ON THE KHOVANOV HOMOLOGY OF CLOSED THREE-BRAIDS

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Abstract. We investigate how the Khovanov homology of closed three-braids behaves under insertion of a number of full twists.

1. Introduction

According to Garside’s solution of the word problem for $B_n$ (see for example Birman [3], Theorem 2.5), each element of $B_n$ has a unique representative of the form $\Delta^n s$ (which can be determined algorithmically), where $s$ is a positive braid and $\Delta$ is the central element of $B_n$, geometrically corresponding to a half twist.

From this point of view, it is natural to study how knot invariants change under insertion and deletion of powers of $\Delta$. In this document, we consider the case of the Khovanov homology [5] of 3-braids. We show (Theorem 6) that if $s$ is a positive 3-braid, then the Khovanov homologies of the closures of $s$ and $\Delta^{2k} s$ are related by a long exact sequence. If in addition $k \geq 0$, we show that the Khovanov homology of the closure of $\Delta^{2k} s$ is determined by the Khovanov homology of the closure of $s$.

The result is a generalization of earlier work of Turner [10], who computed the Khovanov homology of $(3,p)$ torus links.

The following questions remain open; we might address some of these question in the future.

- Can the proof be extended to Khovanov and Rozansky’s $sl(n)$ [6] homology?
- What can be said about Rasmussen’s s-invariant [9]?
- Is there a more explicit relationship when inserting negative twists? What about starting with a braid that is not positive?

2. Khovanov Homology

Following [1] we define the Khovanov chain complex for an n-tangle $T$ as a formal complex over the category $\text{Cob}_{1}(\delta T)$, whose objects are smooth 1-manifolds with endpoints $\delta T$, properly embedded into a disk $D$, and for which morphisms between two such crossingless tangles $A$ and $B$ are given by equivalence classes of $\mathbb{Z}[t]$-linear combinations of smooth decorated surfaces, properly embedded into the cylinder $D \times [0,1]$ and with boundary $A \times \{0\} \cup B \times \{1\} \cup \delta T \times [0,1]$. The decorations are given by dots on the interior of the surface. We will define $S^2_t$ to be a sphere with one dot on it so that adding a dot to a surface can be conveniently expressed as taking the connected sum with $S^2_t$. The equivalence relation is generated by isotopy and the following relations (where $S$ and $T$ are surfaces with a marked component, so that connected sum is well-defined)

\[
S^2_t \cup S = S,
S^2_t \# S^2_t \# S^2_t \cup S = tS,
\]
\[ S \# T = S \# S^2 \cup T + S \cup T \# S^2 \] (neck-cutting).

The category \( \text{Cob}_{/l} \) can be made into a graded category by defining the degree of a morphism \( S \) of \( n \)-tangles to be \( \chi(S) - n \). We call this grading \textit{quantum} grading and we denote its grading shifts by \( \{\cdot\} \). Note that a morphism \( S: T_1\{k_1\} \to T_2\{k_2\} \) has degree \( \chi(S) - n + k_2 - k_1 \). For complexes over \( \text{Cob}_{/l} \), we will denote grading shifts in homological degree by \( [\cdot] \).

We will allow arbitrary orientations (+ or −) to be assigned to each crossing of the tangle. We do not demand that these orientations be induced by an orientation of the tangle. The Khovanov complex of a single positive crossing \( (X^+) \) is given as

\[
\text{Kh}(X^+) = \text{Kh}(X_A)[1] \oplus \text{Kh}(X_B)[2],
\]

where \( X_A \) is the \( A \)-smoothing and \( X_B \) the \( B \)-smoothing of the crossing:

\[
\begin{array}{c}
\otimes \\
\begin{array}{c}
A \\
A \\
\end{array} \\
\begin{array}{c}
\otimes \\
B \\
B \\
\end{array}
\end{array}
\]

The differential on this complex is simply the saddle cobordism between \( X_A \) and \( X_B \). It is easy to check that the differential is of homological degree 1 and quantum degree 0.

Similarly, we define the Khovanov complex of a negative crossing \( (X^-) \) to be

\[
\text{Kh}(X^-) = \text{Kh}(X_A)[-1] \oplus \text{Kh}(X_B)[-1]
\]

with the same differential. Clearly \( \text{Kh}(X^+) = \text{Kh}(X_-)[1][3] \).

