindelöf hypothesis. Recently, the authors proved an analogous result for the eighth moment of the family of holomorphic modular forms with respect to the congruence subgroup \( \Gamma_1(q) \) in [5]. Of course, the basic structure of the two families are radically different.

In §2 to §6 we reduce the moment problem in Theorem 1.1 to the proof of Lemma 4.2, which is the main innovation in this paper. We provide a sketch of the proof of Lemma 4.2 in §4.1.
The initial reduction in §2 to §6 is unexpectedly complicated by the unwieldy Hecke relations for $GL(4)$ and the need for bounds of the form $\sum_{\ell \sim L} |A(1, \ell, 1)|^2 \ll L$. Interestingly, note that the latter bound depends not only on Rankin-Selberg theory, but also the work of Kim on functoriality of the exterior square on $GL(4)$ [13].

2. Initial Setup

We begin by introducing more precise notation. As in definition 9.4.3 in [7], for $\text{Re } s > 1$, the Godement-Jacquet $L$-function associated to $\phi$ is

$$L(\phi, s) = \sum_{n=1}^{\infty} \frac{A(1, 1, n)}{n^s} = \prod_p \left( 1 - \frac{A(1, 1, p)}{p^s} + \frac{A(1, p, 1)}{p^{2s}} - \frac{A(p, 1, 1)}{p^{3s}} + \frac{1}{p^4} \right)^{-1}.$$ 

Here, we normalize the coefficients $A(m_1, m_2, m_3)$ by setting $A(1, 1, 1) = 1$. The dual Maass form $\tilde{\phi}$ is of type $(\nu_3, \nu_2, \nu_1)$ and the Fourier coefficient is $A(m_3, m_2, m_1) = A(m_2, m_1, m_3)$. Hence

$$L(\tilde{\phi}, s) = \sum_{n=1}^{\infty} \frac{A(n, 1, 1)}{n^s} = \prod_p \left( 1 - \frac{A(p, 1, 1)}{p^s} + \frac{A(1, p, 1)}{p^{2s}} - \frac{A(1, 1, p)}{p^{3s}} + \frac{1}{p^4} \right)^{-1}.$$ 

Now let

$$G_{\nu_1, \nu_2, \nu_3}(s) = \pi^{-2s} \prod_{i=1}^{4} \Gamma \left( \frac{s - \alpha_i}{2} \right),$$

where

$$\alpha_1 = \frac{3}{2} - \nu_1 - 2\nu_2 - 3\nu_3$$

$$\alpha_2 = -\frac{3}{2} + 3\nu_1 + 2\nu_2 + \nu_3$$

$$\alpha_3 = -\frac{1}{2} - \nu_1 + 2\nu_2 + \nu_3$$

$$\alpha_4 = \frac{1}{2} - \nu_1 - 2\nu_2 + \nu_3.$$ (2.1)

From Theorem 10.8.6 in [7], the functional equation for $L(\phi, s)$ is

$$G_{\nu_1, \nu_2, \nu_3}(s)L(\phi, s) = G_{\nu_3, \nu_2, \nu_1}(1 - s)L(\tilde{\phi}, 1 - s).$$

Recall that $(u_j)$ forms an orthonormal basis of Hecke-Maass cusp forms on $SL(2, \mathbb{Z})$ with corresponding Laplace eigenvalues $1/4 + t_j^2$. As usual, we write $\lambda_j(n)$ for the Hecke eigenvalue of the $n^{th}$ Hecke operator of the form $u_j$. For $\text{Re } s > 1$, $L(u_j, s)$ is given by

$$L(u_j, s) = \sum_{n=1}^{\infty} \frac{\lambda_j(n)}{n^s}.$$
Following the notation in Chapter 12.3 in [7], the Rankin-Selberg $L$-function is defined by

$$L(u_j \times \phi, s) = \sum_{m,n \geq 1} \sum \lambda_j(n) A(1, m, n) \frac{c(n, u_j \times \phi)}{m^{2s} n^s} := \sum_{n=1}^{\infty} \frac{c(n, u_j \times \phi)}{n^s}. \tag{2.2}$$

For $u_j$ even, the completed $L$-function associated to $L(u_j \times \phi, s)$ is

$$\Lambda(u_j \times \phi, s) = \gamma(s, u_j \times \phi) L(u_j \times \phi, s),$$

where

$$\gamma(s, u_j \times \phi) := \pi^{-4s} \prod_{i=1}^{4} \Gamma\left(\frac{s - it_j - \alpha_i}{2}\right) \Gamma\left(\frac{s + it_j - \alpha_i}{2}\right). \tag{2.3}$$

By Theorem 12.3.6 in [7], the completed $L$-functions satisfies the following functional equation

$$\Lambda(u_j \times \phi, s) = \Lambda(u_j \times \phi, 1 - s).$$

The case where $u_j$ is odd is very similar, the only change being a change in the argument of the Gamma factors appearing in Equation (2) by $1/2$ which corresponds to the change in the Archimedean factors for $L(s, u_j)$. We refer the reader to Chapter 12 of [7] for more details, and specifically to Section 12.3.6 for the functional equation.

We start by reducing our second moment to an appropriate form. Our setup in this section is very similar to the initial setup in Young’s work [25] and we refer to Section 4 - 7 of [25] for a more detailed exposition.

We will first apply the approximate function equation (e.g. Theorem 5.3 in the book [10]). Define a smooth function

$$V(y, s) = \frac{1}{2\pi i} \int_{(3)} y^{-z} \frac{\gamma(s + z, u_j \times \phi) H(z)}{\gamma(s, u_j \times \phi) z} d\zeta,$$

where $\gamma(s, u_j \times \phi)$ is defined in Equation (2.3) and $H(z)$ is an entire function with rapid decay along vertical lines. Moreover, let $\gamma^*(s, u_j \times \phi)$ be a factor in the functional equation

$$\Lambda(u_j \times \phi, s) = \gamma^*(s, u_j \times \phi) L(u_j \times \phi, s),$$

and $V^*(y, s)$ is similar to $V(y, s)$ but replacing $\gamma(s, u_j \times \phi)$ with $\gamma^*(s, u_j \times \phi)$. Then the approximate functional equation gives that for any $Y > 0$

$$L\left(u_j \times \phi, \frac{1}{2} + it_j\right) = \sum_{n} \frac{c(n, u_j \times \phi)}{n^{\frac{1}{2} + it_j}} V\left(\frac{n}{Y}, \frac{1}{2} + it_j\right) + \epsilon_j \sum_{n} \frac{c(n, u_j \times \phi)}{n^{\frac{1}{2} - it_j}} V^*(nY, \frac{1}{2} - it_j),$$

where $|\epsilon_j| = 1$. Due to the Stirling’s formula, the leading term of an asymptotic formula of $V(y, \frac{1}{2} + it_j)$ is

$$V_1(y, t_j) = \frac{1}{2\pi i} \int_{(3)} \left(\frac{t_j^2}{y}\right)^{\frac{1}{2}} h_0(z) \frac{1}{z} ds.$$
where \( h_0(z) \) is a holomorphic function with exponential decay as \( \text{Im}(z) \to \infty \). We also define \( V_2(y, t_j) \) to be the corresponding leading term of an asymptotic formula of \( V^* \), and \( V_2 \) has the same form as \( V_1 \).

It suffices to consider the leading terms of \( V \) and \( V^* \) in what follows. Since there are \( \ll \log T \) dyadic intervals up to \( T^{2+\epsilon} \), we further restrict \( t_j \sim T \) and \( n \sim N \) where \( a \sim A \) is shorthand for \( A < a \leq 2A \). For simplicity, we consider only even forms \( u_j \) - the difference for odd forms is a minor difference in the functional equation.

After applying Cauchy-Schwarz inequality, these terms are

\[
\sum_{t_j \sim T, u_j \text{ even}} \left( \sum_{n \sim P} \frac{c(n, u_j \times \phi)}{n^{\frac{1}{2} + it_j}} w_1 \left( \frac{n}{P}, t_j \right) \right)^2 + \left| \sum_{n \sim P} \frac{c(n, u_j \times \tilde{\phi})}{n^{\frac{1}{2} - it_j}} w_1 \left( \frac{n}{P}, nY, t_j \right) \right|^2,
\]

where \( w_1 \) is a compactly supported smooth function which gives a smooth partition of unity, i.e. \( \sum_{j \geq 1} w_1 \left( \frac{1}{2T}, t_j \right) = 1 \). By the change of variables and combining smooth functions, we need to consider

\[
\sum_{t_j \sim T, u_j \text{ even}} \left| \sum_{n \sim P} \frac{c(n, u_j \times \phi)}{n^{\frac{1}{2} + it_j}} w_2 \left( \frac{n}{P} \right) \right|^2
\]

where \( P \ll T^{2+\epsilon} \) and \( w_2 \) has the same properties as \( w_1 \).

Next we will add the weight \( \alpha_j = |\rho_j(1)|^2 w(t_j) \), where \( \rho_j(1) \) is the Fourier coefficient of the \( u_j(z) \), and the nonnegative smooth function \( w(t_j) \) is defined by

\[
w(t_j) = 2 \frac{\sinh((\pi - \frac{1}{2})t_j)}{\sin(2\pi t_j)}.
\]

We obtain that \( \alpha_j \sim \frac{|\rho_j(1)|^2}{\cosh(\pi t_j)} \exp(-t_j/T) \) when \( t_j \sim T \), and \( t_j^{-\epsilon} \ll \alpha_j \ll t_j^\epsilon \) (see e.g. (5.4) and (5.5) in [25]). The introduction of this weight normalizes our forms for more convenient application of Kuznetsov’s formula. Also, we remove the conditions \( t_j \sim T \) and \( u_j \) is even, allowed by positivity of these terms. Hence it is enough to show that

\[
H := \sum_{t_j} w(t_j) |\rho_j(1)|^2 \left| \sum_{n \sim P} \frac{c(n, u_j \times \phi)}{n^{\frac{1}{2} + it_j}} w_2 \left( \frac{n}{P} \right) \right|^2 \ll T^{2+\epsilon}.
\]

By Equation (2.2), Cauchy-Schwarz inequality and \( P \ll T^{2+\epsilon} \), we have that

\[
H \ll T^\epsilon \sum_{\ell \ll \sqrt{P}} \frac{H_\ell}{\ell}
\]

where

\[
H_\ell = \sum_{t_j} w(t_j) |\rho_j(1)|^2 \left| \sum_{n \sim N} w_2 \left( \frac{n}{N} \right) \frac{A(1, \ell, n) \lambda_j(n)}{n^{1/2 + it_j}} \right|^2,
\]

and \( N = P/\ell^2 \). To prove Equation (2.4), it suffices to show the following two results.
Proposition 2.1. With the above notations and $N = \frac{P}{\ell^2} \ll T^{2+\epsilon}$, we have

$$H_\ell \ll T^{2+\epsilon} \left( 1 + \sum_{n \sim N} \frac{|A(1, \ell, n)|^2}{n} \right).$$

Lemma 2.2. Let $K, L$ be fixed. Then

$$\sum_{k \sim K} \sum_{\ell \sim L} |A(k, \ell, 1)|^2 \ll (KL)^{1+\epsilon}.$$

Thus from Proposition 2.1 and Lemma 2.2, we have that

$$H \ll T^{2+\epsilon} \sum_{\ell \leq \sqrt{P}} \sum_{n \sim \frac{P}{\ell^2}} \frac{|A(1, \ell, n)|^2}{n\ell} \ll T^{2+\epsilon}$$

as desired.

The proof of Lemma 2.2 is more involved than might appear at first sight, and we will prove it in §3. The proof of the main Proposition 2.1 will be the focus of the rest of the paper.

Remarks on notation. As usual, we will use $\epsilon$ to denote a small positive real number, not necessarily the same at each occurrence. At some point in the paper a parameter $\epsilon_1 > 0$ will be fixed once it is introduced and be chosen later. Moreover $e(x) = \exp(2\pi ix)$.

