Local Dynamics of Analytic Functions in the Non-Archimedean Category

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We are interested in the local dynamics of convergent power series which fix a point $A \in F$, where $F$ is a field with $\text{char}(F) = 0$ (complete with respect to some norm). We will assume that $A = 0$. Thus, $f(x) = \sum_{n \geq m} a_n x^n$. 
We are interested in the local dynamics of convergent power series which fix a point $A \in F$, where $F$ is a field with $\text{char}(F) = 0$ (complete with respect to some norm). We will assume that $A = 0$. Thus, $f(x) = \sum_{n \geq m} a_n x^n$.

Two analytic functions $f$ and $g$ are \textit{conjugate} if there is a local map $h$ fixing 0 so that $h \circ f \circ h^{-1} = g$. 

\textbf{Definition of Local Conjugacy}
Assume that \( m \geq 2 \).

**Theorem (Bottcher, 1904)**

Given a \( \mathbb{C} \)-analytic function \( f(z) = a_m z^m + \ldots \) with \( a_m \neq 0 \), there is a unique diffeomorphism of the form \( h(z) = z + \ldots \) conjugating \( f \) to \( g(z) = z^m \).
Assume that \( m = 1 \), and that \( |a_1| \neq 1 \)

**Theorem (Koenigs, 1884)**

Let \( f \) be a \( \mathbb{C} \)-analytic function fixing the origin of the form \( f(z) = a_1z + \ldots \), where \( |a_1| \neq 1 \). Then, there is a unique diffeomorphism of the form \( h(z) = z + \ldots \) fixing 0 so that \( h \circ f \circ h^{-1}(z) = a_1z \).
Maps which are tangent to the identity

We will consider maps of the form $f(z) = z + \ldots$

**Theorem (Formal Equivalence)**

Let $f$ be a $\mathbb{C}$-analytic function fixing the origin of the form $f(z) = z + \ldots$. Then, there exist an integer $m \geq 2$, $\mu \in \mathbb{C}$, and a formal power series of the form $h(z) = cz + \ldots$ so that $h \circ f \circ h^{-1}(z) = g(z) = z + z^m + \mu z^{2m-1}$. The numbers $m$ and $\mu$ provide formal invariants for $f$. 
Sketch of Proof

- Define polynomials \( h_1(z) = c_1z \), and for \( n \geq 2 \),
  \( h_n(z) = z + c_nz^n \). Define \( H_n(z) = h_n(z) \circ \cdots \circ h_1(z) \). Define
  \( f_n(z) = H_n \circ f \circ H_n^{-1}(z) \). For any power series
  \( K(z) = \sum b_nz^n \), define \([K]_n = b_n\).
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We will choose $c_1, c_2, \ldots$ so that $f_n = g$ modulo terms of degree $n + m - 1$. This is accomplished by letting $c_1 = m^{-1}\sqrt{a_m}$, and by letting

$$c_n = \frac{[f_{n-1}]_{n+m-1}}{n-m}.$$
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- Obviously, when $m = n$, this process breaks down. We define $\mu = [f_{m-1}]_{2m-1}$. Letting $h$ be the (formal) limit of the polynomials $H_n$ yields our result.
Some Remarks

The process by which we define the coefficients $c_n$ above is purely algebraic, requiring only basic arithmetic and roots. Thus, this same process may be carried out in any field of characteristic 0, provided that all necessary roots exist.
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• Up to a choice of $(m - 1)$-root of unity, the coefficients $c_1, \ldots, c_{m-1}$ are uniquely determined. Moreover, the above choices allow us to assume that, after a polynomial change of variable, our function $f$ has the form
$$f(z) = z + z^m + \mu z^{2m-1} + O(z^{2m}).$$
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  f(z) = z + z^m + \mu z^{2m-1} + O(z^{2m}).
  \]

- Finally, the coefficient \( c_m \) plays no role in the conjugation process, and may be considered a “free term”. In what follows, we will assume that \( c_m = 0 \).
A natural question to ask is whether or not $h$ converges. When the field in question is $\mathbb{C}$, the analytic classification admits (an uncountable number of) function invariants (Ecalle, Voronin, Malgrange, Il’yashenko).
A natural question to ask is whether or not \( h \) converges. When the field in question is \( \mathbb{C} \), the analytic classification admits (an uncountable number of) function invariants (Ecalle, Voronin, Malgrange, Il’yashenko). However, when the field is non-archimedean, it turns out that formal and analytic equivalence coincide.
Non-archimedean norms

Definition
Let $F$ be a field of characteristic 0. A non-archimedean norm is a map $|·| : F \to \mathbb{R}^+ \cup \{0\}$ satisfying the following criteria:
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- $|xy| = |x||y|
- $|x + y| \leq \max\{|x|, |y|\}$.
The $p$-adic numbers

Fix a prime number $p$. For any integer $n$, recall that $ord_p(n)$ is the highest power of $p$ dividing $n$ ($ord_p(0) = \infty$).
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**Definition**

