A note on relativistic rocketry

Shawn Westmoreland

Department of Mathematics, Kansas State University, Manhattan, Kansas, 66506, USA

Abstract

In the context of special relativity, we discuss the specific impulse of a rocket whose exhaust jet consists of massive and/or massless particles. This work generalizes previous results and corrects some errors of a recently published paper by U. Walter. (The errors stem from the omission of a Lorentz factor.)

Key words: Interstellar travel, Relativistic rocket, Specific impulse

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1. Introduction[1]

In a recently published paper, U. Walter [1] considered the problem of deriving an expression for the specific impulse of a relativistic rocket which utilizes massless and/or massive particles in its exhaust jet. This is an exercise in Special Relativity which does not seem to have been remarked upon before Walter's paper, but unfortunately the solution given by Walter is erroneous due to the omission of a Lorentz factor at a crucial step.

The main purpose of this paper is to fix Walter's solution and to expound on some consequences of the correction. In particular, we find that the antimatter rocket described in Walter's paper can achieve a specific impulse

Email address: `westmore@math.ksu.edu` (Shawn Westmoreland)

[1]Symbols are given in the Appendix.

of about $0.58c$. This is much higher than Walter's figure of $0.21c$, and it agrees with some calculations previously published by Vulpetti [2].

2. Specific impulse

Consider a rocket of mass M accelerating itself along a straight line through resistance-free flat space. ("Mass" always means "rest mass" in this paper.) Choose an inertial frame $\mathcal{F}$ that is instantaneously at rest with respect to the rocket. During an infinitesimal tick $d\tau$ of proper time in $\mathcal{F}$ (which we regard as equivalent to an infinitesimal interval of proper time aboard the rocket), the rocket changes its momentum by an amount $M d\sigma$, where $d\sigma$ denotes the infinitesimal change in the speed of the rocket with respect to $\mathcal{F}$. (Hence $d\sigma$ is the change in the "proper speed" of the rocket.)

By the conservation of linear momentum, the change in the momentum of the rocket must be compensated by the ejection of propellant. As propellant is ejected, the mass of the rocket decreases. Let dM denote the amount of mass lost by the rocket during the infinitesimal time interval $d\tau$. The loss in mass dM is accounted for in terms of the rest masses of any massive exhaust particles together with their kinetic energies, as well as massless exhaust particles and waste (refer to Figure 1). The massive exhaust particles may, but need not, include inert propellants.

We denote by ε the fractional amount of mass that is lost due to the release of energy into space. That is, the amount of energy available for propulsion (during an infinitesimal proper time interval $d\tau$) is $\varepsilon c^2 dM$. The amount of mass lost due to the release of massive exhaust particles is then $(1-\varepsilon)dM$.

Some of the energy available for propulsion is likely to be wasted. Denote by η the fractional amount of available energy that actually gets utilized for propulsion. In other words, the energy utilized by the propulsion system (during an infinitesimal interval $d\tau$ of proper time) is effectively $\eta\varepsilon c^2 dM$ and the amount of energy wasted is $(1-\eta)\varepsilon c^2 dM$. The wasted energy can account for, among other possibilities, a loss in efficiency resulting from an exhaust jet that forms a wide-angled cone instead of a well-collimated beam.

Denote by δ the fractional amount of utilized energy that goes into the effective kinetic energy of the massive exhaust particles. That is, the massive exhaust particles have an effective kinetic energy of $\delta\eta\varepsilon c^2 dM$ and the massless exhaust particles have an effective energy of $(1-\delta)\eta\varepsilon c^2 dM$.