3. Ramanujan on average

We first prove Lemma 2.2, beginning with a Mobius inversion type result which was used by Xiaoqing Li and Young in [17] in an analogous $GL(3)$ case. We will show that

$$(3.1) \quad \sum_{\substack{d|(k,\ell) \, e|(d,k/d) \atop d|d\ell}} \mu(d)\mu(e) A\left(\frac{k}{de}, 1, 1\right) A\left(1, \frac{\ell}{d}, \frac{d}{e}\right) = A(k, \ell, 1).$$

Indeed, the Hecke relations give (for instance, see Theorem 9.3.11 of [7])

$$(3.2) \quad A(k, 1, 1)A(1, \ell, d) = \sum_{c_1,c_2|k \atop c_1|\ell, c_2|d} A\left(\frac{k}{c_1 c_2}, \frac{\ell}{c_1}, \frac{dc_1}{c_2}\right),$$

from which (3.1) follows by standard Mobius inversion type manipulations via

$$\sum_{\substack{d|(k,\ell) \, e|(d,k/d) \atop d|d\ell}} \mu(d)\mu(e) A\left(\frac{k}{de}, 1, 1\right) A\left(1, \frac{\ell}{d}, \frac{d}{e}\right) = \sum_{\substack{d|(k,\ell) \, e|(d,k/d) \atop d|d\ell, e|d, c_2|d/e \atop c_1|\ell/d, c_2|d/e \atop c_1 c_2|k/d}} \mu(d)\mu(e) A\left(\frac{k}{dec_1 c_2}, \frac{\ell}{dc_1}, \frac{dc_1}{ec_2}\right).$$
\[ \sum_{g_1|g_2} A \left( \frac{k}{g_1}, \frac{\ell}{g_1}, g_1 \right) \sum_{d|g_2} \mu(d) = A(k, \ell, 1). \]

Now, by (3.1) and Cauchy-Schwarz,

\[ |A(k, \ell, 1)|^2 = \left| \sum_{d|(k, \ell)} \mu(d) \mu(e) A \left( \frac{k}{de}, 1, 1 \right) A \left( 1, \frac{\ell}{d^2} \right) \right|^2 \]

\[ \leq \left( \sum_{d|(k, \ell)} 1^2 \right) \left( \sum_{(k, \ell)|d, k/d} |\mu(d)| \right) \left( \sum_{(k, \ell)|d, k/d} A \left( \frac{k}{de}, 1, 1 \right) \right)^2 \left( \sum_{(k, \ell)|d, k/d} A \left( 1, \frac{\ell}{d^2} \right) \right)^2 \]

\[ \ll K^\epsilon \sum_{d|(k, \ell)} |\mu(d)| \left| A \left( \frac{k}{de}, 1, 1 \right) \right|^2 \left| A \left( 1, \frac{\ell}{d^2} \right) \right|^2. \]

Hence,

\[ \sum_{k \sim K} \sum_{\ell \sim L} |A(k, \ell, 1)|^2 \ll K^\epsilon \sum_{d \leq \min(L, K)} |\mu(d)| \sum_{d = ef} \left( \sum_{k \sim K/de} |A(k, 1, 1)|^2 \sum_{\ell \sim L/d} |A(1, \ell, f)|^2 \right) \]

\[ \ll K^\epsilon \sum_{d \leq \min(L, K)} |\mu(d)| \sum_{d = ef} \left( \frac{K}{de} \sum_{\ell \sim L/d} |A(1, \ell, f)|^2 \right) \]

since

\[ \sum_{k \sim K/de} |A(k, 1, 1)|^2 \ll \frac{K}{de}, \]

Note that in the sum above, for \( f|d \), we may assume that \( f \) is squarefree, due to the presence of the \( |\mu(d)| \) factor. Thus, Lemma 2.2 follows from the following Lemma.

**Lemma 3.1.** For any positive squarefree integers \( a \) and \( f \)

\[ \sum_{\ell \sim L} |A(a, \ell, f)|^2 \ll afL. \]

Before proving Lemma 3.1, we first state the following Lemma.

**Lemma 3.2.**

\[ \sum_{\ell \sim L} |A(1, \ell, 1)|^2 \ll L. \]
Proof. We note that the exterior square $L$-function $L(s, \phi, \Lambda^2)$ has the Dirichlet series
\[ L(s, \phi, \Lambda^2) = \sum_{\ell \geq 1} \frac{A(1, \ell, 1)}{\ell^s}. \]

Representations of such $L$-functions were studied by Bump and Friedberg [2] as well as Jacquet and Shalika [12]. We refer to the later work of Kontorovich [14] for a relatively elementary proof of (3.3) for $GL(n)$.

Fortunately, we also know that the exterior square is essentially automorphic by the work of Kim. To be precise, Kim’s Theorem A [13] tells us that there is an automorphic $L$-function on $GL(6)$ with the same Euler product as $L(s, \phi, \Lambda^2)$ except possibly the local factors at 2 and 3. Standard contour integration of the Rankin-Selberg $L$-function (see Remark 12.1.8 [7]) then gives us the result of the Lemma. □

Now we turn to the proof of Lemma 3.1. We proceed by induction on $af \lfloor L \rfloor \geq 1$. The base case $af = 1$ is covered by Lemma 3.2 (of course, this also covers the trivial case $af \lfloor L \rfloor = 1$). Thus we assume $af > 1$ and since $a$ and $f$ are in symmetric positions, we may without loss of generality assume that $f > 1$. Then $p \| f$ for some prime $p$ since $f$ is squarefree. By Hecke multiplicativity (Theorem 9.3.11 of [7]) in (3.2) again, we have
\[ A(a, \ell, fp^{-1})A(1, 1, p) = \sum_{c_1 | a \atop c_2 | \ell \atop c_1c_2 | p} A\left(\frac{ac_2}{c_1}, \frac{\ell}{c_2}, \frac{f}{c_1c_2}\right). \]

Rearranging this, we have
\[ \sum_{\ell \sim L} |A(a, \ell, f)|^2 = \sum_{\ell \sim L} A(a, \ell, fp^{-1})A(1, 1, p) - \sum_{c_1 | a \atop c_2 | \ell \atop c_1c_2 = p} A\left(\frac{ac_2}{c_1}, \frac{\ell}{c_2}, \frac{f}{c_1c_2}\right)^2 \]
\[ \leq 48 \sum_{\ell \sim L} |A(a, \ell, fp^{-1})|^2 p^{(1-2/17)} + 3 \sum_{\ell \sim L} \sum_{c_1 | a \atop c_2 | \ell \atop c_1c_2 = p} \left| A\left(\frac{ac_2}{c_1}, \frac{\ell}{c_2}, \frac{f}{c_1c_2}\right)\right|^2, \]

where we have applied Cauchy-Schwarz while noting that the original sum inside the absolute values is of length 3 and have used the bound of Luo, Rudnick and Sarnak of $|A(1, 1, p)| \leq 4p^{1/2-1/17}$ (see Theorem 12.5.1 of Goldfeld’s book [7]). Applying the induction hypothesis, we see that the quantity on the right of (3.4) is
\[ 48 \sum_{\ell \sim L} |A(a, \ell, fp^{-1})|^2 p^{(1-2/17)} + 3 \sum_{\ell \sim L/c_2} \sum_{c_1 | a \atop c_1c_2 = p} \left| A\left(\frac{ac_2}{c_1}, \frac{\ell}{p}, \frac{f}{c_1c_2}\right)\right|^2. \]
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\[ \ll 48p^{(-2/17)}afL + 3 \sum_{c_1||a \atop c_1c_2=p} \frac{afL}{c_1p} \]

(3.5)

In applying our induction hypothesis, we have noted that $\frac{af}{p}|L| < af|L|$ and that $\frac{ac_2f}{c_1p} \leq \frac{af}{c_1p} \leq af|L|$ for $L \geq 1$. For all but finitely many primes $p$,

\[ \frac{3}{p^{2/17}} \left( 16 + \frac{1}{p^{15/17}} \left( 1 + \frac{1}{p} \right) \right) \leq 1, \]

(3.6)

and we have proven the Lemma with no increase in the implied constant. For those finite number of exceptions to (3.6), the quantity in (3.5) is still

\[ \ll afL, \]

which suffices for the Lemma. Note that in these finite number of cases, the implied constant has increased - this is acceptable since $f$ can have only a bounded number of prime factors not satisfying (3.6).

We now proceed to prove a similar result in Lemma 3.5 which will be useful later. To do so, we first prove another Möbius inversion type result.

**Lemma 3.3.** For any prime $p$ and non-negative integers $k, \ell, n$,

\[ A(p^k, 1, 1)A(1, p^\ell, p^n) = A(p^k, p^\ell, p^n) + A(p^{k-1}, 1, 1)A(1, p^\ell, p^{n-1}) + A(p^{k-1}, 1, 1)A(1, p^{\ell-1}, p^{n+1}) - A(p^{k-2}, 1, 1)A(1, p^{\ell-1}, p^n), \]

where we shall follow the convention that $A(a, b, c) = 0$ whenever one of $a, b$ or $c$ is not an integer.

**Proof.** We first show that

\[ A(p^k, 1, 1)A(1, p^\ell, p^n) = A(p^k, p^\ell, p^n) + A(p^{k-1}, 1, 1)A(1, p^\ell, p^{n-1}) + A(p^{k-2}, 1, 1)A(1, p^{\ell-1}, p^n), \]

(3.7)

Indeed, by Hecke multiplicativity (Theorem 9.3.11 of [7])

\[ A(p^k, 1, 1)A(1, p^\ell, p^n) = \sum_{a, b, ab|p^k \atop a|p^\ell, b|p^n} A\left( \frac{p^k}{a}, \frac{p^\ell}{a}, \frac{p^n}{a} \right) \]

\[ = \sum_{a|p^k, p^\ell} A\left( \frac{p^k}{a}, \frac{p^\ell}{a}, \frac{p^n}{a} \right) + \sum_{a, b, ab|p^{k-1} \atop a|p^\ell, b|p^{n-1}} A\left( \frac{p^{k-1}}{a}, \frac{p^\ell}{a}, \frac{p^{n-1}}{a} \right) \]

(3.8)

\[ = \sum_{a|p^k, p^\ell} A\left( \frac{p^k}{a}, \frac{p^\ell}{a}, \frac{p^n}{a} \right) + A(p^{k-1}, 1, 1)A(1, p^\ell, p^{n-1}), \]
where the second line is simply splitting the previous sum into the two cases \( b = 1 \) and \( b \geq p \), and the last line is an application of Hecke multiplicativity. Our claim in (3.7) follows from this and

\[
\sum_{a | (p^k, p^\ell)} A\left(\frac{p^k}{a}, \frac{p^\ell}{a}, p^n a\right) = A(p^k, p^\ell, p^n) + \sum_{a | (p^{k-1}, p^{\ell-1})} A\left(\frac{p^{k-1}}{a}, \frac{p^{\ell-1}}{a}, p^{n+1} a\right).
\]

On the other hand, by (3.8) with \((k, \ell, n)\) replaced by \((k-1, \ell-1, n+1)\),

\[
(3.9)
\]

\[
A(p^{k-1}, 1, 1)A(1, p^{\ell-1}, p^{n+1}) - A(p^{k-2}, 1, 1)A(1, p^{\ell-1}, p^n) = \sum_{a | (p^{k-1}, p^{\ell-1})} A\left(\frac{p^{k-1}}{a}, \frac{p^{\ell-1}}{a}, p^{n+1} a\right),
\]

and this proves the Lemma when combined with (3.7).

This Lemma about prime powers leads directly to the following more general Lemma 3.4, which follows by multiplicativity. Of course, Lemma 3.4 can be proven by Mobius inversion as well at the cost of seeming less motivated.

**Lemma 3.4.**

(3.10) \[
A(k, \ell, n) = \sum_{d,e,f} \mu(d) \mu(e) A\left(\frac{k}{de}, 1, 1\right) A\left(1, \frac{\ell}{d}, \frac{dn}{ef}\right).
\]

Now we are ready to prove the following Lemma.

**Lemma 3.5.** For any \( M > 0 \) and positive integers \( b, c \),

\[
\sum_{m \leq M} |A(m, b, c)|^2 \ll (bcM)^{1+\epsilon}.
\]

**Proof.** We apply Lemma 3.4 and Cauchy-Schwarz to see that

(3.11) \[
\sum_{m \leq M} |A(m, b, c)|^2 \leq (bcM)^r \sum_{d,e,f} \sum_{d|b, e|d, f|c} \left| A\left(1, \frac{b}{d}, \frac{d}{ef}\right) \right|^2 \sum_{m \leq M} \mu^2(df) \left| A\left(\frac{m}{df}, 1, 1\right) \right|^2.
\]

Since \( \mu^2(df) = 0 \) if \( (d, f) > 1 \), we can replace \( \mu^2(df) \) by the condition \( (d, f) = 1 \). Note the conditions \( d|m, e|m/d, f|m \) and \( e|d \) yields that \( de|f|m \). When \( A \) is the Fourier coefficient of \( \phi \), then \( A(1, b, c) = B(c, b, 1) \) where \( B(c, b, 1) \) is the Fourier coefficient of the dual form of \( \phi \), so Lemma 2.2 implies that

(3.12) \[
\left| A\left(1, \frac{b}{d}, \frac{d}{ef}\right) \right|^2 \ll \left(\frac{bc}{ef}\right)^{1+\epsilon},
\]

where \( \epsilon > 0 \) is fixed.
simply by dropping all but one term in the sum, while standard contour integration of the Rankin-Selberg \( L \)-function gives
\[
\sum_{m \leq M} \left| A\left(\frac{m}{df \varepsilon}, 1, 1\right) \right|^2 \leq \sum_{m \leq M/df} |A(m, 1, 1)|^2 \ll \frac{M}{df \varepsilon}.
\]

Using (3.12) and (3.13) in (3.11) gives that
\[
\sum_{m \leq M} |A(m, b, c)|^2 \ll (bcM)^\epsilon \sum_{d, e, f} \frac{b, e}{d, f} |c| \ll (bcM)^{1+\epsilon},
\]
as desired.

\[\square\]

4. Setting up for Fourier analysis

When \( N \ll T \), Proposition 2.1 follows immediately from an application of the large sieve type bound due to Luo (see Theorem 1 of [18]). For the reader’s convenience, we state that bound here. For any sequence of complex numbers \( a_n \), we have that
\[
\sum_{t_j \leq T} (\cosh \pi t_j)^{-1} \left| \sum_{n \leq N} a_n \rho(n) n^i t_j \right|^2 \ll (T^2 + T^{3/2} N^{1/2} + N^{5/4})(NT)^\epsilon \sum_{n \leq N} |a_n|^2.
\]

Actually, the weaker bound claimed in (7) of Luo [18] would suffice, but that bound is cited as Theorem 6 of [6] by Luo, and Theorem 6 of [6] required conditions on the coefficients which are not available for us. We thank one of the anonymous referees for pointing this observation.

For \( N \gg T \), we require the following theorem, which is Theorem 7.1 of Young’s work [25].