The Khovanov complex of the horizontal composition of two tangles \( T_1 \) and \( T_2 \) can be defined as the tensor product \( T_1 \otimes T_2 \) with the evident identifications of vertical boundaries of cobordisms. Note that this means that the exact form of the differential depends on the order in which crossings are glued together, but the isomorphism type of the complex does not depend on this choice.

The isomorphism type of the complex depends on the tangle diagram, but it is shown in [1] that the homotopy type does not. In our setting, this is only true up to a grading shift, and to obtain invariance under a Reidemeister move, we need to require that the orientations of the crossings involved in the move be compatible with an orientation of the tangle.

By abuse of notation, we will often refer to the homotopy type of the Khovanov complex of a tangle as its Khovanov homology.

On links, the chain complex is generated by a union of disjoint circles. In \( \text{Cob}_{/l}(\emptyset) \), a circle is isomorphic to the direct sum of \( \emptyset\{-1\} \) and \( \emptyset\{-1\} \) (see [2]). The Khovanov complex of a link can therefore be viewed as a complex over the full subcategory of \( \text{Cob}_{/l}(\emptyset) \) with \( \emptyset \) as its only object, which is isomorphic to \( \mathbb{Z}[t] \) (equivalently, we may think of this as applying the tautological functor \( \text{Mor}(\emptyset, \cdot) \)). By setting \( t = 0 \), we can recover Khovanov’s original invariant; setting \( t = 1 \) yields Lee’s [7] invariant (see also [4]). Unless otherwise specified, we will work in the most general setting below.

### 3. Simplifying chain complexes

Since we will be studying Khovanov complexes in the more general setting of complexes over a (not necessarily abelian) additive category, we do not have the usual tools of homological algebra at our disposal. Instead of taking homology, we will reduce complexes using a specific type of a strong deformation retract, which allows us to state a result reminiscent of the spectral sequence induced by a double complex.
**Definition 1.** A chain map $G : C \to \hat{C}$ is called a **strong deformation retract** if there is a chain map $F : \hat{C} \to C$ a homotopy map $h : C \to \hat{C}[-1]$ such that $GF = I$, $FG = I - dh - hd$ and $hF = 0 = Gh$. If in addition $h^2 = 0$ we will call $G$ a **special deformation retract** (Note: this is not a standard definition).

**Remark 2.** If the category has the property that idempotents split (as does $\text{Cob}_{1}(\delta T)$), then it is easy to see that this definition is equivalent to $C$ being isomorphic to a direct sum of $\hat{C}$ and a complex whose differential is the identity. The projections onto the three direct summands of $C$ are given by $FG$, $dh$ and $hd$; note that $dh$ and $hd$ are idempotents since $dh = h(I - hd - GF) = h$.

**Proposition 1.** The property of being a special deformation retracts is closed under composition.

**Proof.** It is well-known that strong deformation retracts are closed under composition, so we only need to show that $h^2 = 0$. Let $C_1$, $C_2$ and $C_3$ be chain complexes. For $i = 1, 2$, let $G_i : C_i \to C_{i+1}$, $F_i : C_{i+1} \to C_i$, $h_i : C_i \to C_{i+1}[-1]$ such that $G_iF_i = I$, $F_iG_i = I - d_i h_i - h_i d_i$, $h_i F_i = 0 = G_i h_i$. Then

$$h^2 = h_1^2 - h_1 F_1 h_2 G_1 - F_1 h_2 G_1 h_1 + F_1 h_2 G_1 F_1 h_2 G_1 = F_1 h_2^2 G_1 = 0.$$ 

The following theorem is essentially a homotopy version of the spectral sequence of a double complex, but avoids the problem that it is in general not possible to reconstruct the integral homology from the $E_\infty$ page. We adopt the following conventions. A double complex is an object in a bigraded additive category with a horizontal differential $d$ of bidegree $(1, 0)$ and a “diagonal” differential of bidegree $(0, 1)$, in particular $d^2 = 0$ and $f^2 = 0$. We require that differentials anti-commute, i.e. $df + fd = 0$.

