Elimination of singularities of smooth mappings of 4-manifolds into 3-manifolds

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Abstract

The simplest singularities of smooth mappings are fold singularities. We say that a mapping \( f \) is a fold mapping if every singular point of \( f \) is of the fold type. We prove\(^1\) that for a closed oriented 4-manifold \( M^4 \) the following conditions are equivalent:

1. \( M^4 \) admits a fold mapping into \( \mathbb{R}^3 \);
2. for every orientable 3-manifold \( N^3 \), every homotopy class of mappings of \( M^4 \) into \( N^3 \) contains a fold mapping;
3. there exists a cohomology class \( x \in H^2(M^4; \mathbb{Z}) \) such that \( x \sim x \) is the first Pontrjagin class of \( M^4 \).

For a simply connected manifold \( M^4 \), we show that \( M^4 \) admits no fold mappings into \( N^3 \) if and only if \( M^4 \) is homotopy equivalent to \( \mathbb{C}P^2 \) or \( \mathbb{C}P^2 \# \mathbb{C}P^2 \).

Key words: singularities, cusps, fold mappings, h-principle, jets
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1 Introduction

We study singularities of smooth mappings of an orientable closed 4-manifold \( M^4 \) into an orientable 3-manifold \( N^3 \) and determine a complete obstruction to the existence of a mapping \( M^4 \rightarrow N^3 \) without certain singularities.

Let us begin with review of related results on singularities of mappings into 2-manifolds.

\(^1\) After the paper was written, O. Saeki informed the author that he obtained similar results using a different approach [23].

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1.1 Mappings into surfaces.

Every continuous mapping $M^2 \to N^2$ of surfaces can be approximated by a mapping $f$ with only three types of points: regular points, fold singular points and cusp singular points (Whitney, [31]). By definition, a point $p$ of $M^2$ is a regular point of $f$ if the restriction of $f$ to some neighborhood of $p$ is a diffeomorphism. We say that a point $p$ is a singular point of the fold or cusp type if in some coordinate neighborhoods about the points $p$ and $f(p)$ the mapping $f$ has the form

$$f(x, y) = (x, y^2) \quad \text{or} \quad f(x, y) = (x^3 + xy, y)$$

respectively. Considering local forms, we conclude that the set of regular points of $f$ is an open submanifold of $M^2$ and the set of fold singular points forms a submanifold of dimension 1 with boundary at the discrete set of cusp singular points.

Under general position homotopy (for definition see sections 3 and 4) the number of cusp singular points of the mapping $f$ may change. The representative samples of bifurcations are the homotopies of birth

$$f_t(x, y) = (x^3 \pm xy^2 \mp tx, y), \quad t \in [-1, 1],$$
$$f_t(x, y) = (x^4 + xy - tx^2, y), \quad t \in [-1, 1],$$

under each of which two new cusp singular points appear, and the homotopies of death

$$f_t(x, y) = (x^3 \pm xy^2 \pm tx, y), \quad t \in [-1, 1],$$
$$f_t(x, y) = (x^4 + xy + tx^2, y), \quad t \in [-1, 1]$$

each of which reduces a pair of cusp singular points. We note that under general position homotopy the parity of the number of cusp singular points remains the same. In fact, the parity of the number of cusp singular points is the same as the parity of the Euler characteristic of $M^2$. Thus, for example, $\mathbb{R}P^2$ does not admit a mapping into $\mathbb{R}^2$ with only fold singular points [31].

Singularities of a general position mapping (for definition see section 4) from a manifold of dimension $m$, $m \geq 3$, into a surface are similar to singularities of a general position mapping of surfaces; a general position mapping $f : M^m \to N^2$ has a 1-dimensional submanifold of fold singular points and a discrete set of cusp singular points. As above, the homology class of the cusp singular points does not change under general position homotopy, and hence gives an obstruction to the elimination of cusp singular points by homotopy. If the manifold $N^2$ is orientable, then the homology class of the cusp singular points is Poincaré dual to $w_m(M^m)$ (Thom, [28]) and yields a complete obstruction
In general, the set of singular points $S(f)$ of a smooth mapping $f$ is not a manifold. However, if $f$ is in general position, then $S(f)$ naturally breaks into a union of finitely many manifolds which correspond to Thom-Boardman singularities. The Thom-Boardman singularities and the stratification of $S(f)$ of a general position mapping are defined as follows. Let $TM$ and $TN$ denote the tangent bundles of manifolds $M$ and $N$ respectively. For a smooth mapping $f : M \to N$, the differential $df : TM \to TN$ is a mapping linear on every fiber of $TM$. The dimension of the kernel of $df$ at a point $x \in M$ is called the kernel rank of $f$ at $x$ and is denoted by $kr_x f$. Let $S_i = S_i(f)$ denote the set of points $x \in M$ with $kr_x f = i$. Suppose that $\dim M = m \geq n = \dim N$. Then the points of the set $S_{m-n}$ are called regular. The other points of the manifold $M$ are called singular. If $f$ is in general position, then every set $S_i$ is a submanifold of $M$. Under this assumption, we consider the restriction $f|_{S_i}$ of $f$ to the submanifold $S_i$ and define $S_{i,j}(f|_{S_i})$ as the subset $S_{i,j}(f|_{S_i})$ of $S_i$. Again, for a general position mapping $f$, every set $S_{i,j}$ is a submanifold of $M$ (Boardman, [5]), hence the definition may be iterated. Thus, the set $S_{i,j,k}$ is defined by induction as $S_{i,j,k}(f|_{S_{i,j,k-1}})$. The index $I = (i_1, \ldots, i_k)$ is called the symbol of the singularity. We will write $S_I$ for $S_{i_1,\ldots,i_k}$. If $x$ is a singular point of type $S_I$, then we say that $x$ is an $S_I$-point or $I$-singular point.

We note that the Whitney fold and cusp singular points of a mapping of surfaces are of types $S_{1,0}$ and $S_{1,1,0}$ respectively.

### 1.3 Mappings into 3-manifolds.

As in the case of a mapping into a surface, the types of singularities of a general position mapping $f : M^m \to N^3$, $m \geq 3$, do not depend on the dimension $m$ of the domain manifold. The Boardman Formula for the codimension of the set $S_J(f)$ ([5], formula 6.5) implies that $f$ has a 2-submanifold $S_{m-2,0}(f)$ of fold singular points, a curve $S_{m-2,1,0}(f)$ of cusp singular points and a discrete set $S_{m-2,1,1,0}(f)$ of swallowtail singular points.

Let $f : M^m \to N^3$ be a general position mapping from a closed orientable manifold $M^m$ into an orientable 3-manifold. By the Ando theorem (Ando, [2]), the swallowtail singular points are not essential for the homotopy class of $f$; there is a homotopy of $f$ eliminating all swallowtail singular points. Let $\gamma$ be the closure of the cusp singular points of $f$. As above, the homology class
represented by $\gamma$ obstructs the elimination of the cusp singular points. In contrast with mappings into surfaces, it turns out that if $M^n$ is a 4-manifold, i.e. $m = 4$, then the homology obstruction $[\gamma] \in H_1(M^4;\mathbb{Z}_2)$ may not be complete. O. Saeki [22] (see also [21], [27] and [1]) showed that every general position mapping of the standard complex projective plane $CP^2$ into $\mathbb{R}^3$ has cusp singular points though the homology obstruction is trivial\(^2\), $[\gamma] \in H_1(CP^2;\mathbb{Z}_2) = 0$.

1.4 Results of the paper

In this paper we determine the secondary obstruction to the elimination of the cusp singular points of a mapping from an orientable closed 4-manifold into an orientable 3-manifold and show that the secondary obstruction is complete. In geometric terms this obstruction can be interpreted as follows. Let $f : M^4 \to N^3$ be a general position mapping of a closed connected oriented 4-manifold into an orientable 3-manifold. The set of singular points of $f$ is a surface $S$ embedded into $M^4$. General position homotopy changes $S$ by embedded bordism. However, as we will show in section 3, the normal Euler number $e(S)$ of the surface $S$ depends only on the homotopy class of the mapping $f$. Since the surface $S$ is determined by $f$, we denote $e(S)$ by $e(f)$ as well.

The intersection form of the manifold $M^4$ determines a quadratic form on the free part of $H_2(M^4;\mathbb{Z})$. Let $Q(M^4)$ denote the set of integers taken on by this quadratic form. We note that the set $Q(M^4)$ and the normal Euler number $e(f) \in Q(M^4)$ does not depend on the orientation of $M^4$. In sections 3 and 7 we will prove the main theorem.

**Theorem 1** Let $f : M^4 \to N^3$ be a general position mapping from an orientable closed connected 4-manifold into an orientable 3-manifold. Then the homotopy class of $f$ contains a fold mapping if and only if $e(f) \in Q(M^4)$.

In section 8 we will express the secondary obstruction in terms of the Pontrjagin class $p_1(M^4)$ of the tangent bundle of $M^4$. Namely, we will prove the formula\(^3\) $e(f) = (p_1(M^4), [M^4])$, where $[M^4]$ is the fundamental class of the manifold $M^4$. It allows us to formulate the main theorem in terms of the cohomology ring of $M^4$.

**Theorem 2** Let $f : M^4 \to N^3$ be a continuous mapping from an orientable

\(^2\) Moreover, O. Saeki proved this statement for every manifold $M^4$ whose homology is the same as the homology of $CP^2$. Compare with Theorem 3.

\(^3\) After the paper was written, the author learned that this equality is a special case of a result obtained in [17].
closed connected 4-manifold into an orientable 3-manifold. Then the homotopy class of \( f \) contains a fold mapping if and only if there is a cohomology class \( x \in H^2(M^4; \mathbb{Z}) \) such that \( p_1(M^4) = x^2 \).

Section 9 is devoted to the case where the manifold \( M^4 \) is simply connected. We examine the equation \( p_1(M^4) = x^2 \) and determine when it has a solution.

**Theorem 3** Let \( M^4 \) be an orientable closed connected simply connected 4-manifold and \( N^3 \) be an orientable 3-manifold. Then a homotopy class of a mapping \( f : M^4 \to N^3 \) has no fold mapping if and only if \( M^4 \) is homotopy equivalent to \( \mathbb{C}P^2 \) or \( \mathbb{C}P^2 \# \mathbb{C}P^2 \). Here homotopy equivalence is not supposed to be orientation preserving.

**Remark 4** If two manifolds \( M^4_1 \) and \( M^4_2 \) admit a fold mapping into \( \mathbb{R}^3 \), then the connected sum \( M^4_1 \# M^4_2 \) also admits a fold mapping into \( \mathbb{R}^3 \). In [25] the authors conjectured that the obstruction to the existence of a fold mapping into \( \mathbb{R}^3 \) is additive with respect to connected sum, and the manifold \( k \mathbb{C}P^2 \# l \mathbb{C}P^2 \) admits a fold mapping into \( \mathbb{R}^3 \) if and only if \( k + l \) is odd. Theorem 3 solves the conjecture in the negative.

**Remark 5** Sakuma conjectured (see [13], Remark 2.3) that a closed orientable manifold with odd Euler characteristic does not admit a fold mapping into \( \mathbb{R}^n \) for \( n = 3, 7 \). Saeki [24] presented an explicit counterexample to this conjecture. Theorem 3 shows that there are many manifolds with odd Euler characteristic admitting fold mappings into \( \mathbb{R}^3 \). However, it should be mentioned that Theorem 3 does not suggest a method of an explicit construction of fold mappings.

**Remark 6** A mapping \( f : M^m \to N^n \) is Morin if it has singularities only of types \((m - n + 1, 1, ..., 1, 0)\). If the manifolds \( M^m \) and \( N^n \) are orientable and \( m - n \) is odd, then every Morin mapping \( f : M^m \to N^n \) is homotopic to a mapping with at most cusp singular points [20]. Theorem 3 gives a restriction on further simplification of Morin mappings by homotopy.

The proof of Theorem 2 is based on the Ando–Éliashberg Theorem, which is an analog of the Smale–Hirsch Theorem [11].