**Theorem 4.1.** Let
\[
S(A) = \sum_{t_j} w(t_j) |\rho_j(1)|^2 \left| \sum_{n \sim N} a_n \lambda_j(n) n^i t_j \right|^2
\]
For any \( 1 \leq X \leq T \) and \( N \gg T \), we have
\[
S(A) = S_1(A; X) + O \left( T^2 + \frac{NT}{X} + \frac{N^{3/2}}{T} \right) \sqrt{N} \|A\|^2
\]
where \( \|A\|^2 = \sum_{n \sim N} |a_n|^2 \), and
\[
S_1(A; X) \ll T \sum_{r < X} \frac{1}{r^2} \sum_{0 \neq |k| \ll r^2 T} \min \left\{ \frac{1}{|u|}, \frac{r/|k|}{1 + u^2} \right\} \left| \sum_n a_n S(k, n; r) e\left(\frac{unr}{r^2 T}\right) \right|^2 du.
\]
Here \( S(k, n; r) \) is the usual Kloosterman sum defined by
\[
S(k, n; r) = \sum_{x \pmod{r}} e\left(\frac{kx + nx r}{r}\right),
\]
where $\sum_{x \pmod{r}}^*$ represents the summation restricted to coprime residue mod $r$.

Remark 1. The upper bound for $S_1(A; X)$ stated in Theorem 7.1 of Young’s work [25] is actually

$$S_1(A; X) \ll T \sum_{r<X} \frac{1}{r^2} \sum_{0 \neq |k| \leq rT} \left| \frac{1}{|k|} \int_{-T^{-\epsilon}}^{T^{-\epsilon}} \left| \sum_n a_n S(k, n; r) e\left(\frac{un}{rT}\right) \right| du \right|^2,$$

However this is not enough to obtain the bound $T^{2+\epsilon}$ in Proposition 2.1 especially for the case when $k \sim K$ is small and $r \sim R$ is big. Specifically in that case, Poisson summation over $k$ leads to a long dual sum while the factor of $\frac{1}{|k|}$ is too large. We instead keep the more flexible bound from Equation (8.8) of [25] and truncate the integral to $T^\epsilon$ in the same way as (8.9) and (8.12) of [25].

After we apply Theorem 4.1 to $H_\ell$,

$$S(A) - S_1(A, X) \ll T^{2+\epsilon} \left( \sum_{n \sim N} \left| A(1, \ell, n) \right|^2 \right)$$

upon choosing $X = \min\{T, \frac{N}{T}\}$ and using $N = \frac{P}{T^2}$ and $P \ll T^{2+\epsilon}$. The above bound suffices to upon using Lemma 2.2.

Thus, we find that in order to bound $H_\ell$, we need to bound

$$T \sum_{r<X} \frac{1}{r^2} \sum_{0 \neq |k| \leq rN^\epsilon} \int_{-T^\epsilon}^{T^\epsilon} \min\left\{ \frac{1}{|u|}, \frac{r/|k|}{1 + u^2} \right\} \left| \frac{1}{\sqrt{N}} \sum_n A(1, \ell, n) S(k, n; r) w_3\left(\frac{n}{N}\right) e\left(\frac{un}{rT}\right) \right|^2 du,$$

where $w_3(x) = \frac{w_2(x)}{\sqrt{2}}$, $N = \frac{P}{T^2} \ll T^{2+\epsilon}$ and $1 \leq X = \min\{T, \frac{N}{T}\}$.

Let $R \leq X$ and $K \ll RN^\epsilon \ll RT^\epsilon$. It is sufficient to consider the dyadic sum

(4.2)

$$I(R, K; \ell) = T \sum_{r \sim R} \frac{1}{r^2} \sum_{|k| \leq rK} \int_{-T^\epsilon}^{T^\epsilon} g(u) \left| \frac{1}{\sqrt{N}} \sum_n A(1, \ell, n) S(k, n; r) w_3\left(\frac{n}{N}\right) e\left(\frac{un}{rT}\right) \right|^2 du,$$

where

(4.3)

$$g(u) = g(u, r, k) = \min\left\{ \frac{1}{|u|}, \frac{R/K}{1 + u^2} \right\}.$$

It now suffices to prove the following Lemma.

**Lemma 4.2.** For any fixed $\ell \ll T^{1+\epsilon}$, let $T \ll N \ll \frac{T^{2+\epsilon}}{r^2}$, $R \leq X$, $K \ll RT^\epsilon$, and $X = \min\{T, \frac{N}{T}\}$. Then

$$I(R, K; \ell) \ll T^{2+\epsilon}$$

where the implied constant depends on $\epsilon$. This bound is uniform in $\ell$.

The proof of Lemma 4.2 will be provided Section 6 - 9. To orient the reader, we provide an outline of this important Lemma.
4.1. Outline of the proof of Lemma 4.2. For this outline, we will ignore technical issues and focus on structural features of the proof. First, we consider the sum
\( I(R, K; \ell, U) := \sum_{r \sim R} \frac{1}{U r^2} \sum_{|k| \sim K} \int_U^{2U} \left| \frac{1}{\sqrt{N}} \sum_n A(1, \ell, n) S(k, n; r) w_3 \left( \frac{n}{N} \right) e \left( \frac{un}{rT} \right) \right|^2 du, \)
where we have applied a dyadic subdivision to the integral over \( u \) and replaced \( g(u) \) by \( \frac{1}{U} \). For illustrative purposes, we will assume that \( T^{-100} \leq U \leq T^\epsilon \). Applying Poisson over \( k \) and Voronoi for the sum over \( n \) along with an application of Cauchy-Schwarz results in sums roughly of the form
\( T^{1+\epsilon} \frac{R}{UN} \int_U^{2U} \sum_{r \sim R} \sum_{\alpha \equiv \ell (r)} \left| \sum_{m>0} A(m, 1, 1) \frac{K\mathcal{L}(\alpha_1, m; r, (1,1),(1,1))}{m} F \left( \frac{mx}{r^4} \right) \right|^2 du, \)
where \( K\mathcal{L}(\alpha_1, m; r, (1,1),(1,1)) \) denotes the usual hyper-Kloosterman sum to be defined later and \( F \) is some type of integral transform of \( w_3 \left( \frac{n}{N} \right) e \left( \frac{un}{T} \right) \). In the \( GL(3) \) case in Young’s work [25], Equation (4.5) is already sufficient, since the sum over \( m \) will be nonexistent. In the \( GL(4) \) case, the dual sum over \( m \) can be essentially the same length as the original sum over \( n \) which presents significant difficulties. Here, it is useful to note that the extreme case \( N = T^2, R = T \) is not the most difficult, and indeed follows by an application of the large sieve.

To be more precise, by completing the sum over \( a_1 \) to all \( a_1 \) mod \( r \) and a similar procedure upon opening up the hyper-Kloosterman sum, we get a sum of the form
\( T^{1+\epsilon} \frac{R^3}{UN} \int_U^{2U} \sum_{r \sim R} \sum_{x \equiv \alpha (r)} \left| \sum_{m>0} A(m, 1, 1) \frac{F \left( \frac{mx}{r^4} \right) e \left( \frac{mx}{r} \right)}{m} \right|^2 du. \)
In the actual proof we then proceed to examine two separate cases: when \( m \) is small and when \( m \) is large. However, as a conceptual framework, the reader should think of the following two cases instead: when the phase inside \( F \) is insignificant and when the phase plays a significant role.

The first case includes the case when \( m \) is small but also includes the case when all the parameters are large, specifically when \( N = T^2, R = T \) and the \( m \) of size \( R^4/N = T^2 \). Ignoring all technical complications inside \( F \), we will see that this case can be handled by an application of the large sieve.

The second case is more involved. For motivation, note that in (4.4), one may write
\[ \left( \sum_n e \left( \frac{nu}{rT} \right) \right)^2 = \sum_{n_1, n_2} \int_U^{2U} e \left( \frac{n_1 - n_2}{rT} \right) du, \]
and the integral in \( u \) essentially forces
\( |n_1 - n_2| \ll \frac{RT^{1+\epsilon}}{U} \)
which is significant when \( \frac{RT}{U} \) is smaller than \( N \). (Note that this essentially excludes the extreme case \( N = T^2 \) and \( R = T \).) We thus assume that

\[
RT \ll NU.
\]

The presence of this narrow region type condition on \((n_1, n_2)\) should be no surprise, since our initial sum looked like \( \sum_{t_j} \ldots \left( \frac{n_1}{n_2} \right)^{it_j} \) where \( 1/4 + t_j^2 \) are Laplace eigenvalues, and we expect this average to force \( n_1 \) and \( n_2 \) to be close. (This expectation is obscured after applying Kutznetsov and other tools, but (4.7) is a direct descendant.) We should expect to understand this natural constraint well if we are to derive cutting edge results.

Morally, the main issue is how the condition (4.7) is expressed in (4.6) after Voronoi summation transform the sum over \( n \) into dual sums over \( m \). Of course, when \( n_1 \) and \( n_2 \) correspond to variables of integration \( y_1 \) and \( y_2 \) inside \( F \), this forces \( y_1 \) and \( y_2 \) to be close. Writing the dual sums as

\[
\left| \sum_m \ldots \right|^2 = \sum_{m_1, m_2} \ldots,
\]

a less obvious conclusion is that this in turn forces \( m_1 \) and \( m_2 \) to be somewhat close, which arises from the phases introduced by Voronoi on \( GL(4) \). Actually, this not surprising in hindsight. In particular, one can morally express the condition \( |n_1 - n_2| \ll RT^{1+\epsilon} \) via an integration

\[
\frac{1}{V} \int_{V}^{2V} \left( \frac{n_1}{n_2} \right)^it \, dt
\]

where \( V \approx \frac{NU}{RT} \). The combined conductor of \( n^it \left( \frac{m_1}{m_2} \right) \) is then \( \approx \frac{NU}{T} \) is independent of \( R \).

At this point, one can try to apply the hybrid large sieve, without applying Voronoi in \( n_i \). This will give a bound roughly of the form \( R^2 + \frac{T_2^2}{T} \), which is acceptable only when \( U \gg T^{-\epsilon} \) say. In other words, we lose in the case when \( n_1 \) and \( n_2 \) are not forced to be very close. However, recall that the hybrid conductor mentioned is around size \( \frac{NU}{T} \) and is small when \( U \) is small, and this explains when we should expect Voronoi to give us savings.

Although this specific formula does not appear to be readily available in the literature, we would still expect some formula like

\[
\sum_n n^{it} \left( \frac{xn}{r} \right) g \left( \frac{n}{N} \right) = \sum_m m^{-it} \times \text{hyperkloosterman sum} \times \text{integral transform}
\]

to hold. Thus, we would expect \( \left( \frac{n_1}{n_2} \right)^it \) to transform to \( \left( \frac{m_2}{m_1} \right)^it \) and so the integral over \( t \) forces the dual variables \( m_1 \) and \( m_2 \) to be close also.

In the actual proof, we use in Theorem 5.1. For this reason and due to other suppressed details, the actual computation is rather intricate. After understanding the dual sum over \( m_i \) as described above, we can use the hybrid large sieve to handle both the condition that \( m_1 \) and \( m_2 \) are close, and the resultant linear phases. In our exposition, we choose instead to use a dissection to separate the
interdependence of the \( m_i \). This is purely a matter of preference; for us, this dissection is more direct and easier to visualize, but slightly lengthier. This analysis takes up the bulk of the remaining work in §9.

5. Voronoi summation

In this section, we collect some technical lemmas which arise when using Voronoi summation. We will be using the Voronoi formula over \( GL(4) \) from Miller and Schmid’s work [20] to deal with the sum over \( n \) in \( I(R, K; \ell) \). First, we introduce necessary notations, which are taken from [20] and [21].

Let \( a, n \in \mathbb{Z} \) and \( r \in \mathbb{N} \) and define \( q = (q_1, q_2) \) and \( d = (d_1, d_2) \) to be vectors of positive integers, where \( d_1 | q_1 r \) and \( d_2 | q_1 q_2 r d_1 \). The hyper-Kloosterman sum is defined to be

\[
KL(a, n, r; q, d) = \sum_{x_1 \pmod{q_1 r d_1}}^* \sum_{x_2 \pmod{q_1 q_2 r d_1 d_2}}^* e\left(\frac{d_1 x_1 a r}{q_1} + \frac{d_2 x_2}{q_1 q_2 r d_1 d_2}\right),
\]

where we recall that \( \sum^* \) is defined as in Theorem 4.1.

Next let \( \psi \) be a smooth function on \( \mathbb{R} \) and compactly supported in \((0, \infty)\) and away from 0, and define the integral transform of \( \psi \) to be

\[
\Psi(y) = \int_{\mathbb{R}^4} \psi\left(\frac{x_1 x_2 x_3 x_4}{y}\right) \prod_{j=1}^{4} (e(-x_j)|x_j|^{-\lambda_j} \text{sgn}(x_j)^{\delta_j} \, dx_j),
\]

where \( \bar{\lambda} = (\lambda_1, \ldots, \lambda_4) \), \( \bar{\delta} = (\delta_1, \ldots, \delta_4) \), and \( (\bar{\lambda}, \bar{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n \) is the representation parameter of a cusp form on \( GL(4) \).

The function \( \Psi \) can be reformulated through Mellin transform. This was done in [20] and [21], and we will quote the results here.

Let \( \tilde{\psi}(s) \) is the usual Mellin transform of \( \psi \) defined by

\[
(5.1) \quad \tilde{\psi}(s) = \int_0^\infty \psi(t) t^s dt.
\]

The integral above converges for all \( s \in \mathbb{C} \) since \( \psi \) is compactly supported away from 0. Next we define

\[
\mathcal{G}_\delta(s) := \left\{ \begin{array}{ll}
\frac{\Gamma_R(s)}{\Gamma_R(1-s)} & \text{if } \delta \in 2\mathbb{Z} \\
\frac{i\Gamma_R(s+1)}{\Gamma_R(2-s)} & \text{if } \delta \in 2\mathbb{Z} + 1,
\end{array} \right.
\]

where \( \Gamma_R(s) = \pi^{-s/2} \Gamma(s/2) \). Now, define

\[
G_+(s) = \prod_{j=1}^{4} \mathcal{G}_{\delta_j}(s + \lambda_j)^{-1}, \quad G_-(s) = \prod_{j=1}^{4} \mathcal{G}_{1+\delta_j}(s + \lambda_j)^{-1},
\]
and for $\sigma > 0$ let

$$
\Psi_\pm(x) = \frac{1}{2\pi i} \int_{(-\sigma)} \tilde{\psi}(s)x^s G_\pm(s) \, ds.
$$

Then when $x > 0$ we can write

$$
\Psi(x) = \Psi_+(x) + \Psi_-(x) \quad \text{and} \quad \Psi(-x) = \Psi_+(x) - \Psi_-(x).
$$

Now we are ready to state the Voronoi formula from [20].

