Khovanov-Rozansky Homology and Mutation

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Knots in Washington XXXI
Outline

Conway Mutation
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  Statement of the Main Theorem
  Normalizing Orientation-Preserving Mutation

Khovanov-Rozansky Homology
  A State Model

Mutation Invariance
  Idea
  Key Lemma
  Application of the Lemma
  Proof of the Main Theorem

Future Directions
  Questions and Open Problems

These slides and a draft of my paper are available on my website at http://math.msu.edu/~tjaeger.
Invariants and Mutation

Many invariants cannot distinguish mutants:

- Alexander Polynomial
- (Colored) Jones Polynomial (Morton-Traczyk)
- HOMFLY-PT Polynomial
- Kauffman Polynomial
- Hyperbolic Volume (Ruberman)
- Double-Branched Cover
- $\mathbb{Z}_2$ and odd Khovanov Homology (Bloom, Wehrli)

However, there are some invariants that can tell mutants apart:

- Colored $\mathfrak{sl}(n)$ Polynomial and finite type invariants of order $\geq 11$ (Cromwell-Morton)
- Knot Floer Homology (via Knot Genus)
Statement of the Main Theorem

Theorem (J.)

If two knots are related by positive (“orientation-preserving”) mutation and \( n \) is odd, then their reduced \( \mathfrak{sl}(n) \) homologies are isomorphic.

Positive and Negative Mutation

Of the 16 mutant pairs with 11 crossings, I found 5 that can be realized on the tangle below and thus can be realized by both positive and negative mutation, among them the Kinoshita-Terasaka - Conway pair.

\[
\begin{align*}
R_x & \\
R_y & \\
R_z & 
\end{align*}
\]
Reduction to Braid Form

The two possible orientations of the endpoints of a tangle. Note that there is only one type of positive mutation in each case.

We can use a modification of the proof of Alexander’s Theorem to transform a tangle of type \( \uparrow\uparrow \) into what we call braid form.

The example from the beginning already is in braid form.
Reduction to Tangles of Type $\leftrightarrow$

If the inner tangle is of type $\leftrightarrow$, we can perform a topologically equivalent mutation on a tangle of type $\uparrow\uparrow$.
The $\mathfrak{sl}(n)$ Skein Module of Singular Braids

The $\mathfrak{sl}(n)$ skein module is generated by tangles modulo skein relations

$$\begin{align*}
\begin{array}{c}
\uparrow\downarrow - \downarrow\uparrow &= (q - q^{-1})\begin{array}{c}
\uparrow\downarrow
\end{array}, \\
\begin{array}{c}
\bigcirc
\end{array} &= q^n\begin{array}{c}
\uparrow
\end{array} \quad \text{and} \quad \begin{array}{c}
\bigotimes
\end{array} &= q^{-n}\begin{array}{c}
\uparrow
\end{array}.
\end{array}
\end{align*}$$

Theorem (Murakami, Ohtsuki, Yamada)

Set

$$\begin{align*}
\begin{array}{c}
\bigotimes
\end{array} := q\begin{array}{c}
\uparrow
\end{array} - q^{-1}\begin{array}{c}
\uparrow\downarrow
\end{array} + q^{-2}\begin{array}{c}
\downarrow\uparrow
\end{array} - \cdots
\end{align*}$$

to extend the invariant to singular tangles. Then

$$([k] = q^{-k+1} + q^{-k+3} + \ldots q^{k-1})$$

$$\begin{align*}
\begin{array}{c}
\bigotimes
\end{array} &= [n]\begin{array}{c}
\uparrow
\end{array}, \\
\begin{array}{c}
\bigotimes
\end{array} &= [n-1]\begin{array}{c}
\uparrow
\end{array}, \\
\begin{array}{c}
\bigotimes
\end{array} &= [2]\begin{array}{c}
\uparrow\downarrow
\end{array}, \\
\begin{array}{c}
\bigotimes
\end{array} + \begin{array}{c}
\bigotimes
\end{array} &= \begin{array}{c}
\bigotimes
\end{array} + \begin{array}{c}
\bigotimes
\end{array}.
\end{align*}$$

These relations determine the $\mathfrak{sl}(n)$ polynomial for closed fully singular braids.
Khovanov-Rozansky Homology of Singular Braids