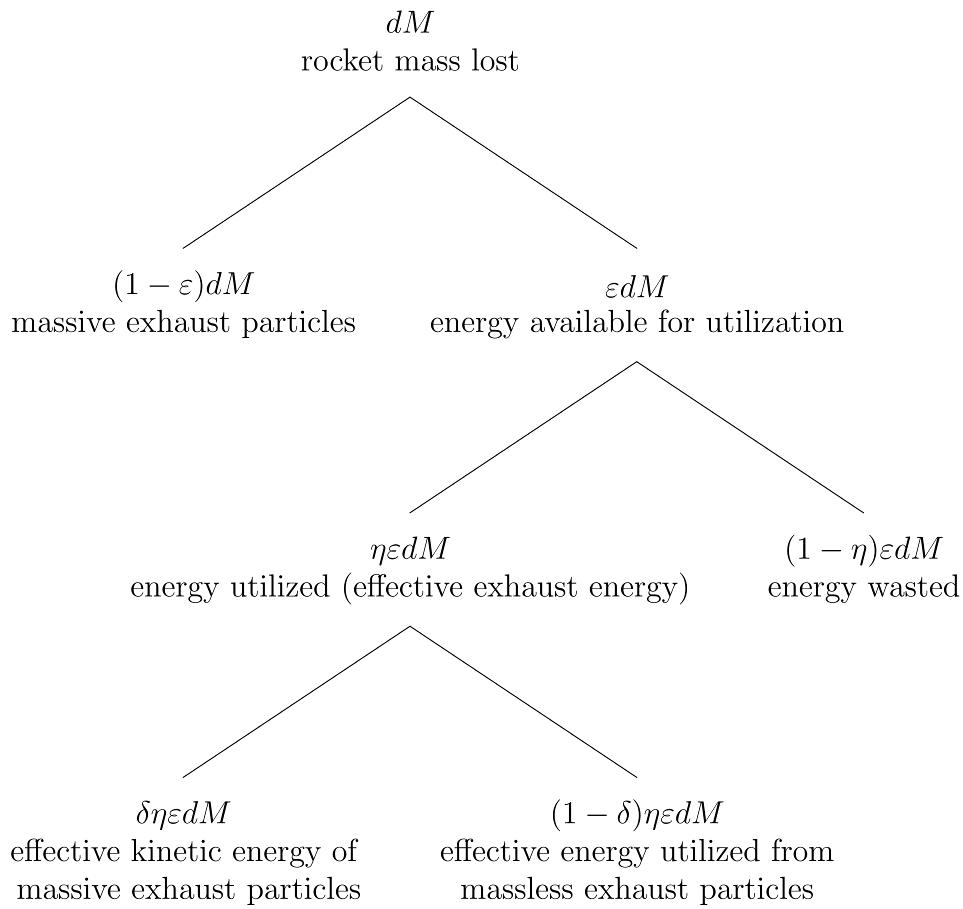

Figure 1: Energy scheme accounting for the total loss of rocket mass dM and its relationship to energized exhaust during an infinitesimal interval of proper time aboard the rocket. (This diagram omits factors of c^2.)

Let u denote the effective relative speed of the massive exhaust particles as they are expelled from the rocket during the infinitesimal proper time interval $d\tau$. (The meaning of "effective relative speed" is that the massive exhaust particles affect the momentum of the rocket as if they were all collected together into a single particle of mass $(1-\varepsilon)dM$ and thrown out of the rocket at a relative speed u in the exactly backward direction.) The conservation of linear momentum gives:

$$M d\sigma = -\frac{(1-\varepsilon) u dM}{\sqrt{1-\frac{u^2}{c^2}}} - (1-\delta)\eta\varepsilon c dM. \tag{1}$$

The minus signs arise because dM represents a loss in rocket mass.

Note that Equation (1) corresponds to the unnumbered equation appearing before Equation (9) in Reference [1], but the referenced paper omitted a Lorentz factor of $1/\sqrt{1-u^2/c^2}$ in the first term. This was the source of an error.

The total effective kinetic energy of the massive exhaust particles is $\delta\eta\varepsilon c^2 dM$. Using the relativistic formula for kinetic energy, we get that:

$$\delta\eta\varepsilon c^2 dM = \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right)(1-\varepsilon) c^2 dM. \tag{2}$$

Equation (2) can be solved for $1/\sqrt{1-u^2/c^2}$ to give:

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1-\varepsilon(1-\delta\eta)}{1-\varepsilon}. \tag{3}$$

Solving Equation (3) for u:

$$u = c\sqrt{1-\left(\frac{1-\varepsilon}{1-\varepsilon(1-\delta\eta)}\right)^2}. \tag{4}$$

Substituting Equations (3) and (4) into Equation (1), and simplifying:

$$M d\sigma = -c\left(\sqrt{\delta\eta\varepsilon(2-2\varepsilon+\delta\eta\varepsilon)} + (1-\delta)\eta\varepsilon\right) dM. \tag{5}$$

Specific impulse w is defined such that (e.g., [3] pp. 28 - 29):

$$M\frac{d\sigma}{d\tau} = -w\frac{dM}{d\tau}. \tag{6}$$

(We warn the reader that "specific impulse" is traditionally defined as w divided by the acceleration of gravity due to Earth at sea-level.)