**Theorem 5.1.** Let $a \in \mathbb{Z}$, $r \in \mathbb{N}$, $(a, r) = 1$, and $\psi$ be a smooth function on $\mathbb{R}$ and compactly supported in $(0, \infty)$ and away from 0. With notations as above,

$$
\sum_{n \neq 0} A(q_2, q_1, n) e\left(\frac{an}{r}\right) \psi(n)
= r \sum_{d_1 | q_1} \sum_{d_2 | q_2} \sum_{m \neq 0} A(m, d_2, d_1) \frac{\mathcal{K}(\mathcal{L}(m, n); r, q_1, d)}{|m|d_1d_2} \psi\left(\frac{md_2d_3}{r^4q_2q_1}\right).
$$

Since $\Psi$ can be written as $\Psi_\pm$ in Equation (5.3), we focus only on studying $\Psi_\pm$. Now we will find an asymptotic formula for $\Psi_\pm$, and this will help in analysing saddle points and main terms. From now on, we will fix $\lambda_j = \alpha_j$ as in Equation (2.1) and $\delta_j = 0$ and prove the following Lemma.

**Lemma 5.2.** Let $N \geq 1$, $\lambda_j = \alpha_j$ as in Equation (2.1) and $\delta_j = 0$. Suppose $\psi(x)$ is a smooth function compactly supported on $[N, 2N]$, and $\Psi_\pm(x)$ be defined as in Equation (5.2). Then for any fixed positive integer $K \geq 1$ and $xN \gg 1$,

$$
\Psi_+(x) = x \int_0^\infty \psi(y) \sum_{j=1}^K \frac{1}{(xy)^{\frac{3}{4}+\frac{s}{2}}} \left[c_j e\left(4(xy)^{\frac{1}{2}}\right) W_j(8\pi(xy)^{\frac{1}{2}}) + d_j e\left(-4(xy)^{\frac{1}{2}}\right) W_j(8\pi(xy)^{\frac{1}{2}})\right] \, dy
+ O\left((xN)^{-\frac{K+3}{4}-\frac{1}{8}}\right)
$$

where function $W_j$ is a finite linear combination of $W_k$ defined in Lemma A.1 and $c_j, d_j$ are suitable constants depending on $\alpha_i$. Moreover, $\Psi_-(x)$ has the same expression except value of constants.

We will focus only on proving $\Psi_+$ as the proof of $\Psi_-$ can be proceeded in the same way. The proof will follow the idea of the proof of Lemma 3 in [9] and Lemma 6.1 in [16], which will be provided in Appendix A.
6. Fourier analysis on \( \mathcal{I}(R, K; \ell) \)

Firstly, we square out the expression of \( \mathcal{I}(R, K; \ell) \) in (4.2), put a smooth weight in \( k \), and use the fact that \( \frac{1}{r^2} \ll \frac{1}{R^2} \). We then obtain that

\[
\mathcal{I}(R, K; \ell) \ll \frac{T}{NR^2} \sum_{r \sim R} \sum_{k} w_1 \left( \frac{k}{K} \right) \int_{-T^r}^{T^r} g(u) \sum_{n_1, n_2} A(1, \ell, n_1) A(1, \ell, n_2) w_3 \left( \frac{n_1}{N} \right) w_3 \left( \frac{n_2}{N} \right)
\]

\[
\times \sum_{n} \sum^{*}_{a_1 \, (\text{mod } r) \, a_2 \, (\text{mod } r)} e \left( \frac{a_1 k - a_2 k}{r} \right) e \left( \frac{\bar{a}_1 n_1 - \bar{a}_2 n_2}{r} \right) e \left( \frac{un_1 - un_2}{rT} \right) du,
\]

where we recall that \( w_1 \) is a smooth compactly supported function defined as above. Then we apply Poisson summation to the sum over \( k \).

\[
\sum_{k} w_1 \left( \frac{k}{K} \right) e \left( \frac{(a_1 - a_2)k}{r} \right) = \sum_{c \, (\text{mod } r)} e \left( \frac{(a_1 - a_2)c}{r} \right) \sum_{k \equiv c \, (\text{mod } r)} w_1 \left( \frac{k}{K} \right)
\]

\[
= \sum_{c \, (\text{mod } r)} e \left( \frac{(a_1 - a_2)c}{r} \right) \sum_{j} \frac{K}{r} e \left( \frac{cj}{r} \right) \hat{w}_1 \left( \frac{Kj}{r} \right) = \sum_{j \equiv a_2 - a_1 \, (\text{mod } r)} \hat{w}_1 \left( \frac{Kj}{r} \right).
\]

Therefore

(6.1)

\[
\mathcal{I}(R, K; \ell) \ll \frac{TK}{NR^2} \sum_{r \sim R} \sum_{j} \hat{w}_1 \left( \frac{Kj}{r} \right) \sum^{*}_{a_1 \, (\text{mod } r)} \int_{-T^r}^{T^r} g(u) \sum_{n_1, n_2} A(1, \ell, n_1) A(1, \ell, n_2) w_3 \left( \frac{n_1}{N} \right) w_3 \left( \frac{n_2}{N} \right)
\]

\[
\times e \left( \frac{\bar{a}_1 n_1 - (a_1 + j)n_2}{r} \right) e \left( \frac{un_1 - un_2}{rT} \right) du
\]

\[
\ll \frac{TK}{NR^2} \sum_{r \sim R} \sum_{j \ll \frac{R}{K}} \sum^{*}_{a_1 \, (\text{mod } r)} \int_{-T^r}^{T^r} g(u) \left( \left| \sum_{n} A(1, \ell, n) w_3 \left( \frac{n}{N} \right) e \left( \frac{\bar{a}_1 n}{r} \right) e \left( \frac{un}{rT} \right) \right|^2 \right)
\]

\[
+ \sum_{n} A(1, \ell, n) w_3 \left( \frac{n}{N} \right) e \left( \frac{(a_1 + j)n}{r} \right) e \left( \frac{un}{rT} \right) \left| e \left( \frac{\bar{a}_1 n}{r} \right) e \left( \frac{un}{rT} \right) \right|^2 du
\]

\[
\ll \frac{T^{1+\epsilon}}{NR} \int_{-T^r}^{T^r} g(u) \sum_{r \sim R \, a_1} \sum^{*}_{(\text{mod } r)} \sum_{n} A(1, \ell, n) w_3 \left( \frac{n}{N} \right) e \left( \frac{un}{rT} \right) e \left( \frac{\bar{a}_1 n}{r} \right) \left| e \left( \frac{un}{rT} \right) \right|^2 du,
\]

where we note that the second term involving \( e \left( \frac{(a_1 + j)n}{r} \right) \) becomes \( e \left( \frac{\bar{a}_1 n}{r} \right) \) by a change of variables on \( a_1 \), and we have extended the sum over \( a_1 \) by positivity.

Note that \( N \gg TR \) since \( R \ll X \leq \frac{N}{r} \). We apply the Voronoi Summation formula (Theorem 5.1) to the sum over \( n \) in Equation (6.1). Let
\[ f(x; u, r) := f(x) = w_3 \left( \frac{x}{N} \right) e \left( \frac{ux}{rT} \right). \]

Then

\[ \sum_n A(1, \ell, n) f(n) e \left( -\frac{a_1 n}{r} \right) \]

\[ = |r| \sum_{d_1 | r \ell} \sum_{d_2 | \frac{r \ell}{d_1}} \sum_{m \neq 0} \frac{A(m, d_2, d_1)}{|m| d_1 d_2} \mathcal{K} \mathcal{L}(\overline{a_1}, m; r, (\ell, 1), (d_1, d_2)) F \left( \frac{md_2^3 d_1^3}{r^4 \ell^2}; r \right), \]

where \( F \) is defined analogously to \( \Psi \), and

\[ \mathcal{K} \mathcal{L}(\overline{a_1}, m; r, (\ell, 1), (d_1, d_2)) = \sum_{x_1 (\mod \frac{r \ell}{d_1})} \sum_{x_2 (\mod \frac{r \ell}{d_1 d_2})} e \left( \frac{d_1 x_1 \overline{a_1}}{r} + \frac{d_2 x_2 x_1}{\frac{\ell}{d_1}} + \frac{m x_2}{\frac{r \ell}{d_1 d_2}} \right). \]

After Cauchy-Schwarz inequality in \( d_1, d_2 \) and considering only positive \( m \) due to symmetry, now we need to bound

\[ \mathcal{I}_1(R, K; \ell) := \frac{T^{1+\epsilon} R}{N} \int_{-T^\epsilon}^{T^\epsilon} g(u) \sum_{r \sim R} \sum_{a_1 (\mod r)} \sum_{d_1 | r \ell} \sum_{d_2 | \frac{r \ell}{d_1}} \frac{1}{d_1^2 d_2^2} \]

\[ \times \left| \sum_{m > 0} \frac{A(m, d_2, d_1)}{m} \mathcal{K} \mathcal{L}(\overline{a_1}, m; r, (\ell, 1), (d_1, d_2)) F \left( \frac{md_2^3 d_1^3}{r^4 \ell^2}; r \right) \right|^2 du. \]

### 7. Simplifying exponential sums

In this section, we deal with the exponential sum in the hyper-Kloosterman sum. Moreover, the bound for \( F_- \) can be evaluated in the same way as \( F_+ \), so we consider only \( F_+ \). By Cauchy-Schwarz inequality, changing variable from \( a_1 \) to \( \overline{a_1} \) and completing
summation over \( a_1 \), we have \( \mathcal{I}_1(R, K; \xi) \) is bounded by

\[
\ll \frac{T^{1+\epsilon} R}{N} \int_{-T}^{T} g(u) \sum_{r \sim R} \sum_{d_1} \sum_{d_2} \sum_{m} \frac{1}{d_1^2 d_2^2} \left( \sum_{m > 0} A(m, d_1, d_2) \chi(m) F_+ \left( \frac{m d_1^2 d_2^3}{r^4 \xi^2}; r \right) \right)^2 du.
\]

\[
= \frac{T^{1+\epsilon} R}{N} \int_{-T}^{T} g(u) \sum_{r \sim R} \sum_{d_1} \sum_{d_2} \sum_{m} \frac{1}{d_1^2 d_2^2} \left( \sum_{m > 0} A(m, d_1, d_2) \chi(m) F_+ \left( \frac{m d_1^2 d_2^3}{r^4 \xi^2}; r \right) \right)^2 du.
\]

Next we sum over \( a_1 \) and see that \( d_1 x_1 \equiv d_1 x_1' \mod r \) by orthogonality, which implies \( x_1 \equiv x_1' \mod \frac{r}{(r, d_1)} \). Thus we may write \( x_1' = x_1 + \frac{r}{(r, d_1)} y \), where \( y \) runs through those residues \( \mod \frac{r}{(r, d_1)} \) such that \( (x_1 + \frac{r}{(r, d_1)} y) \cdot \frac{r}{(r, d_1)} = 1 \). For simplicity, let \( \sum_{y \mod \frac{r}{(r, d_1)}} \) denote the sum over such \( y \). Thus our sum becomes

\[
\frac{T^{1+\epsilon} R}{N} \sum_{r \sim R} \sum_{d_1} \sum_{d_2} \sum_{x_1 \mod \frac{r}{(r, d_1)}} \sum_{y \mod \frac{r}{(r, d_1)}} \left| S_1 S_2 \right| du
\]

\[
(7.1)
\]

where

\[
S_1 = \sum_{m_1 > 0} \frac{A(m_1, d_2, d_1)}{m_1} \chi(m) F_+ \left( \frac{m_1 d_2^3 d_1^3}{r^4 \xi^2}; r \right) \sum_{x_2 \mod \frac{r}{d_1 d_2}} \left( \frac{-d_2 x_2 \bar{x}_1 + \frac{r}{(r, d_1)} y \cdot \frac{r}{d_1}}{r \cdot \frac{r}{d_1}} \right).
\]

and

\[
S_2 = \sum_{m_2 > 0} \frac{A(m_2, d_2, d_1)}{m_2} \chi(m) F_+ \left( \frac{m_2 d_2^3 d_1^3}{r^4 \xi^2}; r \right) \sum_{x_2' \mod \frac{r}{d_1 d_2}} \left( \frac{-m_2 x_2'; d_1 + \frac{r}{(r, d_1)} y \cdot \frac{r}{d_1}}{r \cdot \frac{r}{d_1}} \right).\]

Inside \( S_2 \), we may use the change of variables \( u = x_1 + \frac{r}{(r, d_1)} y \). The condition on \( y \) then becomes that \( (\bar{w} - \frac{r}{(r, d_1)} y) \cdot \frac{r}{d_1} = 1 \). After this change of variables, we extend the sum
over \( y \) to all residues mod \( \frac{(r, d_1) \ell}{d_1} \). Thus,

\[
\sum_{r \sim R_1} \sum_{d_2} \sum_{x_1 \mod \frac{r \ell}{d_1}} \sum_{y \mod \frac{(r, d_1) \ell}{d_1}} |S_2|^2 \leq \sum_{r \sim R_1} \sum_{d_2} \sum_{x_1 \mod \frac{r \ell}{d_1}} |S_2|^2 = \left( \frac{r, d_1}{d_1} \right) \sum_{r \sim R_1} \sum_{d_2} \sum_{x_1 \mod \frac{r \ell}{d_1}} |S_1|^2.
\]