Khovanov-Rozansky homology is a categorification of the $\mathfrak{sl}(n)$ polynomial. It has a particular simple form for fully singular braids: All complexes are supported in homological height 0 and satisfy the following isomorphisms.

\[
\begin{align*}
C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (2,0) {}; \draw[very thick, ->] (a) to (b); \end{tikzpicture}) & \cong C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (2,0) {}; \draw[very thick, ->] (a) to (b); \end{tikzpicture}) \{n-1\} \oplus \ldots \oplus C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (2,0) {}; \draw[very thick, ->] (a) to (b); \end{tikzpicture}) \{-n+1\} \\
C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (1,0) {}; \node (c) at (1,1) {}; \node (d) at (0,1) {}; \draw[very thick, ->] (a) to (b); \draw[very thick, ->] (c) to (d); \end{tikzpicture}) & \cong C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (2,0) {}; \draw[very thick, ->] (a) to (b); \end{tikzpicture}) \{n-2\} \oplus \ldots \oplus C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (2,0) {}; \draw[very thick, ->] (a) to (b); \end{tikzpicture}) \{-n+2\} \\
C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (1,0) {}; \node (c) at (1,1) {}; \node (d) at (0,1) {}; \draw[very thick, ->] (a) to (b); \draw[very thick, ->] (c) to (d); \end{tikzpicture}) & \cong C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (1,0) {}; \node (c) at (1,1) {}; \node (d) at (0,1) {}; \draw[very thick, ->] (a) to (b); \draw[very thick, ->] (c) to (d); \end{tikzpicture}) \{1\} \oplus C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (1,0) {}; \node (c) at (1,1) {}; \node (d) at (0,1) {}; \draw[very thick, ->] (a) to (b); \draw[very thick, ->] (c) to (d); \end{tikzpicture}) \{-1\} \\
C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (1,0) {}; \node (c) at (1,1) {}; \node (d) at (0,1) {}; \draw[very thick, ->] (a) to (b); \draw[very thick, ->] (c) to (d); \end{tikzpicture}) \oplus C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (1,0) {}; \node (c) at (1,1) {}; \node (d) at (0,1) {}; \draw[very thick, ->] (a) to (b); \draw[very thick, ->] (c) to (d); \end{tikzpicture}) & \cong C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (2,0) {}; \draw[very thick, ->] (a) to (b); \end{tikzpicture}) \oplus C^n(\begin{tikzpicture}[baseline, scale=0.5] \node (a) at (0,0) {}; \node (b) at (2,0) {}; \draw[very thick, ->] (a) to (b); \end{tikzpicture})
\end{align*}
\]

These complexes live in the homotopy category of chain complexes over the homotopy category of matrix factorizations with a certain potential associated to the endpoints.
Khovanov-Rozansky Homology of Braids

Using

\[ C^n \left( \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right) \cong C^n \left( \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right) [-1] \rightarrow C^n \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \{1\} \quad \text{and} \quad C^n \left( \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right) \cong C^n \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \{-1\} \rightarrow C^n \left( \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right) [1] \]

we extend the invariant to braids via a “cube of resolutions”:

\[ C^n \left( \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right) \cong C^n \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \{1\} \rightarrow C^n \left( \begin{array}{c} \downarrow \\ \uparrow \end{array} \right) \{2\} \rightarrow C^n \left( \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right) \{1\} \rightarrow C^n \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \{1\} \rightarrow C^n \left( \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right) \{1\} \rightarrow C^n \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \{2\} \]
Khovanov-Rozansky Homology of any tangle in braid form is isomorphic to a chain complex built out of direct sums of shifts of $C^n \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right)$ and $C^n \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right)$. 
Idea of the proof

So far, we have expressed the inner tangle as a chain complex over a particularly simple category: the full subcategory of our category of matrix factorizations on $C^n \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right)$ and $C^n \left( \begin{array}{c} \uparrow \\ \downarrow \\ \downarrow \end{array} \right)$. Reflection/rotation (denoted by $\bar{\cdot}$) also has a simple description in this category.