From Equations (5) and (6), we obtain the following expression for w/c:

$$\frac{w}{c} = \sqrt{\delta\eta\varepsilon(2 - 2\varepsilon + \delta\eta\varepsilon)} + (1 - \delta)\eta\varepsilon. \tag{7}$$

Equation (7) fixes the error in Equation (16) of Reference [1] and generalizes existing results in the literature. In the case where $\eta = \delta = 1$, Equation (7) reduces to $w/c = \sqrt{2\varepsilon - \varepsilon^2}$, which agrees with the corresponding result derived by Sänger in Section 2 of his 1953 paper "Zur Theorie der Photonraketen" [4]. The case where $\eta = 1$ and $\delta = 0$ can be interpreted as a photon rocket that simply jettisons spent fuel at zero relative speed; and here Equation (7) reduces to $w/c = \varepsilon$, in agreement with results derived in Section 3c of Reference [4].

A conceptual distinction should be drawn between the specific impulse and the effective relative speed of the exhaust particles. Consider a propulsion system that utilizes only massive exhaust particles ($\delta = 1$). In this case, Equation (7) gives a specific impulse of $c\sqrt{\eta\varepsilon(2 - 2\varepsilon + \eta\varepsilon)}$. By contrast, setting $\delta = 1$ in Equation (4) yields an effective exhaust speed of $c\sqrt{1 - (1 - \varepsilon)^2/(1 - \varepsilon(1 - \eta))^2}$. These two expressions are not equivalent, and we should not expect otherwise. Specific impulse is defined operationally through Equation (6) as proper thrust divided by the rate at which mass is sacrificed, and not through any direct reference to an independently measurable exhaust speed. The notion of an "effective exhaust speed," which leads to Equations (1) - (5), is independent of and fundamentally distinct from the notion of "specific impulse" introduced by Equation (6).

Specific impulse is still limited by the speed of light, and a practical interstellar rocket needs its specific impulse to be a significant fraction of c. In order to achieve this, one must convert mass into energy with nearly perfect efficiency. It is commonly assumed that the only known way of doing this involves the annihilation of matter with antimatter. However, another possibility involves the quantum mechanical evaporation of a black hole. Recently, Crane *et al.* [5] have argued that a micro-black hole with a Schwarzschild radius on the order of a few attometers would be an excellent power source for an interstellar rocket. Moreover, it is argued that black holes of the requisite size would be safer, easier to handle, and easier to manufacture than the large quantities of antimatter needed to drive an interstellar starship. For details see Reference [5]. In the remainder of this paper, we will use available literature on the more familiar concept of the antimatter rocket as an illustration of the use of Equation (7).

3. An application to antimatter rockets

The purpose of this section is to use Equation (7) to reassess the maximum specific impulse achievable by the antimatter rocket studied in Reference [1] (see also, e.g., [2], [6], [7], [8], [9]). This rocket annihilates hydrogen with antihydrogen and uses electromagnetic fields to collimate charged reaction products into an exhaust beam. Gamma rays, which are also produced in the annihilation, escape into space and their energy is not utilized.

Table 1 describes what happens, on the average, when an atom of hydrogen annihilates with an atom of antihydrogen at rest. The electron from the hydrogen atom annihilates with the positron from the antihydrogen atom and a pair of gamma rays results. The proton from the hydrogen atom annihilates with the antiproton from the antihydrogen atom and the initial result is, on the average, about two neutral pions π^0 and three charged pions (π^+ and π^- particles) [7]. The neutral pion is extremely short-lived and only traverses a microscopic distance before giving rise to its decay products, which are usually (i.e., $98.798 \pm 0.032\%$ of the time [10]) two gamma rays. On the other hand, the charged pions travel a good macroscopic distance (on the order of a couple tens of meters) before giving rise to their decay products, which are usually (i.e., $99.98770 \pm 0.00004\%$ of the time [10]) just a muon μ^+ (or antimuon μ^-) together with a muon neutrino ν_μ (or antimuon neutrino $\bar{\nu}_\mu$). The muons and antimuons travel a distance on the order of a kilometer before decaying (into electrons, positrons and neutrinos).