By a further change of variables from \( x_1 \) to \( x_1' \), the fact that \( S_1 \) is independent of \( y \) and \( \frac{(r, d_1)}{d_1} \leq 1 \), the quantity in (7.1) is bounded by

\[
\ell \frac{T^{1+\epsilon} R^2}{N} \int_{-T^\epsilon}^{T^\epsilon} g(u) \sum_{r \sim R_1} \sum_{d_2} \sum_{x_1 \mod \frac{r \ell}{d_1}} |S_1|^2 du \leq \ell \frac{T^{1+\epsilon} R^2}{N} \int_{-T^\epsilon}^{T^\epsilon} g(u) \sum_{r \sim R_1} \sum_{d_2} \sum_{x_1 \mod \frac{r \ell}{d_1}} \left| \sum_{m > 0} A(m, d_2, d_1) \frac{m d_2^3 d_1^3}{m_1} F_+ \left( \frac{md_2^3 d_1^3}{r^4 \ell^2}; r \right) \sum_{x_2 \mod \frac{r \ell}{d_1 d_2}} e \left( \frac{d_2 x_2 x_1}{\frac{r \ell}{d_1 d_2}} + \frac{m x_2}{\frac{r \ell}{d_1 d_2}} \right) \right|^2 du.
\]

Now we may extend the sum over \( x_1 \) to all residues mod \( \frac{r \ell}{d_1} \) by positivity. Opening the square produces two sums \( x_2, x'_2 \mod \frac{r \ell}{d_1} \). However, by orthogonality, the sum over \( x_1 \) gives the condition \( d_2 x_2 x_1' \equiv d_2 x'_2 \mod \frac{r \ell}{d_1} \), which implies \( x_2 \equiv x'_2 \mod \frac{r \ell}{d_1 d_2} \) because \( d_2 \mid \frac{r \ell}{d_1} \). So the above sum is

\[
\ll \ell^2 \frac{T^{1+\epsilon} R^3}{N} \int_{-T^\epsilon}^{T^\epsilon} g(u) \sum_{r \sim R_1} \sum_{d_2} \sum_{x \mod \frac{r \ell}{d_1 d_2}} \left| \sum_{m > 0} A(m, d_2, d_1) \frac{m d_2^3 d_1^3}{m_1} F_+ \left( \frac{md_2^3 d_1^3}{r^4 \ell^2}; r \right) e \left( \frac{m x}{\frac{r \ell}{d_1 d_2}} \right) \right|^2 du.
\]
where we have used a change of variables $x = \frac{x_2}{2}$. Next we write $r_1 = r\ell$, switch the sums $d_1, d_2$ and $r$ and drop condition $\ell | r_1$. Thus the above expression is

$$< \ell^2 \frac{T^{1+\epsilon} R^3}{N} \int_{-T^e}^{T^e} g(u) \sum_{\substack{d_1 \leq R\ell \; d_2 \leq R\ell \; d_1 \sim d_2 \mod \frac{R\ell}{d_1 d_2} \atop d_1 d_2 | r_1}} \sum_{m>0} A(m, d_1, d_2) \frac{1}{d_1 d_2} \frac{F_+ \left( \frac{m d_2^3 d_1^2 \ell^2}{r_1^2 \ell^2}; r_1 \right) e \left( \frac{m x}{r_1} \right)}{m} \Bigg| \sum_{m>0} A(m, d_1, d_2) \frac{1}{d_1 d_2} \frac{F_+ \left( \frac{m \ell^2}{r_1^2 d_1 d_2}; r_1 \right) e \left( \frac{m x}{r_1} \right)}{m} \Bigg| \, du$$

(7.2)

Now we split $m$ into two ranges. We let $I_{sm}(R, K; \ell)$ be the expression on the right side of (7.2) with $m \leq \frac{R^3 \ell^2}{N d_1 d_2} T^{\epsilon_1}$ and $I_{big}(R, K; \ell)$ be the same expression for $m > \frac{R^3 \ell^2}{N d_1 d_2} T^{\epsilon_1}$, where $\epsilon_1$ is a fixed small constant to be determined later.

Since $|a + b| \leq 2(|a|^2 + |b|^2)$, it now suffices to prove the following Propositions.

**Proposition 7.1.** With notations defined as above,

$$I_{sm}(R, K; \ell) \ll T^{2+\epsilon}$$

where the implied constant depends on $\epsilon$.

**Proposition 7.2.** With notations defined as above,

$$I_{big}(R, K; \ell) \ll T^{2+\epsilon}$$

where the implied constant depends on $\epsilon$.

We prove Proposition 7.1 in Section 8 and Proposition 7.2 in Section 9. Moreover, we note that the presence of $d_1$ and $d_2$ are conceptually unimportant and gives rise to convoluted notation which obfuscates the main ideas. To ease the notational burden, readers may set $d_1 = d_2 = 1$ in Section 8 and 9 below.

### 8. Proof of Proposition 7.1

We would like to apply the usual Large Sieve to Equation 7.2, but we first need to make the inner sum independent of $r$ and $\ell$. To do this, we first note that by the work of Luo, Rudnick and Sarnak [19],

(8.1)

$$|\Re \alpha_j| < \frac{1}{2}.$$

We express $F_+$ as in (5.2), use the change of variable $\frac{1-s}{2} \to s$, and derive that

$$F_+(x; \frac{r_1 d_1 d_2}{\ell}) = \frac{2 \pi^2}{2 \pi i} \int_{(\sigma_1)} \frac{\pi^{-8s} x^{-2s} \Gamma(s - \frac{\alpha_1}{2}) \Gamma(s - \frac{\alpha_2}{2}) \Gamma(s - \frac{\alpha_3}{2}) \Gamma(s - \frac{\alpha_4}{2}) \Gamma(\frac{1}{2} - s + \frac{\alpha_1}{2}) \Gamma(\frac{1}{2} - s + \frac{\alpha_2}{2}) \Gamma(\frac{1}{2} - s + \frac{\alpha_3}{2}) \Gamma(\frac{1}{2} - s + \frac{\alpha_4}{2})}{\Gamma(\frac{1}{2} - s + \alpha_1/2) \Gamma(\frac{1}{2} - s + \alpha_2/2) \Gamma(\frac{1}{2} - s + \alpha_3/2) \Gamma(\frac{1}{2} - s + \alpha_4/2)} f(-2s + 1) \, ds,$$
so we may shift the contour of integration to \( \sigma_1 = 1/4 \) without crossing any poles of the integrand by (8.1). We proceed to further shift the contour to \( \sigma_1 < 1/8 \). We may pick up residues of the form

\[
(8.2) \quad C x^{1-2s_0} \tilde{f}(1-2s_0),
\]

where \( \Re s_0 < 1/4 \) and for some constant \( C \) only dependent on \( \alpha_j \).

Further note that

\[
(8.3) \quad \tilde{f}(-2s+1) = \int_0^\infty w_3 \left( \frac{y}{N} \right) e \left( \frac{u \ell y}{r_1 d_1 d_2 T} \right) y^{-2s} dy \ll N^{1-2s}.
\]

For simplicity, we write

\[
\mathcal{G}(s) = \frac{\Gamma(s - \alpha_1/2) \Gamma(s - \alpha_2) \Gamma(s - \alpha_3) \Gamma(s - \alpha_4/2)}{\Gamma(1/2 - s + \alpha_1/2) \Gamma(1/2 - s + \alpha_2) \Gamma(1/2 - s + \alpha_3) \Gamma(1/2 - s + \alpha_4/2)} \tilde{f}(-2s+1)
\]

and note that

\[
(8.4) \quad \mathcal{G}(s) \ll \frac{N^{1-2\Re s}}{|s|^{1+\epsilon}}
\]

when \( \Re s < 1/8 \) by Stirling’s formula and (8.3). This is the main motivation behind shifting the contour to \( \Re s < 1/8 \). Note that repeated integration by parts on \( \tilde{f}(1-2s) \) will also give sufficiently rapid decay in \( s \) but at the cost of introducing factors like \( \frac{N_{u_j}}{T} \), which can be quite large. This leads us to the examination of the contribution of the residues and the contribution of the remaining contour integral separately.

In both cases, we apply a dyadic subdivision to the sum over \( m \), so that we examine sums \( m \sim M \) for \( M \leq \frac{R_1 R_2^2}{Nd_1 d_2^2} T^{\epsilon_1} \).

8.1. Contribution of residues. For brevity, write \( \lambda = 1 - 2s_0 \), and note that \( \Re \lambda \geq 1/2 \). We need to bound

\[
(8.5) \quad \ell^2 \frac{T^{1+\epsilon} R^3}{N} \int_{-T^*}^{T^*} g(u) \sum_{d_1 \ll R \ell} \sum_{d_2 \ll R \ell} \sum_{d_1^2 d_2^2}^{1} \sum_{r_1 \sim d_1 d_2}^{x \mod r_1} \sum^{*} \left| \sum_{m} A(m, d_2, d_1) \frac{m \ell^2}{r_1^4 d_1 d_2} \tilde{f}(1-2s_0) e \left( \frac{m x}{r_1} \right) \right|^2 du
\]

\[
\ll \ell^2 \frac{T^{1+\epsilon} R^3}{N} \sum_{d_1 \ll R \ell} \sum_{d_2 \ll R \ell} \sum_{d_1^2 d_2^2}^{1} \left( \frac{N M d_2 d_1^2}{R^4 \ell^2} \right)^{2Re \lambda} \frac{1}{M^{2Re \lambda}} \sum_{r_1 \sim d_1 d_2}^{x \mod r_1} \sum_{m} \left| \sum_{m} A(m, d_2, d_1) \frac{m \ell^2}{r_1^4 d_1 d_2} \tilde{f}(1-2s_0) e \left( \frac{m x}{r_1} \right) \right|^2,
\]

using the bound \( \tilde{f}(1-2s_0) \ll N^{1-2\Re s} = N^{Re \lambda} \) and the fact that

\[
\int_{-T^*}^{T^*} g(u) du \ll \log T.
\]
Using that $M \leq \frac{R^4 \ell^2}{N d_2^4} T^{\epsilon_1}$ and $\text{Re } \lambda \geq 1/2$, we see that

$$\left( \frac{NM}{R^4 \ell^2 d_2^2 d_1^3} \right)^{2 \text{Re } \lambda} \ll \left( \frac{NM}{R^4 \ell^2 d_2^2 d_1^3} \right)^T,$$

upon choosing $\epsilon_1$ to be sufficiently small. Substituting this in, we see that the quantity above is bounded by

$$\ll \ell^2 \frac{T^{1+\epsilon} R^3}{N} \sum_{d_1 \ll R} \sum_{d_2 \ll R} \frac{1}{d_1^3 d_2^3} \left( \frac{NM d_2 d_1^2}{R^4 \ell^2} \right)^{1 - 2 \text{Re } \lambda} \sum_{r_1 \sim \frac{R \ell}{M d_2}} \sum_{r_1 \mod r_1} \left| \sum_{m \sim M} \frac{A(m, d_2, d_1)}{m^{1-\lambda}} e \left( \frac{m x}{r_1} \right) \right|^2$$

and since $R \leq N$, $N \ll \frac{T^{2+\epsilon}}{\ell^2}$, and $M \ll \frac{R^4 \ell^2}{N d_2^4} T^{\epsilon_1}$, choosing $\ell_1$ sufficiently small, we conclude that Equation (8.5) is bounded by

$$\ll \frac{T^{1+\epsilon} M^{1-2 \text{Re } \lambda}}{R} \sum_{d_1 \ll R} \sum_{d_2 \ll R} \left( \frac{R \ell}{d_1 d_2} \right)^2 + M \left( \sum_{m \sim M} \left| \frac{A(m, d_2, d_1)}{m^{1-\lambda}} \right|^2 \right)$$

We now apply the usual large sieve to the sum over $r_1$ and $x$ to see that the above is

$$\ll \frac{T^{1+\epsilon} M^{1-2 \text{Re } \lambda}}{R} \sum_{d_1 \ll R} \sum_{d_2 \ll R} \left( \frac{R \ell}{d_1 d_2} \right)^2 + M d_1 d_2$$

upon using Lemma 3.5. Since $R \leq N$, $N \ll \frac{T^{2+\epsilon}}{\ell^2}$, and $M \ll \frac{R^4 \ell^2}{N d_2^4} T^{\epsilon_1}$, choosing $\ell_1$ sufficiently small, we conclude that Equation (8.5) is bounded by

$$\ll \frac{T^{1+\epsilon}}{R} \left( (R \ell)^2 + \frac{R^4 \ell^2}{N} \right) \ll \ell^2 T^{1+\epsilon} \frac{N}{T} + \frac{T^\epsilon N^2 \ell^2}{T^2} \ll T^{2+\epsilon}.$$