The total complex is given by the direct sums $\bigoplus_{i+j=s} C^i_j$ over the columns in the above picture and differential $d + f$. Since we will not be interested in the vertical and diagonal chain complexes, we simply refer to the total complex as the double complex.
Theorem 2. If $C = C^2$ is a (bounded) double complex and $G, C_1 \to \hat{C}$ are special deformation retracts with inverses $F$, and associated homotopy maps $h$, then (the total complex of) $C$ is homotopy equivalent to $\hat{C}$, which is given by $\bigoplus_{i+j=s} C^2$ and has differential $\hat{d} + G(f + fhf + \ldots) F$. In fact, this homotopy equivalence is a special deformation retract in the sense of Definition 1.

Proof. It is convenient to formally define $\frac{1}{\tau x^2} = 1 + x + x^2 + \ldots$. We first need to show that the map $\hat{d} + G f \frac{1}{\tau - \eta^2} F = \hat{d} + G \frac{1}{\tau - \eta^2} F$ does indeed define a differential, i.e. that its square is 0. Clearly, $f d = -df$, $G d = \hat{d} G$, $F \hat{d} = d F$, $F G = I + dh + hd$ and $f^2 = 0$. Therefore

\[
\begin{align*}
(\hat{d} + G f \frac{1}{\tau - \eta^2} F) (\hat{d} + G f \frac{1}{\tau - \eta^2} F) &= G \left( df \frac{1}{\tau - \eta^2} + \frac{1}{\tau - \eta^2} f d + \frac{1}{\tau - \eta^2} (I + dh + hd) f \frac{1}{\tau - \eta^2} \right) F \\
&= G \frac{1}{\tau - \eta^2} ((I - fh) df + f (I - h f) + f (dh + hd) f) \frac{1}{\tau - \eta^2} F \\
&= G \frac{1}{\tau - \eta^2} (df - f h f d + f d + f d h + f h f + f h f d) \frac{1}{\tau - \eta^2} F = 0
\end{align*}
\]

We will show that the following picture defines a homotopy equivalence.

\[
\begin{array}{cccc}
C & \xrightarrow{d+f} & C[1] & \\
\frac{d}{\tau - \eta} F & \xleftarrow{\tau - \eta F} & \xrightarrow{d+G(f+fhf+\ldots) F} & C[1] \\
\hat{C} & \xleftarrow{\hat{d}+G(f+fhf+\ldots) F} & \xrightarrow{\hat{d}+G(f+fhf+\ldots) F} & \hat{C}[1]
\end{array}
\]

The upward-pointing arrows define a chain map since

\[
\begin{align*}
(d + f) \frac{1}{\tau - \eta^2} F - \frac{1}{\tau - \eta^2} F (\hat{d} + G f \frac{1}{\tau - \eta^2} F) &= \frac{1}{\tau - \eta^2} ((I - h f) df + f (I - h f) - f (dh + hd) f) \frac{1}{\tau - \eta^2} F \\
&= \frac{1}{\tau - \eta^2} (d + f - h f d - h f^2 - d + dh f - f - dh f - h d f) \frac{1}{\tau - \eta^2} F = 0
\end{align*}
\]

Similarly, we get a chain map from the downward-facing arrows because

\[
\begin{align*}
G \frac{1}{\tau - \eta^2} (d + f) - (\hat{d} + G \frac{1}{\tau - \eta^2} F) G \frac{1}{\tau - \eta^2} F &= G \frac{1}{\tau - \eta^2} ((d + f) (I - fh) - (I - fh) d - f FG) \frac{1}{\tau - \eta^2} F \\
&= G \frac{1}{\tau - \eta^2} (d + f - dh f h - f h f + f h d - f - dh f - f h d) \frac{1}{\tau - \eta^2} F = 0
\end{align*}
\]

Finally, we need to show that

\[
\begin{align*}
\frac{1}{\tau - \eta^2} FG \frac{1}{\tau - \eta^2} F - I - (d + f) h = \frac{1}{\tau - \eta^2} h (d + f) &= \frac{1}{\tau - \eta^2} (FG - (I - h f) (I - fh) \\
&- (I - h f) (d + f) h - (d + f) (I - fh)) \frac{1}{\tau - \eta^2} F \\
&= \frac{1}{\tau - \eta^2} (I + dh + hd - I + h f + fh - f h f + dh f + h f h) \frac{1}{\tau - \eta^2} F = 0
\end{align*}
\]