Let \( TM \) and \( TN \) be the tangent bundles of smooth manifolds \( M \) and \( N \) respectively, and \( f : M \to N \) a smooth mapping. The celebrated Smale–Hirsch Theorem states that a necessary and sufficient condition for the existence of an immersion \( M \hookrightarrow N \) homotopic to \( f \) is the existence of a bundle homomorphism \( TM \to TN \) of rank \( \dim M \) homotopic to \( df : TM \to TN \). The two bundles \( \xi = TM \) and \( \eta = f^*TN \) over \( M \) give rise to the bundle \( \mathcal{HOM}(\xi, \eta) \) over \( M \), whose fiber is the set of homomorphisms \( \text{Hom}(\xi_x, \eta_x) \) between the fibers \( \xi_x \) and \( \eta_x \) of the bundles \( \xi \) and \( \eta \) over a point \( x \in M \) respectively. If \( f \) is an immersion, then the section \( M \to \mathcal{HOM}(\xi, \eta) \) that sends a point \( p \in M \) to the differential \( df|_p \) does not intersect the singular set \( \{(y, g) \in \mathcal{HOM}(\xi, \eta) | y \in \} \).
$M, \ g \in \text{Hom}(\xi_y, \eta_y)$, and $\text{rank}(g) < \dim M$. An alternative formulation of the Smale–Hirsch Theorem asserts that the mapping $f$ is homotopic to an immersion if and only if there exists a continuous section of the bundle $\mathcal{HOM}(\xi, \eta)$ that does not intersect the singular set.

In [5] Boardman gives a generalization $J^\infty(\xi, \eta)$ of the space $\mathcal{HOM}(\xi, \eta)$ and for every symbol $I$ defines a submanifold $\Sigma_I \subset J^\infty(\xi, \eta)$. Every mapping $f : M \to N$ induces a section $jf$ of the bundle $J^\infty(\xi, \eta)$ over $M$. Moreover, if the mapping $f$ satisfies some general position conditions (see section 4), then the singular sets $S_I(f)$ coincide with the sets $(jf)^{-1}(\Sigma_I)$. We will use the Ando–Éliashberg theorem [2] which states that for a certain class of symbols $I$, a mapping without $I$-singularities exists if and only if there is a section of the bundle $J^\infty(\xi, \eta)$ such that the image of the section does not intersect the singular set $\Sigma_I$ (see section 4).

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2 Singularities of mappings $M^4 \to N^3$

Let $f : M^4 \to N^3$ be a mapping from a closed 4-manifold $M^4$ into a 3-manifold $N^3$. The mapping $f$ is in general position if for every point $p \in M^4$ there are coordinate neighborhoods, $U$ about $p$ and $V$ about $f(p)$, such that the restriction of the mapping $f$ to $U$ has one of the following types:

Regular type
$T_1 = t_1, \ T_2 = t_2, \ Z = t_3$.

Definite fold singularity type
$T_1 = t_1, \ T_2 = t_2, \ Z = q_1^2 + q_2^2$.

Indefinite fold singularity type
$T_1 = t_1, \ T_2 = t_2, \ Z = q_1^2 - q_2^2$.

Cusp singularity type
$T_1 = t_1, \ T_2 = t_2, \ Z = q_1^3 + t_1x + x^3$. 
Swallowtail singularity type
\[ T_1 = t_1, \quad T_2 = t_2, \quad Z = \pm q_2^2 + t_1x + t_2x^2 + x^4. \]

This definition of a general position mapping agrees (see [4]) with the general definition that we recall in section 4. The coordinate neighborhoods \( U \) and \( V \) are called special or standard neighborhoods. We say that a general position mapping from a 4-manifold into a 3-manifold is a fold mapping (respectively cusp mapping) if it does not have cusp (respectively swallowtail) singular points.

It is known that every mapping can be approximated by a general position mapping [5]. Furthermore, as has been mentioned, every mapping of an orientable closed 4-manifold into an orientable 3-manifold is homotopic to a general position mapping \( f \) without swallowtail singular points [2]. Let \( S(f) \), \( S_+(f) \), \( S_-(f) \) and \( \gamma = \gamma(f) \) denote the singular set of \( f \), the set of definite fold points, the set of indefinite fold points, and the set of cusp points respectively. Then the set \( S(f) \) is a 2-dimensional submanifold of \( M^4 \), the sets \( S_-(f) \) and \( S_+(f) \) are 2-submanifolds of \( S(f) \) with boundaries and \( \partial S_-(f) = \partial S_+(f) = \gamma \). It is known that the \( \mathbb{Z}_2 \)-homology class represented by the curve of cusp singular points \( \gamma \) is trivial. We will show that \( \gamma \) bounds an orientable surface.

We denote the union of all components of \( S(f) \) that have cusp points by \( C(f) \) and define \( C_-(f) = S_-(f) \cap C(f) \) and \( C_+(f) = S_+(f) \cap C(f) \).

**Lemma 7** Suppose that \( f : M^4 \to N^3 \) is a cusp mapping from an orientable closed 4-manifold into an orientable 3-manifold. Then the submanifold \( \overline{C_+(f)} \) is an orientable surface bounded by \( \gamma \).

**PROOF.** Let \( U \) be a special coordinate neighborhood of a point \( p \in C_+(f) \). The normal bundle of the image \( f(C_+(f) \cap U) \) has a canonical orientation given by a vector field each vector of which in some special coordinates coincides with \( \frac{\partial}{\partial Z} \). The canonical orientation of the normal bundle of the image of \( C_+(f) \cap U \) leads to an orientation of \( C_+(f) \cap U \). Thus, every point of \( C_+(f) \) has a neighborhood with canonical orientation. These orientations give rise to an orientation of \( C_+(f) \). Therefore, \( C_+(f) \) is an orientable surface bounded by \( \gamma \). \( \square \)

Let us recall that a vector \( v(p) \) at a cusp point \( p \) is called a characteristic vector if there is a standard coordinate neighborhood of \( p \) such that \( v(p) = \frac{\partial}{\partial t_1}(p) \). A vector field on the curve \( \gamma(f) \) in \( M^4 \) is called a characteristic vector field if it consists of characteristic vectors. The existence of a characteristic vector field for an arbitrary cusp mapping of an orientable 4-manifold into an orientable 3-manifold will follow from Lemma 15 bellow.
We adopt the convention to substitute the word ‘bundle’ for the phrase ‘total space of a bundle’. For a vector bundle, we will identify the base with the zero section.

3  Bifurcations of singular set

The singular set $S(f)$ of a general position mapping $f : M^4 \to N^3$ from a closed orientable 4-manifold into an orientable 3-manifold is a submanifold of $M^4$. A homotopy of $f$ gives rise to a deformation of the singular set. If the homotopy of $f$ is in general position, then the singular set changes by isotopy, except for finitely many moments at each of which a stable bifurcation of the singular set occurs. The objective of this section is to describe the stable bifurcations of the singular set and define an invariant of the embedding $S(f) \to M^4$ that does not change under homotopy of $f$.

We say that a homotopy $F : M^4 \times [0,1] \to N^3 \times [0,1]$ joining two general position mappings is a general position homotopy if it is a general position mapping. The definition of a general position mapping is given in section 4. In this section we will use only a classification of singularities of general position mappings and the fact that every mapping has a $C^1$-close approximation by a general position mapping.

Lemma 8 Every homotopy joining two general position mappings can be approximated by a general position homotopy.

PROOF. Let $f_0$ and $f_1$ be two general position mappings, $f_i : M^4 \to N^3$, $i = 0, 1$. Let $F : M^4 \times [0,1] \to N^3 \times [0,1]$ be a homotopy between $f_0 \times \{0\}$ and $f_1 \times \{1\}$. For a small number $\varepsilon$, we may assume that the homotopy $F$ does not change the mapping in intervals $[0,\varepsilon) \subset [0,1]$ and $(1-\varepsilon,1] \subset [0,1]$. As a mapping from a 5-manifold into a 4-manifold, the homotopy $F$ has a $C^1$-close approximation by a general position mapping. Moreover, there is an approximation $\tilde{F}$ that coincides with $F$ on a neighborhood of $M^4 \times \{0\} \cup M^4 \times \{1\}$. Since $\tilde{F}$ is $C^1$-close to $F$, the composition $p \circ \tilde{F} : M^4 \times [0,1] \to [0,1]$, where $p : N^3 \times [0,1] \to [0,1]$ is the projection onto the second factor, has no singular points. Therefore, for every moment $t_0 \in [0,1]$, the inverse image $\tilde{F}^{-1}(N^3 \times \{t_0\})$ is diffeomorphic to $M^4$. Thus, $\tilde{F}$ can be considered as a new homotopy joining $f_0$ and $f_1$, which as a mapping from a 5-manifold into a 4-manifold is in general position. □

Lemma 8 guarantees that any two homotopic general position mappings can be joined by a homotopy in general position. By dimensional reasonings (see,
for example, [5] and [3]) a general position mapping \( f \) from a 5-manifold into a 4-manifold has Morin singularities, and \( D_4 \) singularities with symbol \( J = (2, 2, 0) \). In a neighborhood of a Morin singular point, in some local coordinates, the mapping \( f \) can be written as

\[
f(t_1, t_2, t_3, q, x) = (t_1, t_2, t_3, \pm q^2 + \sum_{i=1}^{k-1} t_i x^i + x^{k+1}), \quad k = 1, 2, 3, 4.
\]  

(1)

For a \( D_4 \) point of \( f \), there are local coordinates in which the mapping \( f \) takes the form

\[
f(t_0, t_1, t_2, u, v) = (t_0, t_1, t_2, u^2v \pm v^3 + t_0u + t_1v + t_2v^2).
\]  

(2)

Suppose that \( F : M^4 \times [0, 1] \to N^3 \times [0, 1] \) is a general position homotopy joining two general position mappings \( f_0 \) and \( f_1 \). From the normal forms (1) and (2), it is easy to verify that the singular set of a general position homotopy \( F : M^4 \times [0, 1] \to N^3 \times [0, 1] \) is a submanifold of \( M^4 \times [0, 1] \). Therefore, \( S(F) \) defines an embedded bordism between the singular sets of \( f_0 \) and \( f_1 \).

Now let us introduce an invariant of a homotopy class of a mapping \( M^4 \to N^3 \).

Let \( K \) be a 2-dimensional submanifold of an oriented closed 4-manifold \( M^4 \), and \( \mathcal{F} \) be an orientation system of local coefficients over \( K \). Then the normal class or the Euler class of the normal bundle over \( K \) in \( M^4 \) is a cohomology class \( e \in H^2(K; \mathcal{F}) \). The number \( (e, [K]) \), where \( [K] \in H_2(K; \mathcal{F}) \) is the fundamental class of \( K \), is an integer called the normal Euler number of the embedded manifold \( K \). We denote this number by \( e(K) \). Note that the sign of the normal Euler number depends on the orientation of \( M^4 \) (see [15]).

For a general position mapping \( f \) from an oriented closed 4-manifold \( M^4 \) into a 3-manifold \( N^3 \), its singular set is an embedded submanifold \( S(f) \) of \( M^4 \). We define the integer \( e(f) \) as the normal Euler number of the embedded 2-dimensional submanifold \( S(f) \). The integer \( e(f) \) turns out to be invariant under homotopy of \( f : M^4 \to N^3 \).

**Lemma 9** If \( f_0 \) and \( f_1 \) are two homotopic general position mappings, then \( e(f_1) = e(f_2) \).

**Proof.** Let \( F : M^4 \times [0, 1] \to N^3 \times [0, 1] \) be a general position homotopy joining \( f_0 : M^4 \times \{0\} \to N^3 \times \{0\} \) and \( f_1 : M^4 \times \{1\} \to N^3 \times \{1\} \). The boundary of the singular set \( B \) of \( F \) is the union of the singular sets \( B_0 \) of \( f_0 \) and \( B_1 \) of \( f_1 \). Let \( i_t : B_t \to B, t = 0, 1 \), denote the inclusion, \( \mathcal{F} \) the orientation system of local coefficients on \( B \), and let \( e \in H^2(B; \mathcal{F}) \) be the Euler class of the normal bundle of \( B \) in \( M^4 \times [0, 1] \). Then

\[
e(f_0) - e(f_1) = (i_0^* e, [B_0]) - (i_1^* e, [B_1]) = (e, i_0^* [B_0] - i_1^* [B_1]) = 0,
\]
since \( i_{0*}[B_0] - i_{1*}[B_1] \) corresponds to the boundary of \( B \) and vanishes in \( H_2(B; \mathcal{F}) \). \( \Box \)

The invariant \( e(f) \) allows us to give a necessary condition for the existence of a fold mapping into \( N^3 \). In the later sections we will prove that this condition is also sufficient.