8.2. Contribution of contour. Recall now that $\sigma_1 < 1$. We get by Cauchy-Schwarz that

$$\left| \int_{(\sigma_1)} G(s) \sum_{m \sim M} \frac{A(m, d_2, d_1)}{m} \left( \frac{m \ell^2}{r_1^4 d_1 d_2^2} \right)^{1-2s} e \left( \frac{m x}{r_1} \right) ds \right|^2$$

$$\ll \int_{(\sigma_1)} |G(s)| \left| \sum_{m \sim M} \frac{A(m, d_2, d_1)}{m} e \left( \frac{m x}{r_1} \right) \left( \frac{m \ell^2}{r_1^4 d_1 d_2^2} \right)^{1-2s} ds \right| \int_{(\sigma_1)} |G(s)| ds$$

$$\ll \left( \frac{N^{1/2} \ell^2}{r_1^4 d_1 d_2^2} \right)^{2 \lambda_1} \int_{-\infty}^{\infty} |G(\sigma_1 + it)| \left| \sum_{m \sim M} \frac{A(m, d_2, d_1)}{m^{1-\lambda_1+2it}} e \left( \frac{m x}{r_1} \right) \right|^2 dt,$$

where $\lambda_1 := 1 - 2\sigma_1$, and since

$$\int_{(\sigma_1)} |G(s)| ds \ll N^{\lambda_1}$$
by (8.4). In order to prove Proposition 7.1, it suffices to prove

$$\ell^2 T^{1+\epsilon} R^3 \frac{1}{N} \int_{-T^\epsilon}^{T^\epsilon} g(u) \int_{-\infty}^{\infty} |G(\sigma_1 + it)| \sum_{d_1 \ll R \ell} \sum_{d_2 \ll R \ell} \frac{1}{d_1^2 d_2^2} \left( \frac{NM d_2 d_3}{R^4 \ell^2} \right)^{2\lambda_1} \frac{1}{(N^{1/2} M)^{2\lambda_1}} \left( \frac{NM d_2 d_3}{R^4 \ell^2} \right)^{2\lambda_1} \frac{1}{(N^{1/2} M)^{2\lambda_1}}$$

(8.8)

$$\times \sum_{r_1 \sim R \ell \text{ mod } r_1} \sum_{x \div m \sim M} \sum_{m \sim M} A(m, d_2, d_1) \frac{m x}{m^{1-\lambda_1+2it}} e \left( \frac{m x}{r_1} \right) \left| \sum_{m \sim M} A(m, d_2, d_1) \frac{m x}{m^{1-\lambda_1+2it}} e \left( \frac{m x}{r_1} \right) \right|^2 du \ll T^{2+\epsilon}.$$

Again since $\frac{NM d_2^2 d_3}{R^4 \ell^2} \ll T^{\epsilon_1}$ and $\lambda_1 = 1 - 2\sigma_1 > 1 - 1/4 > 1/2$,

$$\left( \frac{NM d_2^2 d_3}{R^4 \ell^2} \right)^{2\lambda_1} < \frac{NM d_2^2 d_3}{R^4 \ell^2} T^{\epsilon}.$$ 

Thus the left hand side of the equation above is bounded by

$$\frac{T^{1+\epsilon} M^{1-2\lambda_1}}{RN^{\lambda_1}} \int_{-T^\epsilon}^{T^\epsilon} g(u) \int_{-\infty}^{\infty} |G(\sigma_1 + it)| \sum_{d_1 \ll R \ell} \sum_{d_2 \ll R \ell} \sum_{r_1 \sim R \ell \text{ mod } r_1} \sum_{x \div m \sim M} \sum_{m \sim M} A(m, d_2, d_1) \frac{m x}{m^{1-\lambda_1+2it}} e \left( \frac{m x}{r_1} \right) \left| \sum_{m \sim M} A(m, d_2, d_1) \frac{m x}{m^{1-\lambda_1+2it}} e \left( \frac{m x}{r_1} \right) \right|^2 du$$

Similar to arguments in Section 8.1, we apply the large sieve, noting that

$$\int_{-T^\epsilon}^{T^\epsilon} g(u) du \ll \log T,$$

and obtain that the above is bounded by

$$\ll \frac{T^{1+\epsilon} M^{1-2\lambda_1}}{R} \sum_{d_1 \ll R \ell} \sum_{d_2 \ll R \ell} \left( \frac{R \ell}{d_1 d_2} \right)^2 \left( \sum_{m \sim M} A(m, d_2, d_1) \frac{m x}{m^{1-\lambda_1+2it}} e \left( \frac{m x}{r_1} \right) \right)^2.$$

Thus by the same arguments as in Equation (8.6) and (8.7), we derive Inequality (8.8) as desired.

9. Proof of Proposition 7.2

The proof of Proposition 7.2 is more complex. First of all, we apply a dyadic subdivision to the sum over $m$ and the integral over $u$. So we investigate sums $m \sim M$ for $M \geq \frac{R^4 \ell^2}{NM d_2^2 d_3} T^{\epsilon_1}$ and $u \sim U$ where $T^{-100} < |U| \ll T^\epsilon$. This suffices since there are $\ll \log^2 T$ such subdivisions and since the interval $-T^{-100} \leq u \leq T^{-100}$ is trivially negligible.
From Equation (7.2), it is sufficient to consider

\begin{equation}
J(R, K, M, U) := \int_{u \sim U} g(u) \sum_{d_1, d_2, r_1 \sim R} \frac{1}{d_1^2 d_2^2} \sum_{r_1 \sim \frac{r_1}{d_1 d_2}} x \mod r_1 \sum_{m \sim M} A(m, d_2, d_1) \left( \frac{m \ell^2}{r_1^4 d_1 d_2^2} \right) \frac{r_1 d_1 d_2}{\ell} e \left( \frac{m x}{r_1} \right) \right)^2 du
\end{equation}

\begin{equation}
\leq \int_{-\infty}^\infty g_1 \left( \frac{u}{U} \right) g_2(U) \sum_{d_1, d_2, r_1 \sim R} \frac{1}{d_1^2 d_2^2} \sum_{r_1 \sim \frac{r_1}{d_1 d_2}} x \mod r_1 \sum_{m \sim M} A(m, d_2, d_1) \left( \frac{m \ell^2}{r_1^4 d_1 d_2^2} \right) \frac{r_1 d_1 d_2}{\ell} e \left( \frac{m x}{r_1} \right) \right)^2 du,
\end{equation}

where \( g_1(x) \) is a smooth compactly supported function in \([\frac{1}{2}, \frac{5}{2}]\), and \( g_2(U) = \min\{\frac{1}{|U|}, \frac{R}{U}\} \).

From Lemma 5.2 Equation (5.4), recalling definition of \( f(x) \) in (6.2) and \( \frac{MN_1 d_1^3}{R^4 \ell^2} \geq T^{\kappa_1} \), we have that

\begin{equation}
F_+ \left( \frac{m \ell^2}{r_1^4 d_1 d_2^2} \right) = \frac{m \ell^2}{r_1^4 d_1 d_2^2} \int_0^\infty w_3 \left( \frac{y}{N} \right) e \left( \frac{u \ell y}{r_1^4 d_1 d_2} T \right) \sum_{j=1}^K \frac{1}{j^{\frac{1}{4} + \frac{1}{8}}}
\end{equation}

\begin{equation}
\times \left[ c_j e \left( 4 \left( \frac{m \ell^2 y}{r_1^4 d_1 d_2^2} \right)^{\frac{1}{4}} \right) \right] W \left( 8 \pi \left( \frac{m \ell^2 y}{r_1^4 d_1 d_2^2} \right)^{\frac{1}{4}} \right)
\end{equation}

\begin{equation}
+ d_j e \left( -4 \left( \frac{m \ell^2 y}{r_1^4 d_1 d_2^2} \right)^{\frac{1}{4}} \right) \right] W \left( 8 \pi \left( \frac{m \ell^2 y}{r_1^4 d_1 d_2^2} \right)^{\frac{1}{4}} \right) \right] dy + O(T^{-100}),
\end{equation}

where \( K \) is sufficiently large. Without loss of generality, we consider the term \( j = 1 \) above, the other terms being similar and also visibly smaller. Moreover, since \( U \) can be both negative and positive, we can consider the term \( e \left( 4 \left( \frac{m \ell^2 y}{r_1^4 d_1 d_2^2} \right)^{\frac{1}{4}} \right) \).

Hence pulling out factors of \( d_1, d_2, r_1, \ell \), using \( r_1 \sim \frac{R}{d_1 d_2} \), and opening up the square in (9.1), we see that we need to bound

\begin{equation}
g_2(U) \sum_{d_1, d_2, r_1 \sim R} \sum_{r_1 \sim \frac{r_1}{d_1 d_2}} (d_1^2 d_2^2)^{\frac{1}{4}} J_0(d_1, d_2)
\end{equation}

\begin{equation}
\text{where we examine}
\end{equation}

\begin{equation}
J_0(d_1, d_2) := \sum_{r_1 \sim \frac{R}{d_1 d_2}} \sum_{x \mod r_1} \sum_{m_1, m_2} A(m_1, d_1, d_2) A(m_2, d_2, d_1) \left( \frac{m_1 x - m_2 x}{r_1} \right) J(r_1, x, m_1, m_2; d_1, d_2),
\end{equation}
and
\[ J = \mathcal{J}(r_1, x, m_1, m_2; d_1, d_2) \]
\[ \quad := \int_{-\infty}^{\infty} g_1(u U) \int_0^{\infty} \int_0^{\infty} \frac{1}{y_1^2 y_2^2} w_3(y_1) w_3(y_2) W \left( 8\pi \left( \frac{m_1 \ell^2 y_1}{r_1^4 d_1 d_2^2} \right)^{\frac{1}{2}} \right) \]
\[ \times e \left( \frac{u \ell (y_1 - y_2)}{r_1 d_1 d_2 T} \right) e \left( \frac{4\ell^2}{r_1^4 d_1^2 d_2^2} ((m_1 y_1)^{\frac{1}{2}} - (m_2 y_2)^{\frac{1}{2}}) \right) dy_1 dy_2 du. \]

By the change of variable from \( y_i \) to \( y_i N \) and from \( u \) to \( u U \), and letting
\[ w_4(y_i) = \frac{1}{y_i^3} w_3(y_i) \quad \text{and} \]
\[ w_5(y_i, m_i) = w_5(y_i, m_i, r_1, \ell, d_1, d_2) := w_4(y_i) W \left( 8\pi \left( \frac{m_i \ell^2 N y_i}{r_1^4 d_1 d_2^2} \right)^{\frac{1}{2}} \right). \]

\[ J \] can then be written as
\[ J = N^2 U \int_{-\infty}^{\infty} g_1(u) \int_0^{\infty} \int_0^{\infty} w_5(y_1, m_1) w_5(y_2, m_2) e \left( \frac{u U \ell N (y_1 - y_2)}{r_1 d_1 d_2 T} \right) \]
\[ \times e \left( \frac{4\ell^2 N^2}{r_1^4 d_1^2 d_2^2} ((m_1 y_1)^{\frac{1}{2}} - (m_2 y_2)^{\frac{1}{2}}) \right) dy_1 dy_2 du. \]

Note that the functions \( w_4, w_5 \) are smooth with compact support and satisfy the bound
\[ w_4^{(j)}(x) \ll_j 1, \quad w_5^{(j)}(x, m_i) \ll_j 1. \]

Let \( h(y) = \frac{u U \ell N y}{r_1 d_1 d_2 T} + \frac{4N^2 \ell^2 (m y)^{\frac{1}{2}}}{r_1^2 d_1^2 d_2^2} \). If \( \frac{|U|N}{RT} \ll T^{\epsilon_1/8} \) or \( U \) is positive then the second term dominates and we have \( h'(y) \gg T^{\epsilon_1/4} \) also. Integration by parts many times with respect to \( y_i \) shows that the contribution from these terms are also negligible. So it suffices to consider when \( U \) is negative and \( \frac{|U|N}{RT} \gg T^{\epsilon_1/8} \). Moreover, if \( M \geq \frac{N^3 \ell^2 U^4}{T^4 d_1^3 d_2^2} T^{\epsilon_1} \), then
\[ |h'(y)| = \left| \frac{u U \ell N}{r_1 d_1 d_2 T} + \frac{4N^2 \ell^2 m y^{\frac{1}{2}}}{r_1^2 d_1^2 d_2^2 y^{\frac{3}{2}}} \right| \gg \frac{|U|N}{RT} T^{\epsilon_1/4} \gg T^{3\epsilon_1/8}. \]

Again we can do integration many times and derive that the contribution from these terms are negligible. Therefore we restrict our consideration to
\[ M \leq \frac{N^3 \ell^2 U^4}{T^4 d_1^3 d_2^2} T^{\epsilon_1}. \]

Next we consider the integration over \( u \) in Equation (9.5). Let \( \Delta = y_1 - y_2 \) so by the change of variable \( y_1 = \Delta + y_2 \), we obtain that the integration in Equation (9.5)
where \( \hat{g}_1 \) is the usual Fourier transform of \( g_1 \) and is Schwartz class. Thus,

\[
\hat{g}_1 \left( \frac{-U \ell N \Delta}{r_1 d_1 d_2 T} \right) \ll T^{-100}
\]

whenever

\[
|\Delta| \geq T^\epsilon \frac{RT}{|U|N}
\]

for any \( \epsilon > 0 \). We thus assume that

\[
|\Delta| \leq T^\epsilon \frac{RT}{|U|N} \leq T^{-\epsilon_1/8 + \epsilon}. \tag{9.7}
\]

Applying the Taylor expansion for \( (y_2 + \Delta)^{\frac{i}{2}} \), we have that

\[
(y_2 + \Delta)^{\frac{i}{2}} = y_2^{\frac{i}{2}} (1 + \mathcal{P}(\Delta, y_2))
\]

where \( \mathcal{P}(\Delta, y_2) = \sum_{j=1}^\infty c_j \left( \frac{\Delta}{y_2} \right)^j \). Also let \( \mathcal{K} = \mathcal{K}(R, T, N, U) \) be an interval such that

\[
\mathcal{K} = \left\{ \Delta \geq -y_2 : |\Delta| \leq T^\epsilon \frac{RT}{|U|N} \right\}. \tag{9.8}
\]