□
4. Three-Braids

We will be interested in the effect of adding a number of twists on the Khovanov homology (or more precisely, the homotopy type of the Khovanov complex) of the closure of a 3-braid. One can view braids as links in a standardly embedded thickened annulus. Adding twists then corresponds to switching to a non-standard embedding of the annulus. Any smoothing of a closed 3-braid in an annulus that is not the trivial 3-braid yields isotopic links regardless of the chosen embedding. This fact can be exploited to give a recursive formula for the Jones polynomial, namely

$$V_{\Delta^2}(t) - t^6 V_s = (-\sqrt{t})^{w(s)}(V_{\Delta^2} - t^6 V_s).$$

The goal of this is to establish a similar relationship for Khovanov homology. We will restrict ourselves to positive 3-braids.

4.1. The Khovanov Complex of a 3-braids plus twist. In the following we will always represent a full positive twist $\Delta^2$ by the braid $(\sigma_1 \sigma_2)^3$ and a full negative twist $\Delta^{-2}$ by the braid $(\sigma_1 \sigma_2)^{-3} = (\sigma_1^{-1} \sigma_2^{-1})^3$.

If $s$ is a positive 3-braids of length $n$, we will compare $s$ with $\Delta^{2k}s$ ($k \in \mathbb{Z}$). In light of the previous discussion, $C^k := Kh(\Delta^{2k}s)$ can be viewed as a tensor product of $Kh(\Delta^2)$ and $Kh(s)$, which is a double complex, whose diagonals are $C^n_X := Kh(\Delta^{2k}s_X)[b(X)][2b(X) + n]$ with differential $d$, where $s_X$ is the smoothing corresponding to $X \in \{A, B\}^n$ and $b(X)$ is the number of $B$s occurring in $X$. Horizontal maps between smoothings $X$ and $X'$ are given by $f_{X,X'}^k$, which is induced by a crossing change on $s$. Let $C^k_A$ be the subcomplex of $C^k$ whose underlying graded object is $\bigoplus_{X \in \{A, B\}^n \setminus \{A^n\}} C^n_X$ and $C^k_A$ be the quotient complex $C^k/C^k_A$, which we identify with $C^k_A$. We may think of $C^k_A$ as a cube of partial resolutions for $C^k = Kh(\Delta^{2k}s)$ with the $A^n$-vertex removed.

\begin{figure}[h]
\includegraphics[width=\textwidth]{figure1.png}
\caption{Transforming $\Delta^2 s_X$ into $s_X$. All other cases, negative twist, first B-smoothing on the bottom are analogous.}
\end{figure}

In order to apply Theorem 2, we need to establish special deformation retractions from all the $C^k_X$. If $X = A^n$, then $C^k_A$ retracts to a complex $\tilde{C}^k_A$, which is (for positive $k$) supported in gradings between 0 and $4k$ by Corollary 7 below. If $X$ is not the all-A smoothing, then the braid $\Delta^{2k}s_X$ is isotopic to $s_X$, with an explicit isotopy given by Figure 1. Note that the isotopy consists of only Reidemeister I and II moves, which are easily checked to induce special deformation retracts. In order to be able to perform this isotopy, we need to change $4k$ positive crossings.
to negative crossings, thus we get a retraction \( G : C^k_X \rightarrow \tilde{C}^k_A \), where \( \tilde{C}^k_A := Kh(s) \{ b(X) + 4k \} \{ 2b(X) + n + 12k \} \). Note that \( C^k_X = C^0_X \{ 4k \} \{ 12k \} \).

Theorem 2 now implies that there is a reduced complex \( \tilde{C}^k \), which is the direct sum of the \( \tilde{C}^k_X \)s with differential \( d + \sum_k G(fh)^k fF \). As for \( C^k \), we define a subcomplex \( \tilde{C}^k_A = \tilde{C}^k/C_A \).

**Proposition 3.** Let \( m > 1 \). In the notation of Theorem 2, if \( k > 0 \), then \( G(fh)^m fF = 0 \). If \( k < 0 \), \( G(fh)^m fF = 0 \) only on \( \tilde{C}^k_A \).