With every oriented closed 4-dimensional manifold \( M^4 \) we associate the set \( Q(M^4) \) of integers each of which is the normal Euler number of an orientable surface in \( M^4 \).

**Lemma 10** If \( f : M^4 \to N^3 \) is a fold mapping, then \( e(f) \in Q(M^4) \).

**PROOF.** The singular set of a fold mapping consists of the surfaces \( S_-(f) \) of indefinite fold singular points and \( S_+(f) \) of definite fold singular points. Therefore, \( e(f) = e(S_-(f)) + e(S_+(f)) \). In [21] (see also [1]) it is proved that \( e(S_-(f)) = 0 \). Hence, \( e(f) = e(S_+(f)) \). Since \( S_+(f) \) is orientable (see Lemma 7), we conclude that \( e(f) \in Q(M^4) \). \( \Box \)

**Corollary 11** Suppose that the homotopy class of a general position mapping \( f : M^4 \to N^3 \) contains a fold mapping. Then \( e(f) \in Q(M^4) \).

4 Reduction to an algebraic topology problem

To prove that \( e(f) \) gives a sufficient condition to the existence of a homotopy eliminating the cusp singular points of \( f \) and to calculate the value \( e(f) \) we need some results due to Boardman, Ando, and Éliashberg. In this section we review a definition of singularities given by Boardman and formulate the Ando–Éliashberg theorem.

Let \( M \) and \( N \) be two smooth manifolds. A germ \( f \) at \( x \in M \) is a mapping from a neighborhood about \( x \) in \( M \) into \( N \). Two germs are equivalent if there exists a neighborhood of \( x \) where the germs coincide. A \( k \)-jet is a class of \( \sim_k \)-equivalence of germs. Two germs \( f \) and \( g \) at \( x \) are \( \sim_k \)-equivalent if at the point \( x \) the mappings \( f \) and \( g \) have the same partial derivatives of order \( \leq k \).

The set of all \( k \)-jets \( J^k(M, N) \) is called the \( k \)-jet space. The \( k \)-jet space is a bundle with respect to the projection \( J^k(M, N) \to M \) that takes a \( k \)-jet at \( x \) into the point \( x \). There are natural projections \( J^r(M, N) \to J^{r-1}(M, N) \), which give rise to the inverse limit \( J^\infty(M, N) = \lim J^r(M, N) \) called the jet space. A function on the jet space is smooth if locally it is the composition of the projection onto some \( k \)-jet space and a smooth function on the \( k \)-jet space.
A vector of the tangent bundle of the jet space is a differential operator. We say that a subset of the jet space is a submanifold if it is the inverse image of a submanifold of some \( k \)-jet space.

The set of germs determined by a smooth mapping \( f : M \to N \) defines a jet section \( jf : M \to J^\infty(M, N) \). There is a subbundle \( D \) of the tangent bundle of the jet space such that for every smooth mapping \( f : M \to N \) and every point \( x \) of \( M \) the differential of the section \( jf \) is an isomorphism \( d_x(jf) : T_xM \to D_y \) of the tangent plane at \( x \) to the fiber of the bundle \( D \) over \( y =jf(x) \). We will identify \( D_y \) with \( T_xM \).

Every 1-jet at a point \( x \in M \) determines a homomorphism \( T_xM \to T_{f(x)}N \), where \( f \) is a germ at \( x \) representing the jet. Let \( y \) be a point of the jet bundle and \( K_y \subset D_y \) the kernel of the homomorphism defined by the 1-jet component of \( y \). It is known that for every \( i_1 \) the set

\[
\Sigma_{i_1} = \{ y \in J^\infty(M, N) \mid \dim K_y = i_1 \}
\]  

(3)
is a submanifold of \( J^\infty(M, N) \). Let \( J^r \) denote the set of \( r \) integers \((i_1, \ldots, i_r)\) such that \( i_1 \geq \cdots \geq i_r \). Suppose that the set \( \Sigma_{2^{r-1}} \) has been already defined and \( \Sigma_{2^{r-1}} \) is a submanifold of \( J^\infty(M, N) \). Then define

\[
\Sigma_{2^r} = \{ y \in \Sigma_{2^{r-1}} \mid \dim(K_y \cap T\Sigma_{2^{r-1}}) = i_r \}.
\]  

(4)

Boardman proved that for every symbol \( J^r \) the set \( \Sigma_{2^r} \) is a submanifold of \( J^\infty(M, N) \).

A mapping \( f \) is called a general position mapping if the section \( jf \) is transversal to every submanifold \( \Sigma_{2^r} \). Using the Thom Strong Transversality Theorem (see [4] or [5]), one can prove that every mapping can be approximated by a general position mapping.

The set \( S_r(f) = (jf)^{-1}(\Sigma_{2^r}) \) is called the \( r \)-singular set of \( f \). If \( f \) is in general position, then this definition of \( r \)-singular set coincides with the definition of \( S_r(f) \) given in the introduction.

If \( \zeta \) is a vector space, then \( \zeta^{or} = \zeta \circ \zeta \circ \cdots \circ \zeta \) denotes the vector space defined as the vector space \( \zeta^{or} \) factored by the relation of equivalence: \( v_1 \otimes v_2 \otimes \cdots \otimes v_r \sim w_1 \otimes w_2 \otimes \cdots \otimes w_r \) if and only if there is a permutation of \( r \) elements \( \sigma \) such that \( v_i = w_{\sigma(i)} \) for \( i = 1, \ldots, r \). The space \( \zeta^{or} \) is called the symmetric \( r \)-tensor product of \( \zeta \). As in the example at the end of section 1.4 for every \( r \), the bundles \( \xi \) and \( \eta \) give rise to the bundle \( \text{HOM}(\xi^{or}, \eta) \). The fiber of \( \text{HOM}(\xi^{or}, \eta) \) over a point \( x \in M \) is the set of homomorphisms \( \text{Hom}(\xi^{or}_x, \eta_x) \) between the fibers \( \xi^{or}_x \) and \( \eta_x \) of the bundles \( \xi^{or} \) and \( \eta \) respectively over \( x \). The space \( \text{HOM}(\xi, \eta) \) in the formulation of the Smale–Hirsch Theorem is generalized by the vector
bundle

\[ S^r(\xi, \eta) = \mathcal{HOM}(\xi, \eta) \oplus \mathcal{HOM}(\xi \circ \xi, \eta) \oplus \cdots \oplus \mathcal{HOM}(\xi^r, \eta) \]

over \(M\) (see paper [19] of Ronga). As above we define \(S^\infty(\xi, \eta)\) as the inverse limit \(\lim_{\leftarrow} S^r(\xi, \eta)\).

Let \(kr(g)\) denote the rank of the kernel of a linear function \(g\). A point of \(S^\infty(\xi, \eta)\) over a point \(x \in M\) is a set \(g = \{g_i\}\) that consists of homomorphisms \(g_i \in \text{Hom}(\xi^i \circ x, \eta \circ x)\). We set

\[
\tilde{\Sigma}_{i_1} = \bigcup_{x \in M} \{g \in S^\infty(\xi_x, \eta_x) | kr(g) = i_1\}. \tag{5}
\]

Let \(Kh\) and \(Ch\) respectively denote the kernel and cokernel of a homomorphism \(h \in \text{Hom}(\xi, \eta)\). The composition of natural homomorphisms

\[
\text{Hom}(\xi \circ \xi, \eta) \rightarrow \text{Hom}(\xi, \text{Hom}(\xi, \eta)) \rightarrow \text{Hom}(Kg_1, \text{Hom}(Kg_1, Cg_1))
\]

takes the homomorphism \(g_2 \in \text{Hom}(\xi \circ x, \eta \circ x)\) into some homomorphism \(\tilde{g}_2\). We define

\[
\tilde{\Sigma}_{i_1, i_2} = \bigcup_{x \in M} \{g \in S^\infty(\xi_x, \eta_x) | g \in \tilde{\Sigma}_{i_1} \text{ and } kr(\tilde{g}_2) = i_2\} \tag{6}
\]

and refer the reader to [5] (see also [20]) for the definition of \(\tilde{\Sigma}_{\mathcal{J}} \subset S^\infty(\xi, \eta)\) where \(\mathcal{J}\) is a symbol of length \(\geq 1\).

Suppose we are given a continuous mapping \(f : M \rightarrow N\). Then we can simplify the spaces \(J^k(M, N)\) as follows. The space \(J^k(M, N)\) may be viewed as a bundle over \(M \times N\). Therefore, the mapping \(id \times f : M \rightarrow M \times N\), where \(id\) is the identity mapping of \(M\), induces some bundle \(J^k(M, f, N)\) over \(M\). In what follows we will suppose that a mapping \(f\) is given and we will write simply \(J^k(M, N)\) for \(J^k(M, f, N)\).

The bundle \(J^r(M, N)\) is isomorphic to the bundle \(S^r(\xi, \eta)\), where \(\xi = TM\) and \(\eta = f^*TN\). Moreover, there is an isomorphism of bundles \(S^\infty(\xi, \eta)\) and \(J^\infty(M, N)\) that takes each \(\Sigma_3\) isomorphically onto \(\Sigma_3\) [20]. That is why we identify \(S^r(\xi, \eta)\) with \(J^r(M, N)\), \(r = 1, 2, \ldots, \infty\), and \(\Sigma_3\) with \(\Sigma_3\). Also we will write \(J^r(\xi, \eta)\) for \(S^r(\xi, \eta)\).

Let \(m = \dim M\), \(n = \dim N\), \(i = \max\{1, m - n + 1\}\), \(\xi = TM\), and \(\eta = f^*TN\). Let \(\mathcal{J}_1\) denote the sequence \((i, 0)\) and \(\mathcal{J}_r, r > 1\), denote the sequence \((i, 1, \ldots, 1, 0)\) of length \(r + 1\). The points of the set \(\Sigma_3\) are called Morin singular points. We denote the regular points by \(\Sigma_{i-1}\) and the Morin singular points with index of length at most \(r + 1\) by \(\Omega_r = \Omega_r(\xi, \eta)\). Then \(\Omega_r\) is a bundle over \(M\).
Theorem 12 (Ando–Éliashberg, [2], [8]) Let \( \dim N \geq 2 \). Then for any continuous section \( s : M \to \Omega_r \) there exists a Morin map \( g : M \to N \) such that \( jg : M \to \Omega_r \) becomes a section fiber-wise homotopic to \( s \) in \( \Omega_r \).

In particular the Ando–Éliashberg theorem reduces the question of the existence of a fold mapping to the problem of finding a continuous section of the bundle \( \Omega_1 \). The bundle \( \Omega_1 \) can be induced by an appropriate mapping from the universal bundle, which is defined as follows. Let \( BSO_m \) and \( BSO_n \) denote the Grassmann manifolds. The projections of \( BSO_m \times BSO_n \) onto the first and the second factors induce from the universal bundles over \( BSO_m \) and \( BSO_n \) two bundles over \( BSO_m \times BSO_n \), which we denote by \( E_m \) and \( E_n \) respectively. As above the bundles \( E_m \) and \( E_n \) give rise to a new bundle \( J^\infty(E_m, E_n) \) over \( BSO_m \times BSO_n \). Let \( f : M \to N \) be a smooth mapping from an \( m \)-manifold into an \( n \)-manifold. There are characteristic mappings \( \tau_m : M \to BSO_m \) and \( \tau_n : N \to BSO_n \) inducing the tangent bundles from the universal bundles. It is easily verified that the mapping \( \mu = \tau_m \times (\tau_n \circ f) : M \to BSO_m \times BSO_n \) induces from the bundle \( J^\infty(E_m, E_n) \) the bundle equivalent to the bundle \( J^\infty(\xi, \eta) \) defined above. The bundle \( J^\infty(E_m, E_n) \) is called a universal jet bundle (compare with [10]). As above we can define subbundles \( \Omega_r(E_m, E_n) \). Note that the induced bundle \( \mu^*(\Omega_1(E_m, E_n)) \) is equivalent to the bundle \( \Omega_1(\xi, \eta) \).