Thus

\[
\mathcal{J} = N^{\frac{3}{2}} U \int_0^\infty \int_{\mathcal{K}} w_5(y_2 + \Delta, m_1) w_5(y_2, m_2) \hat{g}_1 \left( \frac{-U \ell N \Delta}{r_1 d_1 d_2 T} \right)
\]

\[
\times e \left( \frac{4 \ell^\frac{1}{2} y_2^{\frac{i}{2}}}{r_1 d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}} (m_1^{\frac{i}{2}} - m_2^{\frac{i}{2}}) \right) \left( \frac{4(N \ell^2 y_2 m_1)^{\frac{1}{2}}}{r_1 d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}} \mathcal{P}(\Delta, y_2) \right) d\Delta d y_2.
\]

Since \( \Delta \ll T^{-3\epsilon_1/32} \) (picking \( \epsilon = \epsilon_1/32 \) in (9.7)) and \( \frac{|U|N \Delta}{RT} \ll T^{\epsilon_1/32} \),

\[
4(N \ell^2 y_2 m_1)^{\frac{1}{2}} \mathcal{P}(\Delta, y_2) \ll \frac{|U| N T^{\epsilon_1/4}}{RT} \mathcal{P}(\Delta, y_2) \ll T^{9\epsilon_1/32}. \tag{9.10}
\]

If \( \frac{4N^{\frac{1}{2}} \ell^\frac{1}{2} y_2^{\frac{i}{2}}}{r_1 d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}} |m_1^{\frac{i}{2}} - m_2^{\frac{i}{2}}| \gg T^{5\epsilon_1/16} \), then we can do integration with respect to \( y_2 \) many times and obtain that the contribution of these terms is negligible. Therefore we consider when

\[
|m_1^{\frac{i}{2}} - m_2^{\frac{i}{2}}| \ll T^{5\epsilon_1/16} \frac{r_1 d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}}{N^{\frac{1}{2}} \ell^{\frac{1}{2}}} \gg T^{5\epsilon_1/16} R M^{1/4},
\]
where
\begin{equation}
R = \frac{R \ell^2}{N^{1/2} \tau_1^{1/2} \tau_2^{1/2}} \ll T^{-\epsilon_1/4},
\end{equation}
upon recalling that we are working in the range \( M \geq \frac{R \ell^2}{N^{1/2} \tau_1^{1/2} \tau_2^{1/2}} T^{\epsilon_1} \). Thus
\begin{equation}
|m_1 - m_2| \ll T^{5\epsilon_1/16} RM := \mathcal{L},
\end{equation}
and (9.10) becomes
\begin{equation}
\mathcal{P}(\Delta, y_2) \ll RT^{9\epsilon_1/32}.
\end{equation}

Next, we divide the range for \( m_1, m_2 \) into intervals \( C_{\eta_1} \) and \( C_{\eta_2} \) of length \( T^{-\epsilon_1} \mathcal{L} \), where \( \eta_1 \) and \( \eta_2 \) are the left endpoints of the intervals \( C_{\eta_1} \) and \( C_{\eta_2} \) respectively. When \( m_i \in C_{\eta_i} \), for some \( \eta_i \) and Equation (9.12) holds, the restriction of the length of the intervals implies that \( |\eta_1 - \eta_2| \ll \mathcal{L} \). Hence for fixed \( C_{\eta_1} \), there are \( O(T^{\epsilon_1}) \) choices for \( C_{\eta_2} \), and so there are \( O(\frac{1}{R} T^{2\epsilon_1/16}) \) relevant pairs of intervals \( (C_{\eta_1}, C_{\eta_2}) \) with end points satisfying Equation (9.12). We let \( \sum_{(C_{\eta_1},C_{\eta_2})} \) denote the sum over such pairs.

From Equations (9.3), (9.4), (9.9), and the trivial bound \( g_1 \left( \frac{U \ell N x}{r_1 d_1 d_2} \right) \ll 1 \), we have that
\begin{equation}
\mathcal{J}_0 \ll g_2(U) UN^{1/2} \int_0^\infty \int_{\mathcal{L}} |w_4(y_2 + \Delta)w_4(y_2)| \sum_{(C_{\eta_1},C_{\eta_2})} S(C_{\eta_1},C_{\eta_2}) d\Delta dy_2,
\end{equation}
where
\begin{equation}
S(C_{\eta_1},C_{\eta_2}) := \sum_{r_1 \sim \frac{R \ell}{d_1 d_2} \mod r_1} \sum_{x \mod r_1} |S_1(C_{\eta_1})S_2(C_{\eta_2})|;
\end{equation}

\begin{equation}
S_1(C_{\eta_1}) := \sum_{m_1 \in C_{\eta_1}} \frac{A(m_1, d_2, d_1)}{m_1^2} W \left( 8\pi \left( \frac{m_1 N \ell^2 (y_2 + \Delta)}{r_1^4 d_1 d_2^2} \right)^{1/2} \right) e \left( \frac{4N^{1/2} \ell^{1/2} y_2 m_1^{1/2}}{r_1 d_1 d_2} \right)
\times e \left( \frac{4(N \ell^2 y_2 m_1)^{1/2}}{r_1 d_1 d_2} \right) \mathcal{P}(\Delta, y_2) e \left( \frac{m_1 x}{r_1} \right);
\end{equation}

and
\begin{equation}
S_2(C_{\eta_2}) := \sum_{m_2 \in C_{\eta_2}} \frac{A(m_2, d_2, d_1)}{m_2^2} W \left( 8\pi \left( \frac{m_2 N \ell^2 y_2}{r_1^4 d_1 d_2^2} \right)^{1/2} \right) e \left( -\frac{4N^{1/2} \ell^{1/2} y_2 m_2^{1/2}}{r_1 d_1 d_2^2} \right) e \left( -\frac{m_2 x}{r_2} \right).
\end{equation}

By the inequality \( 2|ab| \leq |a|^2 + |b|^2 \). We have that
\begin{equation}
|S(C_{\eta_1},C_{\eta_2})| \leq \frac{1}{2} \sum_{r_1 \sim \frac{R \ell}{d_1 d_2} \mod r_1} \sum_{x \mod r_1} |S_1(C_{\eta_1})|^2 + \frac{1}{2} \sum_{r_1 \sim \frac{R \ell}{d_1 d_2} \mod r_1} \sum_{x \mod r_1} |S_2(C_{\eta_2})|^2.
\end{equation}

To bound \( \mathcal{J}_0 \) in Equation (9.14), we first prove the following Lemma.
Lemma 9.1. Let \( S_i(C_{n_i}) \) for \( i = 1, 2 \) be defined as above. We have

\[
\sum_{r_1 \sim \frac{R\ell}{d_1 d_2}} \sum_{x \mod r_1}^* |S_i(C_{n_i})|^2 \ll M^{-3/4} \left( \left( \frac{R\ell}{d_1 d_2} \right)^2 + \mathcal{L} \right) \mathcal{G}(\eta_i),
\]

where

\[
(9.16) \quad \mathcal{G}(\eta_i) := \sum_{m \in C_{n_i}} |A(m, d_2, d_1)|^2.
\]

Proof. We will bound \( \sum_{r_1 \sim \frac{R\ell}{d_1 d_2}} \sum_{x \mod r_1}^* |S_1(C_{n_1})|^2 \) since the proof for \( S_2 \) is similar and somewhat simpler due to the absence of dependence on \( \Delta \).

Define

\[
A(x) = W \left( 8\pi (y_2 + \Delta)^{\frac{1}{4}} \right) e \left( 4y_2^{\frac{1}{2}} \right) e \left( -4\mathcal{P}(\Delta, y_2)^{\frac{1}{2}} \right).
\]

We will take a Taylor expansion of \( A \left( \left( \frac{N\ell^2 m_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \right) \) around

\[
(9.17) \quad x = \left( \frac{N\ell^2 \eta_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \sim \frac{1}{R}
\]

to separate variables \( r_1 \) and \( m_1 \) before applying the Large Sieve. We write

\[
A \left( \left( \frac{N\ell^2 m_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \right) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} A^{(\alpha)} \left( \left( \frac{N\ell^2 \eta_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \right) \left( \left( \frac{N\ell^2 m_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} - \left( \frac{N\ell^2 \eta_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \right)^{\alpha}.
\]

By Equation (9.12), \( m_1, \eta_1 \sim M \) and the Mean Value Theorem, we obtain that

\[
(9.18) \quad |m_1^{\frac{1}{4}} - \eta_1^{\frac{1}{4}}| \leq \frac{|m_1 - \eta_1|}{M^{\frac{3}{4}}} \ll \frac{\mathcal{R}MT^{\frac{91}{16}}}{M^{\frac{3}{2}} T^{\epsilon_1}} \ll \mathcal{R}M^{1/4} T^{-\frac{111}{16}}.
\]

Moreover by the property of \( W(x) \) (e.g. see [24] p.206), Equations (9.17) and (9.13), and \( y_2 \ll 1 \),

\[
A^{(\alpha)}(x) \ll \alpha! c^\alpha x^{-\alpha} + \left( \mathcal{R} T^{9\epsilon_1/32} \right)^{\alpha} \ll \alpha! c^\alpha \left( \mathcal{R} T^{9\epsilon_1/32} \right)^{\alpha}
\]

where \( c \) is some absolute constant. Thus

\[
\frac{1}{\alpha!} A^{(\alpha)} \left( \left( \frac{N\ell^2 \eta_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \right) \left( \left( \frac{N\ell^2 m_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} - \left( \frac{N\ell^2 \eta_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \right)^{\alpha} \ll \left( \mathcal{R} \frac{c}{T^{\frac{9\epsilon_1}{32}}} \right)^{\alpha}.
\]

Recalling (9.11), we can choose \( B \) such that \( \left( \mathcal{R} \frac{c}{T^{\frac{9\epsilon_1}{32}}} \right)^{\alpha} \ll T^{-100} \) and obtain that

\[
A \left( \left( \frac{N\ell^2 m_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \right) = \sum_{0 \leq \alpha \leq B} \frac{1}{\alpha!} A^{(\alpha)} \left( \left( \frac{N\ell^2 \eta_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \right) \left( \left( \frac{N\ell^2 m_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} - \left( \frac{N\ell^2 \eta_1}{r_1^2 d_1 d_2^2} \right)^{\frac{1}{4}} \right)^{\alpha} + O(T^{-100}).
\]
Hence to bound \( \sum_{r_1 \sim \frac{RT}{M^3}} \sum_{x \mod r_1} |S_1(C_{\eta_1})|^2 \), it is enough bound \( S_1(C_{\eta_1}; \alpha) \) for fixed \( \alpha \), which is defined to be

(9.19)

\[
\sum_{r_1 \sim \frac{RT}{M^3}} \sum_{x \mod r_1} \left| \sum_{m_1 \in C_{\eta_1}} \frac{A(m_1, d_2, d_1)}{m_1^3} \right| A^{(\alpha)} \left( \left( \frac{N\ell^2 \eta_1}{r_1^4 d_1^2 d_2^2} \right)^{\frac{1}{4}} \right) \\
\times \left( \left( \frac{N\ell^2 m_1}{r_1^4 d_1 d_2^2} \right)^{\frac{1}{4}} - \left( \frac{N\ell^2 \eta_1}{r_1^4 d_1^2 d_2^2} \right)^{\frac{1}{4}} \right)^\alpha \epsilon \left( - \frac{m_1 x}{r_1} \right)^2
\]

\[
\ll c^\alpha \left( \frac{N^{1/2}d_1^{3/2}d_2}{RT^2} \right)^\alpha \left( \frac{RT^9}{M^{1/2}d_2} \right)^{2\alpha} \sum_{r_1 \sim \frac{RT}{M^3}} \sum_{x \mod r_1} \left| \sum_{m_1 \in C_{\eta_1}} \frac{A(m_1, d_2, d_1)}{m_1^3} \right| \left( \frac{m_1^\frac{1}{2} - \eta_1^\frac{1}{2}}{r_1} \right)^2 \epsilon \left( - \frac{m_1 x}{r_1} \right)^2
\]

\[
= M^{-3/4} \left( \frac{cN^{1/2}d_1^3d_2M^{1/2}R^2R^2}{RT^3} \right)^\alpha \left( \left( \frac{R\ell}{d_1 d_2} \right)^2 + \mathcal{L} \right) \left( \left( \frac{R\ell}{d_1 d_2} \right)^2 + \mathcal{L} \right) \mathcal{S}(\eta_1)
\]

by (9.18) and where \( \mathcal{S} \) is as defined in (9.16). Using that \( \mathcal{R}^2 \gg \frac{R^2}{(MN)^{1/2}d_1^3d_2} \) and \( \mathcal{R} \ll T^{-\epsilon_1/4} \) (see Equation 9.11), we obtain that the quantity in (9.19) is maximized when \( \alpha = 0 \) for sufficiently large \( T \), so that

(9.20)

\[
S_1(C_{\eta_1}; \alpha) \ll M^{-3/4} \left( \frac{R\ell}{d_1 d_2} \right)^2 \mathcal{S}(\eta_1),
\]

for all \( \alpha \).

\[\square\]