Following the proof of mutation invariance for the $\text{sl}(n)$ polynomial, we'd like to be able to say that any such chain complex is invariant under $\bar{\cdot}$, but this is false. Instead, we will try to kill enough information to obtain a complex that is invariant under $\bar{\cdot}$ while retaining enough information to recover the homology of the link — provided that we are promised an endpoint of the tangle and its image under $\bar{\cdot}$ lie on the same component.

To do so, we will consider a certain mapping cone of the complex.
Idea of the proof

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- To do so, we will consider a certain mapping cone of the complex.
Key Lemma

Let $\mathcal{C}$ be an additive category and let $\bar{\cdot} : \mathcal{C} \to \mathcal{C}$ be a functor that is the identity on objects and an involution on morphisms. Furthermore, let $f$ be an element in the center of $\mathcal{C}$ and $\partial : \text{Hom}_\mathcal{C}(A, B) \to \text{Hom}_\mathcal{C}(A, B)$ be a $\mathbb{Z}$-linear operation with the following properties

- For $\phi \in \text{Hom}_\mathcal{C}(A, B)$, $\phi - \bar{\phi} = f_B \partial \phi = \partial \phi f_A$.
- Composable morphisms $\phi \in \text{Hom}_\mathcal{C}(A, B)$ and $\psi \in \text{Hom}_\mathcal{C}(B, C)$ satisfy $\partial(\psi \phi) = \partial \psi \phi + \bar{\psi} \partial \phi = \partial \psi \bar{\phi} + \psi \partial \phi$.

If $C$ is a chain complex over $\mathcal{C}$ with differential $d$, then $f$ gives rise to a chain morphism $f_C : C \to C$. Let $\bar{C}$ be the chain complex obtained by applying $\bar{\cdot}$ to the differential of $C$. Then the mapping cones $\text{Cone}(f_C)$ and $\text{Cone}(f_{\bar{C}})$ are isomorphic.

Proof (mod 2).

$$
\begin{pmatrix} d & f_C \\ f_C & d \end{pmatrix} \cup C[-1] \oplus C \xrightarrow{\begin{pmatrix} I & \partial d \\ I & I \end{pmatrix}} C[-1] \oplus C \cup \begin{pmatrix} \bar{d} & \bar{d} \\ f_{\bar{C}} & \bar{d} \end{pmatrix}
$$
Warmup

One object: \( x = \uparrow \uparrow \bigcirc \uparrow \uparrow \bigcirc \uparrow \uparrow \downarrow = y \) / \( (x^n = y^n = 0) \)

Recall that \( \phi - \bar{\phi} = \partial \phi f \) and \( \partial(\psi \phi) = \partial \psi \phi + \bar{\psi} \partial \phi \).

Reflection is given by \( \bar{x} = \pm y \) and \( \bar{y} = \pm x \). Set \( f = x \mp y \). It follows that

\[
\begin{align*}
f \partial x &= x - \bar{x} = x \mp y \quad \Rightarrow \partial x = 1 \\
\partial y &= y - \bar{y} = y \mp x \quad \Rightarrow \partial y = \mp 1
\end{align*}
\]

We compute

\[
\begin{align*}
\partial(x^2) &= \partial x x + \bar{x} \partial x = x \pm y \\
\partial(y^2) &= \partial y y + \bar{y} \partial y = -(x \pm y) \\
\partial(x^3) &= \partial x x^2 + \bar{x} \partial(x^2) = x^2 \pm y(x \pm y) = x^2 \pm xy + y^2 \\
\partial(y^3) &= \partial y y^2 + \bar{y} \partial(y^2) = \mp y^2 \pm x(- (x \pm y)) = \mp y^2 - xy \mp y^2
\end{align*}
\]
Warmup

One object: \( x = \uparrow \uparrow \bigcirc \uparrow \bigcirc \uparrow \uparrow = y \) \( (x^n = y^n) \)

Recall that \( \phi - \bar{\phi} = \partial \phi f \) and \( \partial(\psi \phi) = \partial \psi \phi + \bar{\psi} \partial \phi \).