5 Corollaries of the Ando–Éliashberg theorem

The proof of Theorem 12 in [2] shows that the relative version of Theorem 12 is valid as well. In other words, suppose that \( U \) is an open set in \( M \) and \( s : M \to \Omega_r \) is a section such that the restriction of \( s \) to a neighborhood of \( M \setminus U \) is the jet section \( jg \) induced by a Morin mapping \( g : M \setminus U \to N \). Then \( g \) admits an extension to a Morin mapping \( \tilde{g} : M \to N \) whose jet section \( j\tilde{g} \) is fiber-wise homotopic in \( \Omega_r \) to \( s \) by homotopy constant over \( M \setminus U \).

The space \( J^\infty(\xi, \eta) \) is infinite dimensional. For almost all our work it will be sufficient to consider finite dimensional jet bundle \( J^2(\xi, \eta) \). Formulas similar to (5) and (6) define subsets \( \Sigma_i^2 \subset J^2(\xi, \eta) \) and \( \Sigma_{i,j}^2 \subset J^2(\xi, \eta) \). We denote the analog of \( \Omega_1 \) by \( \Omega_1^2 \subset J^2(\xi, \eta) \). For fold mappings the Ando–Éliashberg theorem acquires the following form (see the proof of [2, Theorem 1]).

**Theorem 13** The homotopy class of a mapping \( f : M^m \to N^n, m \geq n \geq 2 \), contains a fold mapping if and only if there exists a section \( s : M^m \to \Omega_1^2 \).

As a corollary of the Ando–Éliashberg theorem, we obtain that the existence of a homotopy eliminating the cusp singular points of a mapping \( f \) into an orientable 3-dimensional manifold is independent of \( f \).
Corollary 14 The homotopy class of \( f : M^m \to N^3, m \geq 3 \), contains a fold mapping if and only if there is a fold mapping \( g : M^m \to \mathbb{R}^3 \).

**PROOF.** By Ando–Éliashberg theorem, the homotopy class of \( f \) contains a fold mapping if and only if there is a section \( M^m \to \Omega^2_1 \subset J^2(TM^m, f^*TN^3) \). The latter does not depend on \( f \) or \( TN^3 \) since the tangent bundle of an orientable 3-manifold is trivial. \( \square \)

6 Surgery of \( \gamma(f) \)

In this section we study a surgery of the singular set of a cusp mapping \( f : M^4 \to N^3 \) and find a sufficient condition for the existence of a homotopy of \( f \) realizing a given surgery.

Let \( f : M^4 \to N^3 \) be a cusp mapping of an orientable 4-manifold into an orientable 3-manifold. In general, the restriction of \( f \) to the curve of cusp singular points \( \gamma \) is an immersion. To simplify arguments, we make \( f|_\gamma \) an embedding by a slight perturbation of \( f \) in a neighborhood of \( \gamma \). Let \( \nu \) be a field of characteristic vectors on \( \gamma \). We say that an orientable surface \( H \) is a basis of surgery (see Éliashberg [8]), if

\begin{enumerate}
  \item[(B1)] \( \partial H = \gamma \),
  \item[(B2)] the vector field \( \nu \) is tangent to \( H \) and has an inward direction,
  \item[(B3)] \( H \setminus \partial H \) does not intersect \( S(f) \), and
  \item[(B4)] the restriction \( f|_H \) is an immersion.
\end{enumerate}

We will show that if a basis of surgery \( H \) exists, then we can reduce \( \gamma \) by modification of \( f \) in a neighborhood of \( H \). We assume that \( \gamma \) is connected and \( f \) has no other cusp singular points. The proof in the general case is similar.

We start with a description of the behavior of \( f \) in a neighborhood of \( \gamma \). We recall that in special coordinates the standard cusp mapping \( g : D^3 \to D^2 \) has the form \( g(t, q, x) = (t, q^2 + tx + x^3) \).

**Lemma 15** There are product neighborhoods \( S^1 \times D^3 \) of \( \gamma \) and \( S^1 \times D^2 \) of \( f(\gamma) \) such that \( f \) restricted to \( S^1 \times D^3 \) is the product of the identity mapping of \( S^1 \) and the standard cusp mapping \( g \).

**PROOF.** Let \( S^1 \times D^2 \) be a neighborhood of \( f(\gamma) \) consisting of discs \( D^2_x = f(x) \times D^2 \) each of which is indexed by a point \( x \in \gamma \) and transversally intersects the image of \( \gamma \) at \( f(x) \). The restriction of \( f \) to a neighborhood of \( \gamma \) followed by the natural projection of \( S^1 \times D^2 \) onto \( f(\gamma) \) has rank 1 at every point.
Hence the Inverse Function Theorem implies that there is a neighborhood $S^1 \times D^3$ of $\gamma$ consisting of small discs $D_x^3$ each of which maps under $f$ into the corresponding disc $D^2_x$.

For a point $x \in \gamma$, let $f_x$ denote the restriction $f|_{D^3_x}$. We recall that we write $J^\infty(D^3_x, D^2_x)$ for $J^\infty(D^3_x, f_x, D^2_x)$ and identify this space with $S^\infty(TD^3_x, f^*TD^2_x)$.

**Lemma 16** For every point $x \in \gamma$, the mapping $f_x$ is a general position mapping, i.e. a mapping whose jet section sends $D^3_x$ to $\Omega_2 \subset J^\infty(D^3_x, D^2_x)$ transversally to the submanifolds of $\Sigma_2$ and $\Sigma_2,1$-points.

**Proof of Lemma 16.** We will write $J^\infty([-1,1] \times D^3_x, [-1,1] \times D^2_x)$ for the restriction of $J^\infty([-1,1] \times D^3_x)$ to $[0] \times D^3_x$. Let $s : J^\infty(D^3_x, D^2_x) \rightarrow J^\infty([-1,1] \times D^3_x)$ be the embedding that relates the jet section of a mapping

$$g : D^3_x \rightarrow D^2_x$$

with the jet section of the mapping

$$\text{id} \times g : [-1,1] \times D^3_x \rightarrow [-1,1] \times D^2_x$$

restricted to $\{0\} \times D^3_x$, where $\text{id}$ stands for the identity mapping of $[-1,1]$. More precisely, by definition, $s$ is a unique embedding that makes the diagram

$$
\begin{array}{ccc}
J^\infty(D^3_x, D^2_x) & \xrightarrow{s} & J^\infty([-1,1] \times D^3_x) \\
\downarrow jg & & \downarrow j(id \times g) \\
D^3_x & \rightarrow & [-1,1] \times D^3_x
\end{array}
$$

commutative (see also the definition before Lemma 26). Here the bottom mapping is the embedding identifying $D^3_x$ with the disc $\{0\} \times D^3_x$.

Let us write $\Sigma'_j$ for the set of $J$-points in $J^\infty([-1,1], D^3_x)$ and $\tilde{\Sigma}_j$ for the set of $J$-points in $J^\infty(D^3_x, D^2_x)$.

By [19, Lemma 4.3], for every symbol $J^k$ of length $k \leq 2$, the mapping $s$ is transversal to $\Sigma'_j \cap J^\infty_0$ in $J^\infty_0$ and $s^{-1}(\Sigma'_j) = \tilde{\Sigma}_j$. By definitions (5) and (6), the sets $\Sigma'_j$, $k \leq 2$, are transversal to the fiber $J^\infty_0$ in $J^\infty([-1,1])$. Therefore the mapping $s$ is transversal to $\Sigma'_j$, $k \leq 2$, in $J^\infty([-1,1])$.

Let us prove that $s^{-1}(\Sigma'_j) = \tilde{\Sigma}_j$ for $k = 3$. The mapping $s$ defines a homomorphism $s_*$ of the tangent bundles of $J^\infty(D^3_x, D^2_x)$ and $J^\infty([-1,1])$. Note that

15
for every $y \in J^{\infty}(D^2, D^2)$, the homomorphism $s_*$ bijectively sends the kernel $K_y$ defined in section 4 into the kernel $K_{s(y)}$. For a symbol $T^2$ of length 2, the transversality $s \pitchfork \Sigma_2$ and $s^{-1}(\Sigma_2) = \Sigma_2$ imply that $s^{-1}(T \Sigma_2) = T \Sigma_2$. Therefore

$$s_*(K_y \cap T \Sigma_2) = K_{s(y)} \cap T \Sigma_2.$$  

Hence, by (4), for every symbol $T^3$, the set $\Sigma_{2,1}$ coincides with $s^{-1}(\Sigma_{2,1})$.

Identifying $D^2_3$ with $\{0\} \times D^2_2 \subset [-1, 1] \times D^2_2 \subset S^1 \times D^3$, we obtain $(jf)|_{D^3_2} = s \circ jf_x$. Since the mapping $s$ is transversal to $\Sigma_2, \Sigma_{2,1}$ and the inverse image of the set of $\Omega_2$-points of $J^{\infty}_{[-1,1]}$ under the mapping $s$ is the set of $\Omega_2$-points in $J^{\infty}(D^2_3, D^2_2)$, it remains to show that

(C1) the image of $(jf)|_{D^3_2}$ is in $\Omega_2 \subset J^{\infty}_{[-1,1]}$ and

(C2) $(jf)|_{D^3_2}$ is transversal to the sets of $\Sigma_2$ and $\Sigma_{2,1}$-points.

The condition (C1) holds since $\text{Im}(jf) \subset \Omega_2$. Let us prove (C2). The differential of the jet section $jf$ at $x$ splits into the sum of homomorphisms

$$d(jf) = d(jf)|_{T_xD^3_2} + d(jf)|_{T_x\gamma},$$  

(7)

where $T_xD^3_2$ and $T_x\gamma$ are the tangent spaces of $D^3_2$ and $\gamma$ at $x$ respectively. The differential $d(jf)|_{T_x\gamma}$ sends $T_x\gamma$ into the tangent space of the $\Sigma_{2,1}$-points. Now, since $f$ is in general position, the equation (7) implies that $d(jf)|_{T_xD^3_2}$ is transversal to the sets of $\Sigma_{2,1}$ and $\Sigma_2$-points. This completes the proof of Lemma 16. □

In view of Lemma 16, the collection of the mappings $\{f_x\}$ indexed by the points of an interval of $\gamma$ can be viewed as a homotopy of the standard cusp mapping, which is known to be homotopically stable (for example see Theorem 7.1 in [9]). Therefore, there is a cover $\{I_\alpha\}$ of the curve $\gamma$ by intervals such that for each interval $I_\alpha$ of the cover, the mapping $f$ restricted to $I_\alpha \times D^3$ is equivalent to the product $id_\alpha \times g$ of the identity mapping of $I_\alpha$ and the standard cusp mapping $g$.

The trivializations $\{I_\alpha \times D^3\}$ and $\{I_\alpha \times D^2\}$ lead to bundle structures of $S^1 \times D^3$ and $S^1 \times D^2$ over $S^1$ with common cover $\{I_\alpha\}$ of $S^1$ and with transition mappings consistent with $f$. The latter means that each pair $(\psi, \phi)$ of the corresponding transition mappings belongs to the group stabilizing the standard cusp mapping $g$. Since the normal bundles of $\gamma$ in $M^4$ and of $f(\gamma)$ in $N^3$ are orientable, the transition mappings are elements of the group

$$\text{Aut}(g) = \{ (\psi, \phi) \in \text{Diff}_+(D^3, 0) \times \text{Diff}_+(D^2, 0) \mid \phi \circ g \circ \psi^{-1} = g \}.$$
where $\text{Diff}_+(D^2, 0)$ and $\text{Diff}_+(D^3, 0)$ stand for the groups of orientation preserving auto-diffeomorphisms of $(D^2, 0)$ and $(D^3, 0)$ respectively. The group $\text{Aut}(g)$ reduces to a maximal subgroup $M\text{CAut}(g)$ conjugate to a linear compact subgroup $[12], [29]$. To prove Lemma 15, it remains to show that the group $M\text{CAut}(g)$ is trivial.