**Proof.** The chain morphism \( G(fh)^m fF \) has (homological) degree 1 since it is a differential. Since all the \( \tilde{C}^k_X \) \( (X \neq A^n) \) are supported in grading \( b(X) + 4k \), a degree-1 morphism that travels along more than one edge of the cube is 0 if it starts at any place other then \( \tilde{C}^k_A \). If \( n \geq 0 \), then \( \tilde{C}^k_A \) is supported in gradings \( \leq 4k \), so the morphisms originating in \( \tilde{C}^k \) are also 0 if \( m > 1 \).

**Proposition 4.** Restricted to \( C^k_A \), \( Gf^k_X : X \rightarrow C^k_X \) is an edge belonging to the differential of \( C^k \) as above.

**Proof.** This is trivial if there is a \( B \)-smoothing to the left of the crossing that is being changed, since it does not affect the sequence of Reidemeister moves performed on \( \Delta^{2k} \). If changing the smoothing changes the left-most \( B \)-smoothing from being on the bottom to being on the top, \( Gf^k_X : X \rightarrow C^k_X \) is induced by a morphism \( \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \) of quantum degree 1. The space of such morphisms is one-dimensional, so \( Gf^k_X : X \rightarrow C^k_X \) is an integer multiple of the morphism induced by the saddle \( g = \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \), say \( Gf^k_X : F = mg \). As a special case of Theorem 2, the cone over the closure of \( Gf^k_X : F \) is homotopy equivalent to the cone over \( f^k_{X,X'} \), which computes the homology of the closure of \( \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \), that is an unknot. If \( m \) were neither 1 nor \(-1\), then with coefficients in \( \mathbb{Z}/m\mathbb{Z} \), \( Gf^k_X : F = 0 \), so the (mod-\( m \)) homology of the unknot would have rank \( 4 + 2 = 6 \), a contradiction. The case where \( Gf^k_X : F = \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \) is completely analogous.

If the left-most \( B \)-smoothing stays on the top or on the bottom, say without loss of generality on the top, it is easy to calculate \( Gf^k_X : F = \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \) explicitly. Notice that by neck-cutting, the morphism \( f^k_{X,X'} : \Delta^{2k} \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \) can be written as the sum of two morphisms \( I_{\Delta^{2k} f_1 I \nu} \) and \( I_{\Delta^{2k} f_2 I \nu} \), where \( f_1 = \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \) and \( f_2 = \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \). (The middle part of these cobordisms correspond to adding a 0-handle and adding a 0-handle with a dot, respectively.) Clearly, \( I_{\Delta^{2k} f_2 \circ F} = F \circ f_2 \).

Since sliding a dot across a crossing gives a chain morphism that is homotopic of the negative of the original one, it is easy to see that \( I_{\Delta^{2k} f_1 \circ F} \simeq F \circ f_1 \). Thus \( Gf^k_X : F = G \circ ((I_{\Delta^{2k} f_2 \circ F} I_{\nu}) + (I_{\Delta^{2k} f_1 \circ F} I_{\nu}) I_{\nu}) = G \circ ((F \circ f_2) I_{\nu} + (F \circ f_1) I_{\nu}) = Gf^k_X : F = \pm f^k_0_{X,X'} : \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \) and \( \begin{array}{c} \Xi \rightarrow \Xi' \end{array} \) are supported in a single homological grading, hence the notions of homotopy and equality coincide and \( Gf^k_X : F = \pm f^k_0_{X,X'} \).

**Proposition 5.** The following isomorphisms of complexes hold:

(a) \( \tilde{C}^k_A \cong C^0_A \{ 4k \} \{ 12k \} \).

(b) If \( k > 0 \), then \( \tilde{C}^k \) is isomorphic to the mapping cone

\[
\text{Cone} \left( f : \tilde{C}^k_A \rightarrow C^0_A \{ 4k \} \{ 12k \} \right)
\]
where $f$ is induced by $Gf^n_{A,X}^k : \tilde{C}^k_A \to \tilde{C}^k_X \cong C^k_X[12k]$ for all $X \in \{A,B\}^n$ with $b(X) = 1$.