By Lemma 9.1 and (9.15),

\[
\sum_{(C_{\eta_1}, C_{\eta_2})}^\text{rel} \mathcal{S}(\eta_1, C_{\eta_2}) \ll M^{-3/4} \left( \frac{R\ell}{d_1 d_2} \right)^2 \mathcal{L} \sum_{(C_{\eta_1}, C_{\eta_2})}^\text{rel} \left( \mathcal{S}(\eta_1) + \mathcal{S}(\eta_2) \right)
\]

\[
\ll M^{-3/4} \left( \frac{R\ell}{d_1 d_2} \right)^2 \mathcal{L} T^{\epsilon_1} \sum_{m \sim M} |A(m, d_2, d_1)|^2,
\]

since for each \( \eta_1 \) there are \( \ll T^{\epsilon_1} \) choices for \( \eta_2 \) and vice versa. In the sequel, we shall simplify notation slightly and write our bounds in terms of \( \epsilon > 0 \) which can be made arbitrarily small by taking \( \epsilon_1 \) sufficiently small. Using the bound in Lemma, which is 3.5

\[
\sum_{m \sim M} |A(m, d_2, d_1)|^2 \ll (Md_1 d_2)^{1+\epsilon} \ll Md_1 d_2 T^\epsilon,
\]
and \( \mathcal{L} = \mathcal{R} MT^{5\varepsilon_1/16} \), we see that the above is
\[
\sum_{(C_{\eta_1}, C_{\eta_2})} \text{S}(C_{\eta_1}, C_{\eta_2}) \ll M^{1/4} d_1 d_2 T^\varepsilon (R\ell)^2 + \mathcal{R} \mathcal{M}.
\]
It follows from (9.14) and (9.7) that
\[
J_0 \ll M^{1/4} g_2(U) U N^{\frac{5}{4}} d_1 d_2 T^\varepsilon \left( \frac{RT}{|U|N} \right) \left( \left( \frac{R\ell}{d_1 d_2} \right)^2 + \mathcal{R} \mathcal{M} \right),
\]
and using that \( g_2(U) U \ll 1 \) for all \( U \), and referring to Equations (7.2) and (9.2), it now suffices to bound
\[
\ell^2 \frac{T^{1+\varepsilon} R^3}{N} N^{\frac{5}{4}} \frac{RT}{|U|N} \sum_{d_1 \ll \ell \ell} \sum_{d_2 \ll \ell \ell} \frac{d_1 d_2}{d_1^2 d_2^2} \left( \frac{d_1^2 d_2^2}{R^4 \ell^2} \right)^{\frac{5}{4}} \left( \left( \frac{R\ell}{d_1 d_2} \right)^2 + \mathcal{R} \mathcal{M} \right) M^{\frac{5}{4}}
\]
(9.21) \[
\ll \frac{T^{2+\varepsilon}}{RN^{3/4}T^{1/2}|U|} \sum_{d_1 \ll \ell \ell} \sum_{d_2 \ll \ell \ell} \left( \left( \frac{M^{1/4}}{d_1^2 d_2^2} \right)^2 + \frac{R\ell^1/2 d_1 d_2}{N^{1/4} M^{1/2} d_1^2 d_2^2} \right) M^{\frac{5}{4}}
\]
by (9.11). Using that \( M \ll \frac{N^{3+\ell^2} T^{\varepsilon_1}}{T d_1^2 d_2^2} \) from (9.6) and summing over \( d_1, d_2 \), we obtain that the quantity in (9.21) is
\[
(9.22) \ll \frac{T^{2+\varepsilon}}{RN^{3/4}T^{1/2}|U|} \left( \frac{R^2 \ell^5/2 N^{3/4} |U|}{T} + \frac{RU^4 N^3 \ell^{5/2}}{N^{1/4} T^4} \right) \ll T^{1+\varepsilon} \ell^2 R + T^\varepsilon \frac{N^2 \ell^2}{T^2},
\]
where we have used that \( |U| \ll T^\varepsilon \). Now using that \( R \ll \frac{N}{T} \) and \( N \ll \frac{T^{2+\varepsilon}}{\ell^2} \), we see that (9.22) is \( \ll T^{2+\varepsilon} \) as desired.

**Appendix A. Proof of Lemma 5.2**

The proof requires properties of \( J \)-Bessel function as the following.

**Lemma A.1.** For any integer \( k \geq 0 \),
\[
J_k(2\pi x) = \frac{1}{2\pi \sqrt{x}} \left\{ W_k(2\pi x) e\left( x - \frac{k}{4} - \frac{1}{8} \right) + W_k(2\pi x) e\left( -x + \frac{k}{4} + \frac{1}{8} \right) \right\},
\]
where \( W_k^{(j)}(x) \ll_{j,k} x^{-j} \). Moreover,
\[
J_k(2x) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{x^{2\ell+k}}{\ell!(\ell + k)!}.
\]
These results are standard, e.g. see [24] for the proof.
Proof of Lemma 5.2. We start with evaluating the asymptotic formula for $\Psi_+$. From the definition of $\Psi_+$ in Equation (5.2), for $x > 0$

$$
\Psi_+(x) = \frac{1}{2\pi i} \int_{(-\sigma)} \frac{x^{s-2} \Gamma\left(\frac{1-s-a_1}{2}\right) \Gamma\left(\frac{1-s-a_3}{2}\right) \Gamma\left(\frac{1-s-a_4}{2}\right)}{\Gamma\left(\frac{s+a_1}{2}\right) \Gamma\left(\frac{s+a_3}{2}\right) \Gamma\left(\frac{s+a_4}{2}\right)} \psi(s)x^s ds
$$

$$
= \frac{2x\pi^2}{2\pi i} \int_{(\sigma_1)} \frac{\pi^{-s-2} x^{-2s} \Gamma\left(\frac{1-s-a_1}{2}\right) \Gamma\left(\frac{1-s-a_3}{2}\right) \Gamma\left(\frac{1-s-a_4}{2}\right)}{\Gamma\left(\frac{s+a_1}{2}\right) \Gamma\left(\frac{s+a_3}{2}\right) \Gamma\left(\frac{s+a_4}{2}\right)} \psi(-2s + 1) ds,
$$

where $\sigma_1 = \frac{1+\sigma}{2} > \frac{15}{68}$.

Let

$$
H(s) = 4^{8s-2} \frac{\prod_{j=1}^{\kappa} \Gamma\left(s - \frac{\alpha_j}{2}\right) \Gamma\left(\frac{1}{2} - 4s\right)}{\prod_{j=1}^{\kappa} \Gamma\left(s + \frac{\alpha_j}{2}\right) \Gamma\left(4s - \frac{3}{2}\right)} - 1.
$$

From the Stirling’s formula (e.g. [16]) below

$$
\log \Gamma(s + c) = \left(s + c - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + \frac{K}{s} + O\left(\frac{1}{|s|^{K+1}}\right),
$$

where $c$ is a constant, $a_j$ are suitable constants, $K$ is a fixed positive integer, $|\arg(s)| \leq \pi - \delta$, $\delta > 0$, excluding the points $s = 0$ and the neighborhoods of the poles of $\Gamma(s + c)$, it can be shown that

$$
H(s) = \sum_{j=1}^{\kappa} \frac{b_j}{s_j} + O\left(\frac{1}{|s|^{K+1}}\right),
$$

where $b_j$ are appropriate constants, depending on $\alpha_i$. 

By the definition of $\Psi_+(x)$, we have that

$$
\Psi_+(x) = \frac{2x\pi^2}{2\pi i} \int_{(\sigma_1)} \frac{\pi^{-s-2} x^{-2s} \Gamma\left(\frac{1}{2} - 4s\right)}{\Gamma\left(\frac{s+a_1}{2}\right) \Gamma\left(\frac{s+a_3}{2}\right) \Gamma\left(\frac{s+a_4}{2}\right)} \psi(-2s + 1) ds
$$

$$
+ \frac{2x\pi^2}{2\pi i} \int_{(\sigma_1)} \frac{\pi^{-s-2} x^{-2s} \Gamma\left(\frac{1}{2} - 4s\right)}{\Gamma\left(\frac{s+a_1}{2}\right) \Gamma\left(\frac{s+a_3}{2}\right) \Gamma\left(\frac{s+a_4}{2}\right)} H(s) \psi(-2s + 1) ds
$$

$$
=: I_1 + I_2.
$$

Firstly, let us consider $I_1$. By changing variables $4s - 3/2$ to $w$, we obtain that

$$
I_1 = \frac{x\pi^2}{4\pi i} \int_{(\sigma_2)} \frac{\pi^{-2w-3} 4^{-2w-1} x^{-w} \Gamma(w) \psi\left(-\frac{w}{2} + \frac{1}{4}\right)}{\Gamma(-1 - w) \Gamma\left(-1 - \frac{w}{2}\right)} dw
$$

$$
= \frac{x}{16\pi^2 i x^2} \int_{(\sigma_2)} \frac{\Gamma(w) \psi\left(-\frac{w}{2} + \frac{1}{4}\right)}{\Gamma\left(-1 - \frac{w}{2}\right) (4\pi)^{-2w} x^{-w}} dw,
$$

where $\sigma_2 = 4\sigma_1 - \frac{3}{2} > -\frac{21}{34}$. We can choose $\sigma_2 > 0$. We move countour integration to the left to $\text{Re}(s) = -\infty$, picking up poles of $\Gamma(w)$ at $w = -2, -3, \ldots$. Note that there

\footnote{This is due to the bound for $\lambda_1$ in the work of Luo, Rudnick and Sarnak [19].}
are zeros at \( w = 0, -1 \) from \( \Gamma(-1 - w) \), which are cancelled with the poles at 0, -1. Hence

\[
I_1 = \frac{x}{8\pi x^{1/4}} \sum_{n=2}^{\infty} \frac{(-1)^n}{n!(-1 + n)} (4\pi x^{1/4})^{2n\psi'} \left( \frac{n}{2} + \frac{1}{4} \right)
\]

\[
= \frac{x}{8\pi x^{1/4}} \int_0^\infty \psi(y) \sum_{n=2}^{\infty} \frac{(-1)^n}{n!(n-2)!} (4\pi x^{1/4})^{2n\frac{y}{2} + \frac{1}{4} - 1} dy
\]

\[
= 2\pi x \int_0^\infty \frac{\psi(y)}{(xy)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!n!} (4\pi (xy)^{1/4})^{2n+2} dy
\]

\[
= 2\pi x \int_0^\infty \frac{\psi(y)}{(xy)^{1/4}} J_2 \left( 8\pi (xy)^{1/4} \right) dy
\]

where the last equation comes from the representation of \( J \)-Bessel function in Equation (A.2). By Lemma A.1 Equation (A.1), we then have

\[
I_1 = x \int_0^\infty \frac{\psi(y)}{(xy)^{1/4}} \left[ ce \left( 4(xy)^{1/4} \right) W_2(8\pi (xy)^{1/4}) + de \left( -4(xy)^{1/4} \right) \bar{W}_2(8\pi (xy)^{1/4}) \right] dy
\]

for some constants \( c, d \).

Next we consider \( I_2 \). Since the expansion of \( H(s) \) is of the form in (A.3). Therefore it is sufficient to consider

\[
I_{2,j} = \frac{2x\pi^2}{2\pi i} \int_{(s_j)} \frac{\Gamma(w)}{\Gamma(\frac{1}{2} - 4s)(w + 3/2)^{2s}} \psi(-2s + 1) ds.
\]

By the change of variables \( 4s - 3/2 \to w \), we have

\[
I_{2,j} = \frac{4^j x}{16\pi^2 x^{3/4}} \int_{(s_1)} \frac{\Gamma(w)}{\Gamma(1 - w)(w + 3/2)^{-2s}} (4\pi)^{-2w} x^{-\frac{w}{2}} \psi\left( -\frac{w}{2} + \frac{1}{4} \right) dw.
\]

We illustrate how to find asymptotic formula for \( I_{2,j} \) when \( j = 1 \). Other cases are proceeded similarly.

\[
I_{2,1} = -\frac{x}{4\pi^2 x^{3/4}} \int_{(s_2)} \frac{\Gamma(w)}{\Gamma(1 - w)(1 - w)^{2s}} (4\pi)^{-2w} x^{-\frac{w}{2}} \psi\left( -\frac{w}{2} + \frac{1}{4} \right) dw
\]

\[
+ \frac{x}{8\pi^2 x^{1/4}} \int_{(s_1)} \frac{\Gamma(w)}{\Gamma(1 - w)(1 - w)^{2s}} (4\pi)^{-2w} x^{-\frac{w}{2}} \psi\left( -\frac{w}{2} + \frac{1}{4} \right) dw
\]

\[
=: I_{2,1}^1 + I_{2,1}^2.
\]

By moving the contour integral to the far left for \( I_{2,1}^1 \), we pick up poles at \( n = -1, -2, \ldots \) and obtain that

\[
I_{2,1}^1 = -\frac{x}{2\pi x^{1/4}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\Gamma(n)} (4\pi x^{1/4})^{2n\psi'} \left( \frac{n}{2} + \frac{1}{4} \right) = -2x \int_0^\infty \frac{\psi(y)}{(xy)^{1/4}} J_1 \left( 8\pi (xy)^{1/4} \right) dy.
\]
Next we write $I_{2,1}^2$ as $I_{2,1}^3 + I_{2,1}^4$, where
\[ I_{2,1}^3 = -\frac{x}{8\pi^2 i x^\frac{3}{4}} \int_{(\sigma_2)} \frac{\Gamma(w)}{\Gamma(-w)(-w)} (4\pi)^{-2w} x^{-\frac{w}{2}} \tilde{\psi}\left(\frac{-w}{2} + \frac{1}{4}\right) \, dw \]
\[ I_{2,1}^4 = \frac{3x}{16\pi^2 i x^\frac{7}{4}} \int_{(\sigma_2)} \frac{\Gamma(w)}{\Gamma(-w)(-w)(w+3/2)} (4\pi)^{-2w} x^{-\frac{w}{2}} \tilde{\psi}\left(\frac{-w}{2} + \frac{1}{4}\right) \, dw. \]

Repeating the above arguments, we have that
\[ I_{2,1}^3 = -\frac{x}{4\pi} \int_0^\infty \frac{\psi(y)}{(xy)^\frac{3}{4}} J_0\left(8\pi(xy)^\frac{1}{4}\right) \, dy. \]

We use Equation (A.1) to write the $J$-Bessel function in terms of $W_k$. Finally we then apply the same arguments to $I_{2,1}^4$ and so on, and obtain lower order terms.

\[ \square \]

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References


**Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, United States**

*Email address: chandee@ksu.edu*

**Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, United States**

*Email address: xiannan@math.ksu.edu*