Let $\mathcal{K}$ denote the orientable version of the contact group, i.e. $\mathcal{K}$ is a semiproduct of $\text{Diff}_+(D^2, 0)$ and of the group of germs $(D^2, 0) \to \text{Diff}_+(D^1, 0)$. The group $\text{Aut}_\mathcal{K}(h)$ of a germ $h : (D^2, 0) \to (D^1, 0)$ is defined by

$$\text{Aut}_\mathcal{K}(h) = \{ \varphi \in \mathcal{K} | \varphi(h) = h \}.$$

The standard cusp mapping $g$ is a miniversal unfolding of the germ $g_0 : (D^2, 0) \to (D^1, 0)$, defined by $g_0(q, x) = q^2 + x^3$. Hence by [29, Proposition 3.2], the group $M\text{CAut}(g)$ is isomorphic to $M\text{CAut}_\mathcal{K}(g_0)$. The latter group is isomorphic to a compact subgroup $H$ of the group $\text{Aut} Q_{g_0}$ of automorphisms of the local algebra of $g_0$ (see [18, Theorem 1.4.6]).

Let $\psi \in GL_+(2)$ be a linear automorphism of $\mathbb{R}^2$ that preserves orientation. It defines an action on germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ by sending a germ $h$ into $h \circ \psi$. Suppose that this action factors through an action on local algebra $\text{Aut} Q_{g_0}$. Then $\psi$ defines an element in $\text{Aut} Q_{g_0}$. By the proof of [18, Theorem 1.4.6], we may assume that each element of $H < \text{Aut} Q_{g_0}$ is induced by a linear map of $GL_+(2)$. The linear maps $\psi \in GL_+(2)$ corresponding to elements of $H$ leave invariant the principal ideal $P$ generated by a germ that in some coordinates has the form $(q, x) \mapsto q^2 + x^3$. Note that up to a scalar multiple the vector $\frac{\partial}{\partial x}$ is determined by the property that for every germ $h \in P$, the partial derivative $\frac{\partial h}{\partial x}(0, 0) = 0$. Consequently, if a map $\psi \in GL_+(2)$ corresponds to an element in $H$, then $\psi$ leaves the direction of the vector $\frac{\partial}{\partial x}$ invariant, i.e. $\psi(\frac{\partial}{\partial x}) = \alpha \frac{\partial}{\partial x}$ with $\alpha \neq 0$. Moreover, since $\psi$ leaves the ideal $P$ invariant, we conclude that $\alpha > 0$. Hence, the group $H$ is contractible. This completes the proof of Lemma 15. □

We need one more preliminary observation. Let $I$ denote the closed interval $[-1, 1]$ and $g : I \times D^2 \to I \times I$ be the standard cusp mapping defined by $g(t, q, x) = (t, q^2 + tx + x^3)$, where $t$ and $q, x$ are the coordinates of the first and the second factors of the domain $I \times D^2$ respectively. Then $g$ can be considered as a homotopy $g_t : D^2 \to I, t \in I$, defined by $g_t(q, x) = q^2 + tx + x^3$. Note that $g_{-1} : D^2 \to I$ is a Morse function. Since Morse functions are homotopically stable, there are coordinates in which $g_s = g_{-1}$ for each $s \in [-1, -1 + \varepsilon)$.

**Lemma 17** Suppose that there is a basis of surgery $H_1$. Then there is a fold mapping $\tilde{f} : M^4 \to N^3$, which differs from $f$ only in a neighborhood of $H_1$. 

17
PROOF. Let \( U(\gamma) = S^1 \times D^3 \) be a neighborhood of \( \gamma \) in \( M^4 \) given by Lemma 15. Let \( t_2 \) be a cyclic coordinate on the circle \( S^1 \), \((t_1, q, x)\) be coordinates on \( D^3 \) and \((T_1, T_2, Z)\) be coordinates in a neighborhood of \( f(\gamma) \) with \( T_2 \) cyclic such that \( f|_{U(\gamma)} \) is given by
\[
T_1 = t_1, \quad T_2 = t_2, \quad Z = q^2 + t_1 x + x^3.
\]
We may assume that
\[
H_1 \cap U(\gamma) = \{ (t_1, t_2, q, x) \mid x = q = 0, \ t_1 \in [0, 1] \}.
\]
We define
\[
H_0 = \{ (t_1, t_2, q, x) \mid x = q = 0, \ t_1 \in [-1, 0] \}
\]
and set \( H = H_0 \cup H_1 \).

We regard a tubular neighborhood of a submanifold as a disc bundle. The properties (3) and (4) of the definition of a basis of surgery guarantees that the submanifold \( H \) has a tubular neighborhood \( A \) such that the restriction of \( A \) to \( H \cap U(\gamma) \) is in \( U(\gamma) \), the intersection \( S(f) \cap \partial A \) is in the restriction of the bundle \( A \) to \( \partial H \) and the set \( A \setminus U(\gamma) \) contains no singular points of the mapping \( f \). To simplify explanations we assume that \( f|H \) is an embedding.

Then the image of \( A \), which we denote by \( B \), is a line bundle over \( f(H) \).

In the following, for a manifold \( X \) with boundary \( \partial X \), let \( CX \) denote a collar neighborhood of \( \partial X \) in \( X \), and let \( I \) denote \([-1, 1]\).

First, by the remark preceding the lemma, the manifolds \( B_1 = B|_{f(CH)} \) and \( A_1 = f^{-1}(B_1) \cap A \) have product structures \( A_1 = CH \times I \times I \) and \( B_1 = f(CH) \times I \) such that \( f|_{A_1} \) is a product of a diffeomorphism \( CH \rightarrow f(CH) \) and a Morse function. We may assume that the restriction of this Morse function to \( CI \times I \cup I \times CI \) is the projection onto the factor corresponding to the coordinate \( x \), \( CI \times I \rightarrow I \), \( I \times CI \rightarrow CI \), and that \( A_1 = A|_{CH} \).

Next, we extend the product structure of \( B_1 \) to a product structure \( f(H) \times I \) of \( B \). Then we restrict this product structure to \( B_2 = f(H) \times CI \) and define \( A_2 = f^{-1}(B_2) \cap A \). The mapping \( f|_{A_2} \) is regular and therefore we may assume that \( A_2 = H \times I \times CI \) is a trivial line bundle over \( B_2 \) with projection \( f|_{A_2} \) along the second factor.

Finally, we can find \( A_3 \subset A \) and a product structure \( H \times CI \times I \) of \( A_3 \) such that \( f|_{A_3} \) is a trivial \( CI \)-bundle over \( f(H) \times I \) and \( A_1 \cup A_2 \cup A_3 \) is a collar neighborhood of \( \partial A \).

The connected components of \( A_2 \) and \( A_3 \) are orientable 1-dimensional bundles with bundle mappings given by the restrictions of \( f \). Since the structure group of orientable line bundles reduces to the trivial group, we can make the third
coordinates of $A_1, A_2$ and $A_3$ agree on intersections. We fix an extension of the product structures of $A_1, A_2$ and $A_3$ to a product structure $H \times I \times I$ of $A$.

Let $p \in \partial H$ and $f_p$ denote the restriction of $f$ to the fiber of the bundle $A$ over $p$. We let $\tilde{f}(x) = f(x)$ for $x \in M^4 \setminus A$ and

$$\tilde{f}(u, v, w) = f(u) \times f_p(v, w)$$

for $x = (u, v, w) \in A = H \times I \times I$. It is easily verified that $\tilde{f}$ is a smooth mapping and $\tilde{f}$ satisfies the requirements of the lemma. □

7 Sufficient condition

The objective of this section is to prove that the condition $e(f) \in Q(M^4)$ is sufficient for the existence of a fold mapping homotopic to a general position mapping $f : M^4 \to \mathbb{R}^3$. In view of Corollaries 8 and 14 this completes the proof of Theorem 1.

Lemma 18 Let $f$ be a general position mapping from a connected closed oriented 4-manifold $M^4$ into $\mathbb{R}^3$. Suppose $e(f) \in Q(M^4)$. Then $M^4$ admits a fold mapping into $\mathbb{R}^3$.

PROOF. The condition $e(f) \in Q(M^4)$ guarantees the existence of an orientable 2-submanifold $S$ of $M^4$ with normal Euler number $e(f)$.

Let us prove that in the complement $M^4 \setminus S$, there is an orientable possibly disconnected embedded surface $\tilde{S}$ such that

(P) every orientable surface embedded in $M^4 \setminus S$ with non-trivial normal bundle intersects $\tilde{S}$.

If $M^4 \setminus S$ admits no orientable embedded surface with non-trivial normal bundle, then the property (P) holds for any orientable embedded surface $\tilde{S}$. Suppose that in $M^4 \setminus S$ there is an orientable embedded surface with non-trivial normal bundle and that a surface with property (P) does not exist. Then for any positive integer $k$ there is a family of oriented embedded surfaces $\{F_i\}_{i=1,...,k}$ such that each of the surfaces has a non-trivial normal bundle and does not intersect the other surfaces of the family. Let $\text{Tor } H_2(M^4 \setminus S; \mathbb{Z})$ denote the subgroup of $H_2(M^4 \setminus S; \mathbb{Z})$ that consists of all elements of finite order. The group $H_2(M^4 \setminus S; \mathbb{Z})/\text{Tor } H_2(M^4 \setminus S; \mathbb{Z})$ is finitely generated. Fix a set of generators $e_1, ..., e_s$. Every surface $F_i$ represents a class $[F_i]$ in $H_2(M^4 \setminus S; \mathbb{Z})$. While...
There is a general position mapping

Let

Lemma 19 Let \( \alpha \) of the surfaces is greater than the number \( s \) of the generators, then there is a combination

\[
\alpha_1[F_1] + \alpha_2[F_2] + \cdots + \alpha_k[F_k] = 0
\]

with \( \alpha_1^2 + \cdots + \alpha_k^2 \neq 0 \). Multiplication of both sides by \([F_i] \), \( i = 1, \ldots, k \), gives \( \alpha_i[F_i] \cdot [F_i] = 0 \). Therefore, \( \alpha_i = 0 \) for every \( i = 1, \ldots, k \). Contradiction. Thus a surface \( \tilde{S} \) with property \((P)\) exists.

Let us construct a mapping for which the set \( \tilde{S} \cup S \) is the part of the singular set. We recall that we identify the base of a vector bundle with the zero section.

**Lemma 19** There is a general position mapping \( h : NS \to \mathbb{R}^3 \) from the normal bundle \( NS \) of \( S \) in \( M^4 \) such that the set \( S \) is the set of definite fold singular points of \( h \) and \( h \) has no other singular points.

**PROOF.** The fiber of the bundle \( NS \) is diffeomorphic to the standard disc \( D^2 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \} \). Let \( m : D^2 \to [-2, 2] \) be the mapping defined by the formula \( m(x, y) = x^2 + y^2 \). Then \( m \) is a Morse function on \( D^2 \) with one singular point.

For every open disc \( U_\alpha \) in \( S \), the restriction of the normal bundle \( NS \) to \( U_\alpha \) is a trivial bundle \( U_\alpha \times D^2 \to U_\alpha \). Let \( I_3 \) denote the segment \((-3, 3)\). We define the mapping \( g_\alpha : U_\alpha \times D^2 \to U_\alpha \times I_3 \) by \( g_\alpha(u, z) = (u, m(z)) \), where \( u \in U_\alpha \), and \( z \in D^2 \). Note that rotations of \( D^2 \) do not change the function \( m \). We may assume that the fiber bundle \( NS \to S \) is an \( SO_2 \)-bundle. Then the mappings \( g_\alpha \) give rise to a mapping \( g : NS \to S \times I_3 \). The open oriented manifold \( S \times I_3 \) admits an immersion into \( \mathbb{R}^3 \). We define \( h : NS \to \mathbb{R}^3 \) as the composition of \( g \) and this immersion. \( \square \)

**Lemma 20** Let \( N\tilde{S} \) be the normal bundle of \( \tilde{S} \) in \( M^4 \). Then there is a general position mapping \( h : N\tilde{S} \to \mathbb{R}^3 \) such that the set \( \tilde{S} \) is the singular set of \( h \) and every component of \( \tilde{S} \) has at least one cusp singular point of \( \tilde{h} \).

