Continuity, Intermediate Value Theorem (2.3)

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**Intuitive Definition:** A function \( f(x) \) is **continuous** at \( a \) if you can draw the graph of \( y = f(x) \) without lifting your pen when \( x = a \).

**Definition (continuity)**

A function \( f(x) \) is **continuous** at a number \( a \) if

\[
\lim_{x \to a} f(x) = f(a).
\]

This means that a function \( f(x) \) is **continuous** at \( a \) provided that:

1. \( f(a) \) is defined
2. \( \lim_{x \to a} f(x) \) exists
3. \( \lim_{x \to a} f(x) = f(a) \)
Definition (continuity)

A function \( f(x) \) is **continuous** at a number \( a \) if

\[
\lim_{x \to a} f(x) = f(a).
\]

Each of the plots above are **discontinuous** at \( x = 2 \) but **continuous** at \( x = 1 \).
Classification: We classify discontinuities of a function $f(x)$ as one of removable, jump, infinite or other.

Each of the plots above are discontinuous at $x = 2$.

1. For removable discontinuities, the limit exists, but is not equal to the function value.
2. For jump discontinuities, the left and right limits exist but are not equal (so the limit does not exist).
3. For infinite discontinuities, left or right limits are $\pm \infty$.
Classification: We classify discontinuities of a function $f(x)$ as one of removable, jump, infinite or other.

$$\lim_{x \to 0^-} f(x), \lim_{x \to 0^+} f(x), \text{ and } \lim_{x \to 0} f(x) \text{ do not exist}$$

This familiar plot is discontinuous at $x = 0$. This oscillating discontinuity is classified as "other".
We say that a function \( f(x) \) is continuous on an interval if it is continuous at every point in the interval.

**Example:** Where is \( f(x) = \frac{1}{x-2} \) continuous/discontinuous?

- **Continuous:**
- **Discontinuous:**

![Graph of \( y = \frac{1}{x-2} \)](image-url)
Definition

1. A function $f(x)$ is continuous from the left at a number $a$ if
\[ \lim_{x \to a^-} f(x) = f(a). \]

2. A function $f(x)$ is continuous from the right at a number $a$ if
\[ \lim_{x \to a^+} f(x) = f(a). \]

Note, $f(x)$ is continuous at $a$ if and only if $f(x)$ is continuous from both the right and left at $a$. 
The function above is discontinuous at $x = 2$, continuous from the right at $x = 2$, and discontinuous from the left at $x = 2$.

The function is continuous on $(0, 2)$ and $(2, 4)$. 
From earlier in the lecture, we know that wherever they are defined, polynomials, rational functions, and root functions are all continuous. This is what we really were using when we said for example that
\[
\lim_{x \to 2} (x^2 - 3x + 1) = -1.
\]

**Theorem**

The following types of functions are continuous at every number in their domains:

- polynomials (ex., \(x^3 + x + 1\))
- rational functions (ex., \(\frac{x^3+x+1}{x^2+1}\))
- root functions (ex., \(\sqrt[3]{x}\))
- trigonometric functions (ex., \(\cos(x)\))
- inverse trigonometric functions (ex., \(\arcsin(x)\))
- exponential functions (ex., \(e^x\))
- logarithmic functions (ex., \(\ln(x)\))
Theorem

If \( f(x) \) and \( g(x) \) are continuous at \( a \) and \( c \) is a constant, then the following functions are also continuous at \( a \):

- \( f(x) + g(x) \)
- \( f(x) - g(x) \)
- \( c \cdot f(x) \)
- \( f(x) \cdot g(x) \)
- \( \frac{f(x)}{g(x)} \) if \( g(a) \neq 0 \).

Applying these rules multiple times, we see that

\[
h(x) = \frac{\sqrt[3]{x} + x^{47}\cos(x)}{e^x}
\]

is continuous wherever it is defined (which is everywhere).

Thus, \( \lim_{x \to 3} h(x) = \ldots \)
Continuity also works well with the composition of functions.

**Theorem**

If \( g(x) \) is continuous at \( a \) and \( f(x) \) is continuous at \( g(a) \), then the function \( (f \circ g)(x) = f(g(x)) \) is continuous at \( a \).

Hence, the following functions are continuous wherever they are defined:

- \( w(x) = \sqrt[3]{x^2 + 1} \) so \( w(x) = f(g(x)) \) where
  
  \[
  f(x) = \quad \text{and} \quad g(x) =
  \]

  \[
  \lim_{{x \to 4}} w(x) =
  \]

- \( v(x) = \ln(x^2 + x) \) so \( \lim_{{x \to 3}} v(x) = \)
Let \( f(x) = \begin{cases} 
\sin(x) & \text{if } x < 0 \\
4 & \text{if } x = 0 \\
x + 4 & \text{if } x > 0 
\end{cases} \)

Where is \( f(x) \) continuous/discontinuous? Is \( f(x) \) ever discontinuous but continuous from the left or right?
Let $f(x) = \begin{cases} 
\sin(x) & \text{if } x < 0 \\
4 & \text{if } x = 0 \\
x + 4 & \text{if } x > 0 
\end{cases}$
Theorem (Intermediate Value Theorem)

Suppose that $f(x)$ is a continuous function on the closed interval $[a, b]$ and that $f(a) \neq f(b)$. Let $M$ be any number strictly between $f(a)$ and $f(b)$. Then, there exists a $c$ in $(a, b)$ with $f(c) = M$.

- $f(x)$ above is continuous on $[0, 2]$, and since 3 falls between $f(0)$ and $f(2)$, there exists a $c$ in $(0, 2)$ with $f(c) = 3$. 
Old joke:
Q: Why did the chicken cross the road?
A: To get to the other side!

**IVT variant:**
Q: How do you know the chicken crossed the road?
A: Because it started on one side and got to the other side!
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Q: How do you know the chicken crossed the road?
A: Because it started on one side and got to the other side!

**More precise version:**
Q: Suppose $g(x)$ is a continuous function on $[0, 2]$ that goes through the points $(0, 1)$ and $(2, 5)$. How do you know it must cross the line $y = 3$?
A: Because it goes from $y = 1$ to $y = 5$. When it crosses $y = 3$ at the point $c$ in the interval $(0, 2)$, $g(c)$ must be 3.
Theorem (Intermediate Value Theorem)

Suppose that \( f(x) \) is a continuous function on the closed interval \([a, b]\) and that \( f(a) \neq f(b) \). Let \( M \) be any number strictly between \( f(a) \) and \( f(b) \). Then, there exists a \( c \) in \((a, b)\) with \( f(c) = M \).

Show that \( x^7 + x^2 = -x + 1 \) has a solution in \((0, 1)\).
Intermediate Value Theorem

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Show that \( 2^x = 5x \) has a solution.