Concepts: The First and Second Derivative

The 1st Derivative: What does \( f'(x) \) tell us?

a) Tells us about the increasing/decreasing behavior of \( f(x) \)

b) We can find local minima/maxima of \( f(x) \) with the first derivative.

1) Example: \( f(x) = x^3 - 9x^2 - 48x + 52 \), (this is the example done in class, and in the text)

The first derivative is: \( f'(x) = 3x^2 - 18x - 48 \)

Critical points: Set \( f'(x) = 0 \):

It follows that:

\[
3x^2 - 18x - 48 = 0 \\
3(x^2 - 6x - 16) = 0 \\
3(x-8)(x+2) = 0
\]

Therefore, \( x = -2 \) and \( x = 8 \) are our critical points.
(Note: You may not always be able to factor nicely as in this example. Make sure you review alternative methods of finding zeros of functions.)

How do critical points help us determine where \( f(x) \) is increasing or decreasing? This is where a sign chart is useful. You want to make a sign chart for the 1st derivative, so when you test your intervals, you are testing values in the 1st derivative!!

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\begin{array}{c|cccccccc}
\end{array}
\]

What does this sign chart tell us? It tells us that \( f(x) \) is increasing on \(( -\infty, -2 ) \cup (8, \infty)\), and \( f(x) \) is decreasing on \( ( -2, 8 )\).

In this example, -2 is a local max of \( f(x) \), and 8 is a local min of \( f(x) \). \( (\text{Why??}) \) Now, a question for you: Are critical points always points of local minima/maxima? The answer is NO. A simple example is \( x^3 \). This function has a critical point at 0, but we do not have a local max/min at \( x=0 \) on the graph of \( x^3 \).

In short, the 1st derivative is a good tool that helps us find local maxima/minima. It is very useful in word problems, such as maximizing profit functions, or minimizing average cost functions.

The 2nd Derivative: What does \( f''(x) \) tell us?

a) Tells us about the concavity of \( f(x) \)

b) We can find inflection points of \( f(x) \) with the second derivative.
1) Example: \( f(x) = x^3 - 9x^2 - 48x + 52 \), (this is the example done in class, and in the text)

The second derivative is: \( f''(x) = 6x - 18 \)

Now, find the zeros of the second derivative: Set \( f''(x) = 0 \). You will discover that \( x = 3 \) is a zero of the second derivative. Now, this \( x \)-value could possibly be an inflection point. How do we know? We use a sign chart for the 2nd derivative. When you test values in the intervals, you plug them into the 2nd derivative function!!

| \( f''(x) \) | \( - \) | \( + \) | \( + \) | \( + \) |
|----------------|-------|-------|-------|
| \( x \) | \( 3 \) |

What does this sign chart tell us? It tells us that \( f(x) \) is concave up on \( (3, \infty) \), and \( f(x) \) is concave down on \( (-\infty, 3) \). Since there is a sign change at 3, we have an inflection point there. When there is no such sign change, you cannot conclude that you have an inflection point!! (You should think of an example to convince yourself of this.)

Think about it: 1) Can you think of a function \( f(x) \) that is concave up for \( x > 3 \), concave down for \( x < 3 \), but with \( f''(x) > 0 \) everywhere? (E-mail me if you want the answer!)

2) A baseball is hit and its distance above the ground as a function of the time is given by the following: \( f(t) = -.78t^2 + 24t + 19 \), \( t \geq 0 \). What is the maximum height of the ball? Use calculus for this one! (E-mail me if you want the answer)