**PROOF.** For a closed disc \( D \subset \tilde{S} \), the restriction of the bundle \( N\tilde{S} \to \tilde{S} \) to \( \tilde{S} \setminus D \) is a trivial bundle \((\tilde{S} \setminus D) \times D^2 \to (\tilde{S} \setminus D)\), where \( D^2 \) is the disc as in Lemma 19. Let \( I_3 \) denote the segment \((-3, 3)\). The function \( m_1 : D^2 \to I_3 \) defined by \( m_1(x, y) = x^2 - y^2 \) is a Morse function with one singular point at the origin. Let \( id_1 : \tilde{S} \setminus D \to \tilde{S} \setminus D \) be the identity mapping. Put \( E_1 = (\tilde{S} \setminus D) \times D^2 \) and \( B_1 = (\tilde{S} \setminus D) \times I_3 \). Then \( id_1 \times m_1 : E_1 \to B_1 \) is a fold mapping.

Set \( E_2 = S^1 \times (-1, 1) \times D^2 \) and \( B_2 = S^1 \times (-1, 1) \times I_3 \). There is a mapping \( g : E_2 \to B_2 \) such that
(1) the set \( S^1 \times (-1, 0) \times \{(0, 0)\} \) is the set of all indefinite fold singular points of \( g \),
(2) the set \( S^1 \times (0, 1) \times \{(0, 0)\} \) is the set of all definite fold singular points of \( g \),
(3) the curve \( S^1 \times \{0\} \times \{(0, 0)\} \) is the set of all cusp singular points of \( g \).

Let \( U \) denote the intersection of \( \hat{S} \setminus D \) and a collar neighborhood of \( \partial(\hat{S} \setminus \hat{D}) \) in \( \hat{S} \setminus \hat{D} \). Then \( U \) is diffeomorphic to \( S^1 \times (-1, -1/2) \). We can identify the subset \( U \times D^2 \) of \( E_1 \) with the subset \( S^1 \times (-1, -1/2) \times D^2 \) of \( E_2 \) and the subset \( U \times I_3 \) of \( B_1 \) with the subset \( S^1 \times (-1, -1/2) \times I_3 \) of \( B_2 \) so that the obtained sets \( E_1 \cup E_2 \) and \( B_1 \cup B_2 \) are manifolds and the mapping \( id_1 \times m_1 \) coincides with the mapping \( g \) on the common part of the domains \( E_1 \cap E_2 \subset E_1 \cup E_2 \). Thus, \( id_1 \times m_1 \) and \( g \) define a cusp mapping \( c : E_1 \cup E_2 \to B_1 \cup B_2 \). Note that \( E_1 \cup E_2 \) is diffeomorphic to \( (\hat{S} \setminus D) \times D^2 \) and \( B_1 \cup B_2 \) is diffeomorphic to \( (\hat{S} \setminus D) \times I_3 \).

Let \( m_3 : D^2 \to I_3 \) be the Morse function, defined by \( m_3(x, y) = x^2 + y^2 \), and \( id_3 : \hat{D} \to \hat{D} \) be the identity mapping of the open 2-disc \( \hat{D} = D \setminus \partial D \). Then \( id_3 \times m_3 : \hat{D} \times D^2 \to \hat{D} \times I_3 \) is a fold mapping. Let \( V \) be the intersection of \( \hat{D} \) and a tubular neighborhood of \( \partial D \) in \( \hat{S} \). Then \( V \) is diffeomorphic to \( S^1 \times (1/2, 1) \). We identify the part \( V \times D^2 \) of \( E_3 \) = \( \hat{D} \times D^2 \) with the part \( S^1 \times (1/2, 1) \times D^2 \) of \( E_2 \subset E_1 \cup E_2 \) and the part \( V \times I_3 \) of \( B_3 = \hat{D} \times I_3 \) with the part \( S^1 \times (1/2, 1) \times I_3 \) of \( B_2 \subset B_1 \cup B_2 \) so that

(1) the obtained sets \( E = E_1 \cup E_2 \cup E_3 \) and \( B = B_1 \cup B_2 \cup B_3 \) are manifolds,
(2) the mapping \( id_3 \times m_3 \) coincides with \( c \) on the common part of the domains,
(3) the manifold \( E \) is diffeomorphic to \( NS \).

The condition (3) can be achieved since the mapping \( m_3 \) does not change under rotations of the fiber \( D^2 \).

Then \( id_3 \times m_3 \) and \( c \) define a cusp mapping \( NS \to B \). Note that \( B \approx \hat{S} \times I_3 \) is an open orientable 3-manifold. Thus, it admits an immersion into \( \mathbb{R}^3 \). The composition of \( NS \to B \) and the immersion \( B \to \mathbb{R}^3 \) is a cusp mapping satisfying the conditions of the lemma. \( \square \)

We identify \( NS \) and \( NS \) with open tubular neighborhoods of \( S \) and \( \hat{S} \) in \( M^4 \) respectively. There is a general position mapping \( g : M^4 \to \mathbb{R}^3 \) which extends \( h : NS \to \mathbb{R}^3 \) and \( \tilde{h} : NS \to \mathbb{R}^3 \). In general the extension \( g \) has some swallowtail singular points. Let us prove that we may choose \( g \) to be a cusp mapping.
Ando (see Section 5 in [2]) showed that for any general position mapping \( f : M^4 \to \mathbb{R}^3 \), the obstruction to the existence of a section of the bundle \( \Omega_2(TM^4, f^*T\mathbb{R}^3) \) over the orientable closed 4-manifold \( M^4 \) coincides with the number of the swallowtail singular points of \( f \) modulo 2. Also Ando calculated that this obstruction is trivial. Since the mapping \( h \cup \tilde{h} \) does not have swallowtail singular points, the obstruction to the existence of an extension of the section \( j^3(h \cup \tilde{h}) \), defined over \( NS \cup N\tilde{S} \), to a section of \( \Omega_2 \) over \( M^4 \) is trivial. Therefore, the relative version of the Ando–Éliashberg theorem (see section 5) implies the existence of an extension to a cusp mapping \( g : M^4 \to \mathbb{R}^3 \).

The singular set \( S(g) \) consists of \( S \cup \tilde{S} \) and probably of some other connected submanifolds \( A_1, \ldots, A_k \) of \( M^4 \). We have

\[
e(f) = e(S(g)) = e(S) + e(\tilde{S}) + e(A_1) + e(A_2) + \cdots + e(A_k). \tag{8}
\]

The normal Euler number of the submanifold \( S \) equals \( e(f) \). Hence the sum of the normal Euler numbers \( e(A_1) + \cdots + e(A_k) \) equals \( -e(\tilde{S}) \).

Let \( A_t \) be a component of \( \cup A_i \). Suppose \( A_t \) is a surface of definite fold singular points. The surface \( A_t \) is orientable (see Lemma 7) and does not intersect \( S \cup \tilde{S} \). By definition of \( \tilde{S} \), this implies \( e(A_t) = 0 \). Suppose \( A_t \) is a surface of indefinite fold singular points. Then again \( e(A_t) = 0 \) (see [21] or [1]). Therefore, \( e(A_t) \) is nontrivial only if the surface \( A_t \) contains cusp singular points. Let us recall that the union of those components of the singular submanifold \( S(g) \) that contain cusp singular points is denoted by \( C = C(g) \). The equation (8) implies that \( e(C) = e(\tilde{S}) + e(A_1) + \cdots + e(A_k) = 0 \).

It remains to prove the following lemma.

**Lemma 21** If \( g : M^4 \to \mathbb{R}^3 \) is a cusp mapping and \( e(C) = 0 \), then there exists a homotopy of \( g \) eliminating all cusp singular points.

**PROOF.** If the curve of cusp singular points is not connected, then there exists a homotopy of \( g \) to a mapping with one component of the curve of cusp singular points. We may require that the homotopy preserves the number \( e(C) \). We omit the proof of these facts since the reasonings are similar to those in section 6.

We will assume that the curve of cusp singular points \( \gamma(g) \) is connected and hence so is \( C(g) \).

**Lemma 22** Let \( \nu(x) \) be a characteristic vector field on \( \gamma(g) \). If \( e(C) = 0 \), then \( \nu(x) \) can be extended on \( C(g) \) as a normal vector field.
PROOF. For a general position mapping \( g: M^4 \to \mathbb{R}^3 \), the set
\[
F = f^{-1}(f(C_- (g))) \hookrightarrow M^4
\]
is an immersed 3-manifold. The self-intersection points of \( F \) correspond to the points of the surface \( C_- (g) \).

We say that two vectors \( v_1 \) and \( v_2 \) of a vector space have the same direction if \( v_1 = \lambda v_2 \) for some scalar \( \lambda \neq 0 \). There is an unordered pair of directions \((l_1(p), l_2(p))\) over \( C_- (g) \) [1] with the following property. For every point \( p \) of \( C_- (g) \) there are a neighborhood \( U \) about \( p \) with coordinates \((x_1, x_2, x_3, x_4)\) and a coordinate neighborhood about \( g(p) \) such that the restriction \( g|_U \) has the form \((x_1, x_2, x_3^2 - x_4^2)\) and the directions of the vectors \( \partial/\partial x_3 \) and \( \partial/\partial x_4 \) coincide with \( l_1(p) \) and \( l_2(p) \) respectively. An \( \mathcal{L} \)-pair is a pair \((l_1(p), l_2(p))\) that satisfies this property.

Let \( F_1 \subset C_- (g) \) denote the complement of a regular neighborhood of the curve \( \gamma(g) \) in \( C_- (g) \). The proof of Lemma 3 in [1] shows that there is a vector field \( v(p) \) in the normal bundle over \( F_1 \) with directions \( l_1(p) + l_2(p) \) or \( l_1(p) - l_2(p) \) over the boundary \( \partial F_1 \) for some \( \mathcal{L} \)-pair \((l_1(p), l_2(p))\).

We say that a direction at a cusp singular point is an \( x \)-direction if it is tangent to the surface \( S(f) \) and transversal to the curve \( \gamma(g) \). Note that for a special coordinate neighborhood about a cusp singular point the direction of the vector \( \partial/\partial x \) has an \( x \)-direction.

It is easily verified that for an \( \mathcal{L} \)-pair \((l_1(p), l_2(p))\), the directions \( l_1(p) \pm l_2(p) \) are tangent to \( F \) at every point \( p \) in \( C_- (f) \). Furthermore the directions \( l_1(p) + l_2(p) \) and \( l_1(p) - l_2(p) \) approach the same \( x \)-direction as \( p \) approaches \( \gamma(g) \). It implies that the vector field \( v(p) \) over \( F_1 \) has an extension to \( \overline{C_- (g)} \) such that \( v(p) \) is transversal to \( C_- (g) \) at every point of \( C_- (g) \) and has an \( x \)-direction at every point of \( \gamma(g) \). If necessary, we multiply the vector field \( v(p) \) by \(-1\) to get a vector field which points toward \( C_- (g) \) over \( \gamma(g) \). Now the vector field \( v(p) \) can be modified in a neighborhood of \( \gamma(g) \) so that a new \( v(p) \) is normal to \( C_- (g) \) at every point of \( C_- (g) \) and the restriction of \( v(p) \) to \( \gamma(g) \) is the characteristic vector field \( v(p) \).