The antimatter rocket design in Reference [1] achieves its thrust by collimating (via electromagnetic fields) the charged pion products into an exhaust jet. The gamma rays simply escape into space as waste. A negligible amount of reaction products are absorbed by the spacecraft.

Referring to Table 1, and assuming that the charged pion products are collimated with perfect efficiency, we get that:

$$\begin{aligned} 1 - \varepsilon &= \frac{\text{massive exhaust particles utilized}}{\text{rocket mass lost}} \\ &= \frac{418.8}{1877.6}, \end{aligned} \tag{8}$$

which yields (in agreement with Reference [1]):

$$\varepsilon = 0.7769. \tag{9}$$

species	rest mass (MeV)	kinetic energy (MeV)
inital reactants:		
p^+	938.3	0
e^-	0.5	0
p^-	938.3	0
e^+	0.5	0
initial products:		
$2.0\pi^0$	269.9	439.1
$1.5\pi^+$	209.4	374.3
$1.5\pi^-$	209.4	374.3
2γ (from $e^- + e^+$)	0	1.0
decay products:		
4γ (from $2\pi^0$)	0	709.1
$1.5\mu^+$ (from $1.5\pi^+$)	158.5	288.5
$1.5\nu_\mu$ (from $1.5\pi^+$)	0	136.8
$1.5\mu^-$ (from $1.5\pi^-$)	158.5	288.5
$1.5\bar{\nu}_\mu$ (from $1.5\pi^-$)	0	136.8

Table 1: Reaction products arising from the annihilation of hydrogen with antihydrogen at rest (data taken from Reference [7]).

Moreover, since only the kinetic energy of the charged pion products are utilized for propulsion, we get that:

$$\begin{aligned} \eta\varepsilon &= \frac{\text{energy utilized}}{\text{rocket mass lost}} \\ &= \frac{748.6}{1877.6}, \end{aligned} \tag{10}$$

yielding (also in agreement with Reference [1]):

$$\eta = 0.5132. \tag{11}$$

Furthermore, since this example utilizes only massive particles as exhaust, we have that $\delta = 1$.

Plugging the values $\varepsilon = 0.7769$, $\eta = 0.5132$, and $\delta = 1$ into Equation (7) gives:

$$\frac{w}{c} = 0.5804. \tag{12}$$

Thereby we find that the ideal specific impulse of the rocket is $0.5804c$. This is significantly higher than the value of $0.2082c$ obtained in Reference [1].

Vulpetti [2] supports our $0.5804c$ result. Indeed, Equation (6) of Vulpetti's paper implicitly defines an expression for specific impulse which is equivalent to our Equation (7) when $\delta = 1$ (Vulpetti assumes a drive that utilizes only massive exhaust particles). The fact that Vulpetti's equation is implicit in Equation (7) may not be obvious at first glance because Vulpetti's equations are expressed in terms of variables that are quite different from ours.

4. Utilizing gammas

The propulsion design discussed in Section 3 utilizes, at best, only $\varepsilon\eta = 39.87\%$ of the annihilation energy as exhaust energy (cf. Morgan [8], p. 536). A large amount of energy is uselessly carried off into space by gamma rays. In the present section, we entertain briefly the contentious speculation that these gammas can somehow be utilized.

Sänger famously proposed that a gamma reflector could be made from an extremely dense pure electron gas [11]. A parabolic reflector of this kind, with the annihilation point at its focus, would steer gamma rays into a well-collimated exhaust beam. However, the feasibility of Sänger's proposal remains unclear (see, e.g., Forward [9]).

Let us denote by α the fractional amount of gamma ray energy utilized by some hypothetically modified pion drive (we will not specify how this modification is to be achieved). Assuming that no reaction products are permanently absorbed by the spacecraft, we have:

$$\begin{aligned}(1-\delta)\eta\varepsilon &= \frac{\text{massless exhaust particles utilized}}{\text{rocket mass lost}} \\ &= \frac{710.1\alpha}{1877.6}.\end{aligned} \tag{13}$$

Assuming that the pions are collimated with perfect efficiency, we have that:

$$\delta\eta\varepsilon = \frac{748.6}{1877.6}. \tag{14}$$

As before, we have from Equation (8) that $1 - \varepsilon = 418.8/1877.6$.