Given $x = m/n \in \mathbb{Q}$, define $|x|_p = (1/p)^{\text{ord}_p(m)-\text{ord}_p(n)}$. The map $| \cdot |_p$ satisfies all of the properties of a non-archimedean norm; we call this map the $p$-adic norm.
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**Definition**

The topological completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted $\mathbb{Q}_p$, and is called the set of $p$-adic numbers.
Some simple non-archimedean facts

Properties of non-archimedean norms

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- The series $\sum a_n$ converges if and only if $a_n \to 0$.
- Given an integer $n$, we have $|n| \leq 1$.
- Fix a non-archimedean field $F$. Let $a, b \in F$ with $|b| < |a|$. Then, $|a + b| = |a|$.
Theorem (Spallone, 2010)

Let $K$ be an algebraically-closed non-archimedean field, and let

$$f(x) = x + ax^m + \cdots$$

be an analytic function defined in some neighborhood $U$ of 0 with coefficients in $K$. Then, $f$ is analytically equivalent to $g(x) = x + x^m + \mu x^{2m-1}$. In fact, there exist $q \in K$, $0 < |q| \leq 1$, and a conjugating series $h$ so that $h$ satisfies the estimate

$$|q^{4n}[h]_n| \leq 1$$
Theorem (Spallone, — , 2010)

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The fact that formal conjugacy implies analytic conjugacy was stated in an earlier paper of Rivera-Letelier (2003). In an unpublished dissertation (2005), Vieugué uses methods similar to ours to show that the two classifications coincide when the coefficients of $f$ satisfy the estimate $[f]_n \leq 1$. 
A Sharper Estimate

The estimate given in the theorem is not the one we use. Rather, we consider the set of integer-valued functions, indexed by $m \geq 2$, given by

$$\sigma_m(n) = (n - 1) + m \left\lfloor \frac{n - 2}{m - 1} \right\rfloor.$$

Our final estimate then becomes $|(n - m)!q^{\sigma_m(n)}[h]_n| \leq 1$. 

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Our final estimate then becomes $|(n - m)!q^{\sigma_m(n)}[h]_n| \leq 1$. This estimate was improved by students Harold Blum and Hank Ditton, as part of the Kansas State Research Experience for Undergraduates. We will not recount this improvement here.
Understanding compositions

Let us consider two maps $f$ and $g$ which are tangent to the identity. We would like to understand the terms which appear in the composition $f \circ g$. 
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We fix the following notation. Given a finite sequence of positive integers \( \underline{i} = (i_1, i_2, \ldots, i_l) \), we define \( |\underline{i}| = i_1 + \cdots + i_l \), and we define the “length” of \( \underline{i} \) to be \( l = l(\underline{i}) \).
The Main Lemmas

Lemma

Let \( f(x) = x + \sum a_n x^n \) and \( g(x) = x + \sum b_n x^n \). Write the composition \( f \circ g(x) = x + \sum c_n x^n \). Then, we have

\[
c_n = \sum \alpha_{k_i} a_k b_i
\]

where \( \alpha_{k_i} \) is an integer (possibly 0), and the sum is taken over those sequences \( i' = (i, k) \) satisfying \( n = |i'| - l(i') + 1 \).
The Main Lemmas (con’t)

Lemma

Let \( f(x) = x + \sum a_nx^n \), and write \( f^{-1}(x) = x + \sum c_nx^n \). Then, we have

\[
c_n = \sum \alpha_i a_i
\]

where \( \alpha_i \) is an integer (possibly 0), and the sum is taken over those sequences \( i \) satisfying \( n = |i| - l(i) + 1 \).
Lemma

Fix \( m \geq 2 \), and for all \( n \geq 2 \), define

\[
\sigma_m(n) = n - 1 + \left\lfloor \frac{n - 2}{m - 1} \right\rfloor.
\]

Let \( \mathbf{i} = (i_1, \ldots, i_l) \), and define \( N = |\mathbf{i}| - l(\mathbf{i}) + 1 \). Then, we have the estimate

\[
\sum_{j=1}^{l} \sigma_m(i_j) \leq \sigma_m(N).
\]
Theorem (Blum, Ditton, —)

Let $F$ be a complete non-archimedean field, fix $q \in F$ with $0 < q \leq 1$, and consider the set of power series

$$G = \{ f(x) = x + \sum_{n \geq 2} a_n x^n : |q^{\sigma_m(n)} a_n| \leq 1 \}.$$  

Then, $G$ is a group.
Putting It Together

Theorem (Blum, Ditton, —)

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Then, $G$ is a group.

In fact, the function $\sigma_m$ may be replaced with any function satisfying the above inequality. For this reason, we call such functions “dynamical”.

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Non-Archimedean Conjugacies
The same questions can be posed in higher dimensions. In recent work with Steven Spallone, we have extended these results to so-called “semi-hyperbolic” maps in two dimensions. However, we currently have no results for maps which are tangent to the identity.
Thank you!