The obstruction to the existence of an extension of \( v(p) \) to a vector field over \( C(g) \) is the normal Euler number \( e(C) \). Since \( e(C) = 0 \), such an extension exists. \( \square \)

The ends of the vectors \( \nu(p), p \in C_+(g) \), define an embedding of an orientable surface \( H \) diffeomorphic to \( C_+(g) \) into \( M^4 \). We modify the embedding in a neighborhood of the boundary \( \partial H \) so that the new embedding defines a basis of surgery. Now Lemma 21 follows from Lemma 17. \( \square \)
8 Computation of $e(f)$

In this section we will calculate the number $e(f)$ for a general position mapping $f$ from a closed oriented 4-manifold $M^4$ into an orientable 3-manifold $N^3$.

**Lemma 23** Let $p_1(M^4)$ denote the first Pontrjagin class of $M^4$ and $[M^4]$ the fundamental class of $M^4$. Then $e(f) = (p_1(M^4), [M^4])$.

A smooth mapping $f : M^4 \to N^3$ induces a section of the 2-jet bundle $J^2(TM^4, f^*TN^3)$ over $M^4$. To calculate the invariant $e(f)$ we consider sections $M^4 \to J^2(\xi, \eta)$, where $\xi$ is an arbitrary orientable 4-vector bundle over $M^4$ and $\eta$ is an arbitrary orientable 3-vector bundle over $M^4$.

The singular set $\Sigma$ in the bundle $J^2(\xi, \eta)$ over $M^4$ is a manifold with singularities. By dimensional reasonings, the image of a general position section $j : M^4 \to J^2(\xi, \eta)$ does not contain singular points of the manifold with singularities $\Sigma$. Consequently, the singular set $j^{-1}(\Sigma)$ of the section $j$ is a submanifold of $M^4$. We define the normal Euler number $e(j)$ of the section $j$ as the normal Euler number $e(j^{-1}(\Sigma))$.

A regular neighborhood $E$ of $\Sigma$ in $J = J^2(\xi, \eta)$ is an open manifold. There is a system of local coefficients $\mathcal{F}$ over $E$, the restriction $\mathcal{F}|_\Sigma$ of which gives a $\mathbb{Z}$-orientation of $\Sigma$. The Poincaré homomorphism for cohomology and homology with twisted coefficients takes the fundamental class $[\Sigma]$ onto some class $\tau \in H^2(E, E \setminus \Sigma; \mathcal{F})$. Note that $\tau \cup \tau$ is in $H^4(E, E \setminus \Sigma; \mathbb{Z})$. Let $i$ be the composition

$$H^4(E, E \setminus \Sigma; \mathbb{Z}) \to H^4(J, J \setminus \Sigma; \mathbb{Z}) \to H^4(J, \emptyset; \mathbb{Z})$$

of the excision isomorphism and the homomorphism induced by the inclusion. We define $h(\xi, \eta) = i(\tau \cup \tau)$. Then we claim that

$$e(j) = (j^*h(\xi, \eta), [M^4]).$$

**Lemma 24** For every general position section $j : M^4 \to J^2(\xi, \eta)$, the normal Euler class of the surface $j^{-1}(\Sigma)$ is given by (9). In particular, for every mapping $f : M^4 \to N^3$, we have $e(f) = ((j^2 f)^* h(TM^4, f^*TN^3), [M^4])$.

**Proof.** Let $A \subset M^4$ denote the singular set $j^{-1}(\Sigma)$ and $B$ denote a tubular neighborhood of $A$. The tubular neighborhoods $E$ of $\Sigma$ and $B$ of $A$ may be viewed as vector bundles. Since $j : B \to E$ is transversal to $\Sigma$, there is a
commutative diagram of vector bundles

\[
\begin{array}{ccc}
B & \longrightarrow & E \\
\downarrow & & \downarrow \\
A & \longrightarrow & \Sigma,
\end{array}
\]

from which it follows that the Thom class of the bundle \(B \to A\) is \(j^*(\tau)\). We have a commutative diagram

\[
\begin{array}{ccc}
& i : H^4(E, E \setminus \Sigma; \mathbb{Z}) & \longrightarrow & H^4(J, J \setminus \Sigma; \mathbb{Z}) & \longrightarrow & H^4(J; \mathbb{Z}) \\
j^* & j^* & j^* & j^* & j^* & j^* \\
H^4(B, B \setminus A; \mathbb{Z}) & \longrightarrow & H^4(M^4, M^4 \setminus A; \mathbb{Z}) & \longrightarrow & H^4(M^4; \mathbb{Z}),
\end{array}
\]

which completes the proof. \(\square\)

Lemma 24 shows that the number \(e(j)\) depends only on the bundles \(\xi\) and \(\eta\). That is why we will denote this number by \(e(\xi, \eta)\).

In the following, for an arbitrary manifold \(V\), we denote the trivial line bundle over \(V\) by \(\tau(V)\) or simply by \(\tau\).

**Lemma 25** There is an integer \(k \neq 0\) such that for any orientable 4-vector bundle \(\xi\) over any closed oriented 4-manifold \(M^4\), the equality \((p_1(\xi), [M^4]) = ke(\xi, 3\tau)\) holds.

**PROOF.** We recall (see section 4) that the bundle \(J^2(\xi, 3\tau)\) over \(M^4\) is induced by an appropriate mapping \(\mu : M^4 \to BSO_4 \times BSO_3\) from some bundle \(J^2(E_4, E_3)\) over \(BSO_4 \times BSO_3\). As above we define a cohomology class \(h(E_4, E_3) \in H^4(J^2(E_4, E_3); \mathbb{Z})\). Let \(\alpha\) be an arbitrary section of the bundle \(J^2(E_4, E_3)\). Together with \(\mu\), the section \(\alpha\) defines a section \(j : M^4 \to J^2(\xi, 3\tau)\) such that the diagram

\[
\begin{array}{ccc}
J^2(\xi, 3\tau) & \longrightarrow & J^2(E_4, E_3) \\
\downarrow j & & \downarrow \alpha \\
M^4 & \longrightarrow & BSO_4 \times BSO_3
\end{array}
\]

commutes. We have

\[
j^* h(\xi, 3\tau) = j^* \tilde{\mu}^* h(E_4, E_3) = \mu^* \alpha^* h(E_4, E_3),
\]

where \(\tilde{\mu}\) denotes the upper horizontal homomorphism of the diagram. Consequently, the class \(j^* h(\xi, 3\tau)\) is induced by \(\mu\) from some class \(\alpha^* h(E_4, E_3)\) in
There is a cohomology class \( s \) afforded by the trivial line bundle \( 2 \mathbb{Z} \). Hence \((j^* h(\xi, 3\tau), [S^4]) = e(f) = 0.\) Since \( p_1(TS^4) = 0 \) and \( W_4(TS^4) = 2 \), we conclude that \( l = 0 \).

Finally, \( k \neq 0 \) follows from \( p_1(\xi) \neq 0 \) for some \( \xi \). \( \square \)

To find the number \( k \) of Lemma 25 we need another description of the invariant \( e(j) \). Let \( \xi, \eta \) and \( \zeta \) be vector bundles over a manifold \( M^4 \). There are natural projections \( p_r : (\xi \oplus \zeta)^r \rightarrow \xi^r \) and inclusion \( i : \eta \rightarrow \eta \oplus \zeta \). A point of \( J^n(\xi, \eta) \) is a set of \( n \) homomorphisms \( \{g_i\}_{i=1, \ldots, n} \) (see section 4). Define the embedding

\[ s_n : J^n(\xi, \eta) \rightarrow J^n(\xi \oplus \zeta, \eta \oplus \zeta) \]

by

\[ s_n(g_1, \ldots, g_k) = (g_1 \oplus id, i \circ g_2 \circ p_2, \ldots, i \circ g_n \circ p_n). \]

The homomorphism \( s_n, n \geq 1 \), is called the stabilization homomorphism afforded by \( \zeta \).

**Lemma 26** *(Ronga [19])*  

(1) \( s_2^{-1}(\Sigma_i^2(\xi \oplus \zeta, \eta \oplus \zeta)) = \Sigma_i^2(\xi, \eta) \),  
(2) \( s_2^{-1}(\Sigma_{i,j}^2(\xi \oplus \zeta, \eta \oplus \zeta)) = \Sigma_{i,j}^2(\xi, \eta) \), and  
(3) the embedding \( s_2 \) is transversal to the submanifolds \( \Sigma_i^2(\xi \oplus \zeta, \eta \oplus \zeta) \) and \( \Sigma_{i,j}^2(\xi \oplus \zeta, \eta \oplus \zeta) \).

Let \( \xi \) be an orientable 4-vector bundle, \( \eta \) the trivial 3-vector bundle over \( M^4 \), and \( s_2 \) the stabilization homomorphism afforded by the trivial line bundle \( \tau \) over \( M^4 \). Lemma 26 allows us to give a definition of \( e(j) \) in terms of some cohomology class of \( H^4(J^2(\xi \oplus \tau, \eta \oplus \tau); \mathbb{Z}) \).

**Lemma 27** There is a cohomology class \( h \in H^4(J^2(\xi \oplus \tau, \eta \oplus \tau); \mathbb{Z}) \) such that for a section \( j : M^4 \rightarrow J^2(\xi, \eta) \), we have \( e(j) = ((s_2 \circ j)^*(h), [M^4]) \).

The proof of Lemma 27 is the same as that of Lemma 24.
Let $\xi$ and $\eta$ be the vector bundles of dimensions 4 and 3 respectively over the standard 4-disc $D^4$. The set of regular points in $J^2(\xi \oplus \tau, \eta \oplus \tau)$ is homotopy equivalent to $SO_5$. Therefore, each section $j : S^3 \to J^2(\xi \oplus \tau, \eta \oplus \tau)$ that sends $S^3$ into the set of regular points defines an element $\tilde{j}$ in the set of homotopy classes $[S^3, SO_5]$. The space $SO_5$ is an $H$-space; hence $\tilde{j}$ is an element of $\pi_3(SO_5) = \mathbb{Z}$. Since $J^2(\xi \oplus \tau, \eta \oplus \tau)$ is contractible, the section $j$ admits an extension to a section over $D^4$ transversal to the singular set of $J^2(\xi \oplus \tau, \eta \oplus \tau)$. We obtain a mapping $e : \pi_3(SO_5) \to \mathbb{Z}$ that sends the homotopy class $j$ of a section $j$ to the normal Euler number of the singular set of the section $j$ extended over $D^4$.

**Lemma 28** The mapping $e : \pi_3(SO_5) \to \mathbb{Z}$ is a well defined homomorphism.

**PROOF.** Let $j_1$ and $j_2$ be two sections of the bundle $J^2(\xi \oplus \tau, \eta \oplus \tau)$ over $D^4$ whose restrictions $j_1|_{\partial D^4}$ and $j_2|_{\partial D^4}$ map $S^3 = \partial D^4$ into the set of regular points of $J^2(\xi \oplus \tau, \eta \oplus \tau)$ and represent the same homotopy class $\tilde{j} \in [S^3, SO_5]$. The arguments similar to those in the proof of Lemma 9 show that the normal Euler numbers of the submanifolds $j_1^{-1}(\Sigma)$ and $j_2^{-1}(\Sigma)$, where $\Sigma$ is the singular set of $J^2(\xi \oplus \tau, \eta \oplus \tau)$, are equal. Therefore, the number $e(\tilde{j})$ does not depend on the choice of representative of the homotopy class $\tilde{j}$.

We need to verify that the equality

$$e(\tilde{j}_1 + \tilde{j}_2) = e(\tilde{j}_1) + e(\tilde{j}_2) \tag{11}$$

holds for every pair of elements $\tilde{j}_1, \tilde{j}_2$ of $\pi_3(SO_5)$.