Plugging these into Equation (7) gives:

$$\frac{w}{c} = 0.5804 + 0.3782\alpha. \tag{15}$$

If all of the pions and gammas are utilized, then the specific impulse can be nearly $0.96c$. If half of the gamma ray energy is utilized ($\alpha = 0.5$), then the specific impulse can be nearly $0.77c$. *Caveat lector* — utilizing gamma rays for propulsion is a tricky proposition.

5. Conclusions and closing remarks

In this paper, we deduced an equation, Equation (7), which expresses (in terms of parameters ε, η and δ) the specific impulse of a rocket which utilizes massive and/or massless particles as exhaust. The analysis was done in the context of Special Relativity. We solved a problem which was considered in a previous paper entitled "Relativistic rocket and space flight" by U. Walter [1], but Walter unfortunately omitted a Lorentz factor which lead him to obtain erroneous results.

When Equation (7) is applied to the case of a particular example, as in Section 3, we find that our corrections to Walter's calculations are significant. Walter considered the problem of calculating the best specific impulse that could theoretically be achieved by an antimatter pion drive. He calculated a specific impulse of about $0.21c$, whereas we calculated a specific impulse of about $0.58c$. Vulpetti [2] agrees with our $0.58c$ result.

The application of Equation (7) to the black hole stardrive [5] will be saved for future work.

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Appendix

Symbols

α = fractional amount of gamma ray energy that is effectively utilized for propulsion

γ = photon
δ = fractional amount of propulsive energy that goes into the effective kinetic energy of massive exhaust particles
ε = fractional amount of lost rocket mass that is accounted for by mass converting into energy
η = fractional amount of available energy that is utilized for propulsion
μ^+ = antimuon
μ^- = muon
ν_μ = muon neutrino
$\bar{\nu}_\mu$ = antimuon neutrino
π^0 = neutral pion
π^+ = positive pion
π^- = negative pion
σ = proper speed
τ = proper time
c = the speed of light
d = differential operator/"infinitesimal" prefix
e^+ = positron
e^- = electron
$\mathcal{F}$ = inertial reference frame instantaneously at rest with respect to the rocket
M = instantaneous mass of rocket
p^+ = proton
p^- = antiproton
u = effective relative speed of massive exhaust particles with respect to the rocket
w = specific impulse

References

[1] U. Walter, (2006) "Relativistic rocket and space flight," Acta Astronautica, Vol. 59, pp. 453 - 461.

[2] G. Vulpetti, (1985) "Maximum terminal velocity of relativistic rocket," Acta Astronautica, Vol. 12, No. 2, pp. 81 - 90.

[3] G. P. Sutton, O. Biblarz, *Rocket Propulsion Elements*, 7th Edition, John Wiley and Sons, New York, 2001.

[4] E. Sänger, (1953) "Zur Theorie der Photonraketen," Ingenieur-Archiv, Band 21, pp. 213 - 226 (in German).

[5] L. Crane and S. Westmoreland, (2009) "Are black hole starships possible?" arXiv:0908.1803 [gr-qc].

[6] G. Vulpetti, (1983) "A concept of low-thrust relativistic-jet-speed high-efficiency matter-antimatter annihilation thruster," International Astronautical Federation 34th Congress, Budapest, October 1983.

[7] R. H. Frisbee, (2003) "How to build an antimatter rocket for interstellar missions," 39th AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit, Huntsville, Alabama, July 20 - 23, 2003.

[8] D. L. Morgan, (1988) in *Antiproton Science and Technology*, ed. R. W. Augenstein *et al.*, pp. 530 - 565.

[9] R. L. Forward, (1985) "Antiproton annihilation propulsion," J. Propulsion, Vol. 1, 370 - 374.

[10] C. Amsler, *et al.*, (2008) "Review of Particle Physics," Physics Letters B, Vol. 667, 1.

[11] E. Sänger, "Photon propulsion," in *Handbook of Astronautical Engineering*, First Edition, H. H. Koelle (Ed.), McGraw-Hill, New York, 1961.