Reflection is given by \( \bar{x} = -y \) and \( \bar{y} = -x \). Set \( f = x + y \). It follow that

\[
\begin{align*}
    f \partial x &= x - \bar{x} = x + y & \Rightarrow \partial x &= 1 \\
    f \partial x &= y - \bar{y} = y + x & \Rightarrow \partial y &= +1
\end{align*}
\]

We compute

\[
\begin{align*}
    \partial(x^2) &= \partial x x + \bar{x} \partial x = x - y \\
    \partial(y^2) &= \partial y y + \bar{y} \partial y = -(x - y) \\
    \partial(x^3) &= \partial x x^2 + \bar{x} \partial(x^2) = x^2 - y(x - y) = x^2 - xy + y^2 \\
    \partial(y^3) &= \partial y y^2 + \bar{y} \partial(y^2) = + y^2 - x(-(x - y)) = + y^2 - xy + y^2
\end{align*}
\]
The Category Associated to 2-Tangles

\[ R = \mathbb{Q}[a, b, c, d]/(a + b = c + d), \quad q \in R \text{ a certain polynomial of degree } n - 1. \]

**Objects**

\[
\begin{array}{c}
R \xleftarrow{\begin{array}{c}
\frac{c-a}{c-b} q
\end{array}} R \\
R \xleftarrow{\begin{array}{c}
\frac{(c-a)(c-b)}{q}
\end{array}} R
\end{array}
\]

**Morphisms**

\[
\begin{array}{ccc}
R & \xleftarrow{\ast} & R \\
\downarrow & \ast & \downarrow \\
R & \xleftarrow{\ast} & R
\end{array}
\]

(modulo homotopy)

**Reflection**

Reflection \( R_y \) is given by the ring homomorphism \( \tilde{\cdot} : R \rightarrow R, \ a \mapsto -b, \ b \mapsto -a, \ c \mapsto -d, \ d \mapsto -c. \) Here we use that \( n \) is odd.

**Differential**

\( f = a + b = c + d, \ \partial a = \partial b = \partial c = \partial d = 1, \) hence \( \partial(c - a) = 0 \) and \( \partial(c - a)(c - b) = 0 \) and also \( \partial q = 0, \) so we may define \( \partial(x, y) = (\partial x, \partial y) \) for any morphism \( (x, y). \) This operation also descends to homotopy.
Wrapping it up

There is an action of the “edge ring” on $C_n(T)$ such that $a \simeq b$ if edges $a$ and $b$ lie on the same component of the tangle $T$. For a link $L$, reduced Khovanov-Rozansky homology (wrt. the component that $a$ lies on) can be defined by $H_n(L) = H^\ast \left( \text{Cone} \left( C_n(L) \xrightarrow{a} C_n(L) \right) \right)$.

Let $K_1 = T \cup T'$ and $K_2 = \bar{T} \cup T'$ be mutant knots. Applying the Lemma to $T$, we see that

$$\text{Cone} \left( C_n(T) \xrightarrow{a+b} C_n(T) \right) \cong \text{Cone} \left( C_n(\bar{T}) \xrightarrow{a+b} C_n(\bar{T}) \right)$$

Closing up (by taking $\otimes C_n(T')$), we get

$$\text{Cone} \left( C_n(K_1) \xrightarrow{a+b} C_n(K_1) \right) \cong \text{Cone} \left( C_n(K_2) \xrightarrow{a+b} C_n(K_2) \right)$$

$$\text{Cone} \left( C_n(K_1) \xrightarrow{2a} C_n(K_1) \right) \cong \text{Cone} \left( C_n(K_2) \xrightarrow{2a} C_n(K_2) \right)$$
Questions

- Is it possible to lift some of the restrictions of the Theorem? In particular,
  - Can the proof be extended to unreduced Homology?
  - Can we extend the argument to negative mutation?
  - How about even $n$, particularly $n = 2$, i.e. Khovanov Homology?
- Can the method be applied to other homological knot invariants?
  - Rasmussen’s $s$-invariant?
  - HOMFLY-PT Homology? (Answer: Yes)
  - Rasmussen’s spectral sequence from $\mathfrak{sl}(n)$ to HOMFLY-PT Homology?
  - Can Bloom’s invariance result for odd Khovanov homology be re-proved using Putyra’s formalism?
- Can a generalization of the Lemma be used to show invariance of $\mathbb{Z}_2$-Khovanov Homology under genus-2 mutation?