For $i = 1, 2$, let $j_i : \partial D^4_i \to J^2(\xi \oplus \tau, \eta \oplus \tau)$ be a section that leads to $\tilde{j}_i$. We can modify $j_1$ by homotopy that does not intersect the singular set of $J^2(\xi \oplus \tau, \eta \oplus \tau)$ so that the sections $j_1$ and $j_2$ agree on some non-empty open subset of $\partial D^4_1 = \partial D^4_2$. Then $j_1$ and $j_2$ determine a section $j_3 : \partial D^4_1 \# \partial D^4_2 \to J^2(\xi \oplus \tau, \eta \oplus \tau)$, which leads to an element $\tilde{j}_1 + \tilde{j}_2$ of $\pi_3(SO_5)$. Extensions of $j_1$ and $j_2$ to $D^4_1$ and $D^4_2$ respectively give rise to an extension of $j_3$ to $D^4_1 D^4_2$ the singular set of which is the union of the singular sets of the extensions of $j_1$ and $j_2$. Therefore, (11) holds. \(\square\)

We have defined the normal Euler number of a general position section of the bundle $J^2(\xi, \eta)$ in the case where $\xi$ is an orientable 4-vector bundle and $\eta$ is an orientable 3-vector bundle. If $\xi$ is an orientable 5-vector bundle and $\eta$ is an orientable 4-vector bundle, then again the singular set of a general position section $j$ of $J^2(\xi, \eta)$ is a 2-submanifold of $M^4$ and therefore we can define the normal Euler number $e(j)$ and the number $e(\xi, \eta)$ in the same way as above.
Let $g : S^4 \to SO_5$ be a generator of $\pi_3(SO_5)$ and $\vartheta$ the number $e(g)$. Let us calculate $e(\delta \oplus \tau, 4\tau)$, where $\delta$ is the 4-vector bundle associated with the Hopf fibration $S^7 \to S^4$.

**Lemma 29** $e(\delta, 3\tau) = e(\delta \oplus \tau, 4\tau) = \vartheta$, up to sign.

**PROOF.** The sphere $S^4$ is a union of two discs $D_1$ and $D_2$ with $\partial D_1 = \partial D_2$. A choice of trivializations of $\delta$ over $D_1$ and $D_2$ defines a gluing homomorphism

$$\alpha : \delta \oplus \tau|_{\partial D_1} \longrightarrow \delta \oplus \tau|_{\partial D_2},$$

which being identified with a mapping $S^3 \to SO_5$ represents a generator $[\alpha] \in \pi_3(SO_5)$.

Let $J^1$ and $J^0$ respectively denote the space $J^1(\delta \oplus \tau, 4\tau)$ and the complement to the singular set $\Sigma$ in $J^1$. To prove Lemma 29, it suffices to determine the normal Euler number of $j^{-1}(\Sigma)$ for a particular section $j : S^4 \to J^1$. We regard a section of $J^1$ as a bundle homomorphism $\delta \oplus \tau \to 4\tau$. If $j$ is given over $D_1$, then the diagram

$$
\begin{array}{ccc}
\delta \oplus \tau|_{\partial D_1} & \longrightarrow & \delta \oplus \tau|_{\partial D_2} \\
\downarrow j|_{\partial D_1} & & \downarrow \text{id}(j|_{\partial D_1}) \circ \alpha^{-1} \\
4\tau|_{\partial D_1} & \longrightarrow & 4\tau|_{\partial D_2}
\end{array}
$$

shows that in the trivialization of $\delta$ over $D_2$ the section $j|_{\partial D_2}$ is $\text{id}(j|_{\partial D_1}) \circ \alpha^{-1}$, where $\text{id}$ is the identity mapping. If we choose $j$ to be constant over $D_1$, then in the trivialization of $\delta$ over $D_2$ the section $j|_{\partial D_2}$ induces a mapping $S^3 \to J^0 \approx SO_5$ representing the homotopy class $-[\alpha] \in \pi_3(SO_5)$. Thus the normal Euler number of $j$ extended over $D_2$ is $\vartheta$ up to sign. □

**Lemma 30** There is an integer $q$ such that $e(T\mathbb{C}P^2, 3\tau) = 1 + q\vartheta$.

**PROOF.** There is a mapping $\tilde{f}$ of a regular neighborhood $E$ of $\mathbb{C}P^1 \subset \mathbb{C}P^2$ into $\mathbb{R}^3$ such that the singular set of $\tilde{f}$ is $\mathbb{C}P^1$ (see Lemma 19). Let $f$ be a general position extension of $\tilde{f}$ on $\mathbb{C}P^2$. The number $e(f)$ is the sum of the normal Euler number of $\mathbb{C}P^1$ and the normal Euler number of the surface of singular points that lies in the disc $D^4 = \mathbb{C}P^2 \setminus E$. The latter number is a multiple of $\vartheta$. Hence for some $q$, $e(f) = 1 + q\vartheta$. □

To calculate the exact value of $e(T\mathbb{C}P^2, 3\tau)$ we use the notion of the connected sum of two bundles.
For \( i = 1, 2 \), let \( M_i^4 \) be a closed oriented 4-manifold and \( \xi_i \) an orientable 4-vector bundle over \( M_i^4 \). Identifying the fiber of \( \xi_1 \) over some point in \( M_1^4 \) with the fiber of \( \xi_2 \) over some point in \( M_2^4 \), we obtain a bundle over \( M_1^4 \cup M_2^4 \), which is transferred to a bundle over \( M_1^4 \# M_2^4 \) by a natural mapping \( M_1^4 \# M_2^4 \to M_1^4 \cup M_2^4 \). We denote the resulting bundle over \( M_1^4 \# M_2^4 \) by \( \xi_1 \# \xi_2 \). It follows that the additivity properties

\[
(p_1(\xi_1 \# \xi_2), [M_1^4 \# M_2^4]) = (p_1(\xi_1), [M_1^4]) + (p_1(\xi_2), [M_2^4])
\]

and

\[
e(\xi_1 \# \xi_2, 3\tau(M_1^4 \# M_2^4)) = e(\xi_1, 3\tau(M_1^4)) + e(\xi_2, 3\tau(M_2^4))
\]

take place.

**Lemma 31** \( e(T\mathbb{C}P^2, 3\tau) = 3 \).

**PROOF.** Let \( \delta \) be the 4-vector bundle over \( S^4 \) with \( p_1(\delta) = 2 \). Lemma 29 implies that \( e(\delta, 3\tau) = \pm \vartheta \). For \( K = 2\mathbb{C}P^2 \), we have \( p_1(TK \# \delta^{#3}) = 0 \), where \( \delta^{#3} \) stands for \( \delta \# \delta \# \delta \). Lemma 25 shows that \( e(TK \# \delta^{#3}, 3\tau) \) is also zero. By additivity,

\[
0 = e(TK \# \delta^{#3}, 3\tau) = -2(1 + q\vartheta) \pm 3\vartheta.
\]

Since \( q \) and \( \vartheta \) are integers, we conclude that \( e(TK, 3\tau) = \pm 6 \), which implies \( e(T\mathbb{C}P^2, 3\tau) = \pm 3 \). On the other hand it is known [26] that \( e(T\mathbb{C}P^2, 3\tau) \equiv 3 \pmod{4} \). Therefore, \( e(T\mathbb{C}P^2, 3\tau) = 3 \). \( \square \)

Lemma 31 shows that the integer \( k \) in Lemma 25 equals 1. Thus, for every oriented 4-manifold \( M^4 \) and a general position mapping \( f : M^4 \to \mathbb{R}^3 \), we have \( e(f) = (p_1(M^4), [M^4]) \). This completes the proof of Lemma 23. \( \square \)

Theorem 2 is proved. \( \square \)

**9 Proof of Theorem 3**

As has been shown, a homotopy class of a general position mapping \( f \) from a connected closed oriented 4-manifold \( M^4 \) into an orientable 3-manifold \( N^3 \) has a fold mapping if and only if \( e(f) = (p_1(M^4), [M^4]) \in \mathcal{Q}(M^4) \). That is the number \( (p_1(M^4), [M^4]) \) is a value of the intersection form of \( M^4 \). First, let us consider the case where the intersection form of \( M^4 \) is indefinite. If \( p_1(M^4) = 0 \), then for every \( f \), \( e(f) = 0 \in \mathcal{Q}(M^4) \). Suppose \( p_1(M^4) \neq 0 \).

**Lemma 32** If the intersection form of a closed simply connected manifold \( M^4 \) with \( p_1(M^4) \neq 0 \) is indefinite odd, then \( \mathcal{Q}(M^4) \) contains every integer. In particular, \( (p_1(M^4), [M^4]) \in \mathcal{Q}(M^4) \).
PROOF. Since the intersection form of \( M^4 \) is odd, in \( H_2(M^4; \mathbb{Z}) \) there exists a basis \( g_1, g_2, \ldots, g_s, g_k \) such that the value of the intersection form at \( \alpha_1 e_1 + \cdots + \alpha_k e_k \) is \( \alpha_1^2 + \cdots + \alpha_s^2 - \alpha_{s+1}^2 - \cdots - \alpha_k^2 \). Therefore, the number \( e(f) \) is in \( Q(M^4) \) if and only if \( e(f) \) can be represented in the form \( \alpha_1^2 + \cdots + \alpha_s^2 - \alpha_{s+1}^2 - \cdots - \alpha_k^2 \) for some integers \( \alpha_i, i = 1, \ldots, k \). Since the intersection form is indefinite, this sum has at least one positive square and at least one negative square. Since the signature \( \sigma(M^4) = \frac{1}{2} p_1(M^4) \neq 0 \), the number \( k \) of squares is at least 3. Suppose that the number \( e(f) \) is odd. Then it can be represented as the difference of two squares. Suppose that \( e(f) \) is even. Then the odd number \( e(f) \pm 1 \) can be represented as the difference of two squares and the third square of the sum can be used to add \( \mp 1 \) to the difference to get \( e(f) \). Hence \( Q(M^4) = \mathbb{Z} \).

Suppose that the intersection form of \( M^4 \) is indefinite even. Being even, it is isomorphic to a direct sum of some copies of the forms \( \pm E_8 \) and some copies of the form with matrix \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]. Consequently, the number \( (p_1(M^4), [M^4]) = 3\sigma(M^4) \) is even. Every even indefinite intersection form contains a subform isomorphic to \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]. Since this subform takes every even value, we have \( (p_1(M^4), [M^4]) \in Q(M^4) \).

Thus, every closed simply connected 4-manifold with indefinite or trivial intersection form admits a fold mapping into \( \mathbb{R}^3 \).

To treat the case where the intersection form of \( M^4 \) is definite, we need the Donaldson Theorem. Let \( kJ, k \neq 0 \), denote the form of rank \( |k| \) given by the diagonal matrix with eigenvalues 1 if \( k > 0 \) and \(-1 \) if \( k < 0 \).

**Theorem 33 (Donaldson, [6])** If the intersection form of a closed oriented smooth 4-manifold is definite, then the form is isomorphic to \( kJ \) for some integer \( k \neq 0 \).

**Lemma 34** Suppose that the intersection form of a connected closed simply connected manifold \( M^4 \) with \( p_1(M^4) \neq 0 \) is definite. Then \( (p_1(M^4), [M^4]) \in Q(M^4) \) if and only if the intersection form is isomorphic to \( kJ, |k| \geq 3 \).

**PROOF.** It suffices to consider only the case where \( k > 0 \). If \( k = 1 \), then the intersection form is isomorphic to that of \( CP^2 \) and \( (p_1(M^4), [M^4]) = 3\sigma(M^4) \) is not in \( Q(M^4) \). For \( k = 2 \), the set \( Q(M^4) \) consists only of integers that can be represented as the sum of at most two squares. Hence the number \( (p_1(M^4), [M^4]) = 6 \) is not in \( Q(M^4) \). If \( k = 3 \), then \( (p_1(M^4), [M^4]) = 9 \in Q(M^4) \).
Finally, by the Lagrange theorem, every positive integer can be represented as a sum of four squares. Thus for \( k \geq 4 \), we have \((p_1(M^4), [M^4]) \in Q(M^4)\). □

In view of Theorem 1, Lemmas 32 and 34 imply that \( M^4 \) admits a fold mapping into \( \mathbb{R}^3 \) if and only if the intersection form of \( M^4 \) is different from \( \pm J \) and \( \pm 2J \). By the J. H. C. Whitehead Theorem about the oriented homotopy type of a simply connected 4-manifold (see [16], [30]), this completes the proof of Theorem 3. □

References