**Proof.** We prove both parts in parallel. In each case, the two complexes agree up to signs. More precisely, the complexes on the left are given as cubes with vertices $\tilde{C}^k_X$ (and with the $A^n$ vertex removed in case (a)), whose edges are the chain morphisms $Gf^n_{A,X}^k$. By Proposition 3, the corresponding morphisms on the right hand side are $\epsilon_{X,X'}Gf^n_{A,X}^kF$ with $\epsilon_{X,X'} \in \mathbb{Z}^* = \{\pm 1\}$. We claim that when viewing this cube (possibly with a vertex and all its adjacent cells removed) as a simplicial complex, $\epsilon$ defines a 1-cocycle. This requires us to show that if a face of the cube has vertices $X, X', X''$ and $X'''$ that $\epsilon_{X,X'}\epsilon_{X',X''} = \epsilon_{X,X''}\epsilon_{X'',X'}$, which is clear if $Gf^n_{X,X'}^kF \circ Gf^n_{X',X''}^kF \neq 0$. If $X \neq A^n$, then this morphism is a shift of $\pm f^n_{X,X'}^k,X''F$, which cannot be zero as a composition of two saddle cobordisms. If $X = A^n$, we will argue by contradiction, so assume that $Gf^n_{X,X'}^kF \circ Gf^n_{X',X''}^kF = Gf^n_{X,X'}^kFg^n_{X',X''}^kF = 0$. Extend $s$ by $\sigma_1$ on the right and note that the above equality implies that $Gf^n_{X',X'',X''}Fg^n_{X',X''}F \simeq 0$. Again, up to grading shift this morphism is given by $f^n_{X',X'',X''}Fg^n_{X',X''}F : C^n_B \to C^n_{X''}$. As before, this morphism is not zero, and it cannot be homotopic to zero either, since $C^n_A$ and $C^n_{X''}$ are supported in a single homological grading, which is the desired contradiction.

Since the simplicial complex is contractible in both cases and thus has trivial first cohomology, $\epsilon$ is a coboundary and there exists a 0-cochain $\eta$ such that $\partial \eta = \epsilon$ and thus $\epsilon_{X,X'} = \eta_X\eta_{X'}$. Hence $\eta$ gives the desired isomorphism of complexes. \qed

5. The main statement

**Theorem 6.** Let $\Delta = \sigma_1\sigma_2\sigma_1 \in B_3$ be a half-twist and let $\hat{s}$ denote the closure of the 3-braid $s$. Suppose that all 3-braids are oriented in the natural way and define

$$r(s) = \begin{cases} 
\epsilon & \text{if } s = \epsilon \\
\sigma_i & \text{if } s \text{ only contains } \sigma_i \text{'s} \\
\sigma_1\sigma_2 & \text{if } s \text{ contains both } \sigma_1 \text{'s and } \sigma_2 \text{'s}
\end{cases}$$

(a) There is a bigraded module $Y_s$ such that

$$Y_s \oplus Kh(\hat{r}(s))\{|s| - |r(s)|\} \cong Kh(\hat{s})$$

(b) For any $k \in \mathbb{Z}$, there is a long exact sequence

$$\cdots \longrightarrow Y_s[4k] \{12k\} \longrightarrow Kh(\hat{\Delta}^{2k}s) \longrightarrow$$

$$Kh(\hat{\Delta}^{2k}r(s))\{|s| - |r(s)|\} \longrightarrow Y_s[4k + 1] \{12k\} \longrightarrow \cdots$$

(c) If $k \geq 0$, then

$$Kh(\hat{\Delta}^{2k}s) \cong Y_s[4k] \{12k\} \oplus Kh(\hat{\Delta}^{2k}r(s))\{|s| - |r(s)|\}$$
smoothings of braid diagrams. We will decompose
represent smoothings of the closure of the braid even though they are depicted as
smoothings of braid diagrams. We will decompose

\[ C := C^0 \]

into \( A \oplus B \) as follows.

\[
\begin{align*}
A_0 &= C_0 \\
A_1 &= \{ ((x, \ldots, x), (y, \ldots, y)) \in [\mathbb{R}]^m \oplus [\mathbb{R}]^n \} \\
A_2 &= \{ ((0, \ldots, 0), (y, \ldots, y), (0, \ldots, 0)) \in [\mathbb{R}]^m \oplus [\mathbb{R}]^n \oplus [\mathbb{R}]^m \} \\
A_i &= 0 \quad (i > 2) \\
B_0 &= 0 \\
B_1 &= \{ ((x_1, \ldots, x_{m-1}, 0), (y_1, \ldots, y_{m-1}, 0)) \in [\mathbb{R}]^m \oplus [\mathbb{R}]^n \} \\
B_2 &= \{ ((x_1, \ldots, x_{m}), (y_1, \ldots, y_{mn-1}, 0), (z_1, \ldots, z_{(n^2)})) \in [\mathbb{R}]^m \oplus [\mathbb{R}]^n \oplus [\mathbb{R}]^m \} \\
B_i &= C_i \quad (i > 2)
\end{align*}
\]

Now, \( z \in C_0 = [\mathbb{R}] \) implies \( dz = ((f_1 z, \ldots, f_1 z), (f_2 z, \ldots, f_2 z)) \in A_1 \), similarly
\( d((x, \ldots, x), (y, \ldots, y)) = ((g_1 x - g_1 x, \ldots), (h_1 x - h_2 y, \ldots), (g_2 y - g_2 y, \ldots)) \in A_2 \). We
claim that \( d|_{A_1} : A_1 \to A_2 \) is surjective, which implies \( dA_2 \subseteq A_3 = 0 \) since \( d^2 = 0 \).
It is clearly sufficient to consider the case \( n = m = 1 \), which computes the homology
of an unknot. Since \( A_1 = C_1 \) for \( n = m = 1 \) and the second homology of the unknot
is trivial, \( \text{im}(d|_{A_2}) = \ker(d|_{A_2}) = A_2 \), which implies the claim. Hence \( dA \subseteq A \).

If \( z = ((x_1, \ldots, x_m), (y_1, \ldots, y_m)) \in B_1 \), then \( dz = ((\ldots, (h_1 x - h_2 y, \ldots)) \in B_2 \). Thus \( dB \subseteq B \). Set \( Y_x = H^*(B) \). It follows that \( H^*(C) = H^*(A) \oplus H^*(B) \cong K h(r(s)) \oplus Y_x \).

For part (b), notice that we can find a similar subcomplex \( \tilde{B} \) in \( \tilde{C} := \tilde{C}^k \). By
Proposition 5, \( \tilde{B} \cong B[4k]\{12k \} \), furthermore \( \tilde{C}/\tilde{B} \cong A[|s| - |r(s)|] \), where \( A \) is
the reduced complex corresponding to \( \Delta^{2k} r(s) \). Thus the short exact sequence of
complexes \( 0 \to B[4k]\{12k \} \to \tilde{C} \to \tilde{A}[|s| - |r(s)|] \to 0 \) gives rise to the desired
long exact sequence in homology.
For part (c), we can construct \( \tilde{A} \) in the same way as \( A \) in part (a), and we get a similar decomposition \( \tilde{C} = \tilde{A}\{s - |r(s)|\} \oplus B[4k]\{12k\} \) that carries over to homology. □

![Figure 2. (Partial) tree of resolutions of \( \Delta^2 \)](image)

**Corollary 7.** For \( k \geq 1 \), the homotopy type of the Khovanov complex of \( \Delta^{2k} \) is supported in homological gradings between 0 and \( 4k \).

**Proof.** For \( k = 1 \), the proof follows from Figure 2 and the fact that a tangle with \( n \) positive crossings is supported in gradings between 0 and \( n \) by repeated application of Theorem 2. For the induction step, notice that \( (\sigma_1 \sigma_2)^4 \) is supported in gradings between 0 and \( 8 \leq 4k + 4 \), so for \( s = \Delta^{2k} \), \( Y_s \) is supported between 0 and \( 4k \) by induction hypothesis, so \( Y_s \) lies between 4 and \( 4k + 4 \) and \( \Delta^{2k} r(s) \) lies between 0 and 4, thus by Theorem 6(c), \( \Delta^{2k+1} \) is supported in gradings between 0 and \( 4k + 4 \). □

**Remark 3.** We are using Corollary 7 to show Theorem 6, which might appear to be circular reasoning. However, since we only use Theorem 6 for \( k - 1 \) to show Corollary 7 for \( k \), we can use induction to show Theorem 6 and Corollary 7 simultaneously.

**References**


