VECTOR BUNDLES.

1. THE DEFINITION OF A VECTOR BUNDLE

The definitions below can be applied to any geometric category that contains vector spaces (topological spaces, smooth varieties, algebraic varieties, etc). We need to be able to say what is a variety and a morphism of varieties, we need topology on our varieties, and we need direct product.

A vector bundle over a variety $X$ is a variety $Y$ with a map $p : Y \to X$, plus the structure of a vector space on every $p^{-1}(x)$ for $x \in X$.

We say that two vector bundles $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ are isomorphic, if there is an isomorphism $Y_1 \to Y_2$ that commutes with $p_1$ and $p_2$, and induces a linear map on every fiber: $p_1^{-1}(x) \to p_2^{-1}(x)$.

Define a trivial vector bundle as $X \times V$ with the direct product projection for some vector space $V$. We say that a vector bundle is locally trivial, if for every $x \in X$ there is an open neighborhood $x \in U \subset X$ such that $p : p^{-1}(U) \to U$ is isomorphic to a trivial vector bundle on $U$. On a connected component of $X$ all fibers of a locally trivial vector bundle have the same dimension, which is called the rank of this vector bundle. From now on, we will assume that all our vector bundles are locally trivial.

Example 1. Let $V = k^{n+1}$ and let $X = \mathbb{P}^n = \mathbb{P}(V)$. For $x \in X$ denote by $l_x \subset V$ the line in $V$ that corresponds to $x$. Then the set of pairs $(x,v)$ where $x \in X$ and $v \in l_x$ forms a line bundle on $X = \mathbb{P}^n$. This line bundle is called the tautological line bundle on $\mathbb{P}^n$. It is a subbundle of the trivial bundle $X \times V$.

Example 2. For a smooth variety $X$, the set of pairs $(x,v)$ with $x \in X$ and $v \in T_xX$ forms a vector bundle that is called the tangent bundle of $X$ and denoted by $TX$.

2. TRANSITION FUNCTIONS

Let $p : Y \to X$ be a vector bundle over $X$ with fiber $V$ of dimension $n$. Choose a covering $X = \bigcup U_i$ of $X$ by open sets with trivializations $\phi_i : U_i \times V \sim \to p^{-1}(U_i)$. Then on each intersection $U_{ij} = U_i \cap U_j$ we have a composition

$$U_{ij} \times V \xrightarrow{\phi_i} p^{-1}(U_{ij}) \xrightarrow{\phi_j^{-1}} U_{ij} \times V.$$  

Since we required trivializations to be linear maps on every fiber, these maps are also linear on every fiber. These compositions may be viewed either as $n \times n$ matrices whose entries are functions on $U_{ij}$, or as functions $g_{ij} : U_{ij} \to GL_n(k)$.

The functions $g_{ij}$ that are obtained in this manner from a vector bundle satisfy two conditions: first, $g_{ii} = Id$, then, $g_{jk}g_{ij} = g_{ik}$ on $U_i \cap U_j \cap U_k$ for all $i, j, k$. They are not defined uniquely by the vector bundle and the covering: we can choose different trivializations $\psi_i : U_i \times V \sim \to p^{-1}(U_i)$ and get a different system $f_{ij} = \psi_j \psi_i^{-1}$.
Let us find a set of transition functions for the tautological line bundle on $\mathbb{P}$ such that

\[
\bigcup_{i} U_i \times V \to U_i \times V \text{ is another function with values in } GL_n(k),
\]

This time on $U_i$. Then

\[
f_{ij} = \psi_j^{-1} \psi_i = A_j^{-1} \phi_j^{-1} \phi_i A_i = A_j^{-1} g_{ij} A_i.
\]

The above conditions are enough to define a vector bundle: whenever we have a covering $X = \bigcup U_i$ of $X$ by open subsets, and functions $g_{ij} : U_i \cap U_j \to GL_n(k)$, such that $g_{ii} = Id$ and $g_{jk}g_{ij} = g_{ik}$ on $U_i \cap U_j \cap U_k$ for all $i,j,k$, there is a vector bundle that produces $g_{ij}$ as transition functions. We can define this vector bundle by factorizing the union $\bigcup (U_i \times V)$ by an obvious equivalence relation.

3. Tautological line bundle on $\mathbb{P}^n$

(See the definition and the notation for the tautological bundle in Example 1). Let us find a set of transition functions for the tautological line bundle on $\mathbb{P}^n$. Let $\mathbb{P}^n = \mathbb{P}(V)$, where $V = k^{n+1}$ with coordinates $x_0, \ldots, x_n$. Let $U_i$ be the affine chart $x_i \neq 0$. Then over each $U_i$ the tautological line bundle is trivial: we can write a map $U_i \times k \to \{(x,v) | x \in U_i, v \in V_x\}$ by sending each $(x_0 : \ldots : x_n) \times c$ to $(x_0 : \ldots : x_n) \times c x_i$. In other words, we choose a basis of $k$ consisting of an element 1, and over each point $(x_0 : \ldots : x_n)$ we send this basis vector to the point on the corresponding line where $x_i = 1$. We can do that since for all points in $U_i$ we have $x_i \neq 0$. The map $\phi_i^{-1}$ then maps $((y_0 : \ldots : y_n),(x_0, \ldots, x_n))$ to $(y_0 : \ldots : y_n) \times x_i$. (here $(x_0 : \ldots : x_n) = (y_0 : \ldots : y_n)$ when $(x_0 : \ldots : x_n)$ is defined; the notation with $y$’s is only employed to treat the fact that $(x_0, \ldots, x_n)$ might be the origin).

Using these trivializations, we find that $g_{ij}(c) = \frac{x_j}{x_i}$, so $g_{ij} = \frac{x_j}{x_i} \in GL_1(k) \cong k$.

**Definition 1.** We denote by $\mathcal{O}(k)$ the line bundle on $\mathbb{P}^n$ that is trivial on each affine chart $U_i$ and whose transition functions are $g_{ij} = \left(\frac{x_i}{x_j}\right)^k$. Thus, the tautological line bundle is (isomorphic to) $\mathcal{O}(-1)$.

4. Operations on vector bundles

For two vector bundles $E$ and $F$ over $X$, we can define a dual vector bundle $E^\vee$, a direct sum $E \oplus F$, a tensor product $E \otimes F$, and the bundle of homomorphisms $\text{Hom}(E,F)$ by performing the constructions for every fiber over every $x \in X$. Another way to describe this is to find an open covering such that both $E$ and $F$ are trivial over each open subset, perform the operations over each set, and introduce transition functions using the transition functions for $E$ and $F$.

5. Sections of vector bundles

Let $E \xrightarrow{f} X$ be a vector bundle. A regular section of this vector bundle is a map $X \xrightarrow{s} E$ such that $f \circ s = \text{Id}_X$. In other words, for every $x \in X$ we choose a point $s(x) \in f^{-1}(x)$. Then for every open $U \subset X$ such that $E|_U$ is trivial, once we choose a trivialization $\phi : U \times V \cong f^{-1}(U)$, the section $s$ is given by regular functions $\phi^{-1}(s(x))$. A rational section is a section described likewise by rational functions: we can view it either as a rational function $X \to E$ such that $f(s(x)) = x$ for all $x$ for which $s$ is well-defined, or as a set of maps $s_i : U_i \to V$ with rational components and
the condition \(g_{ij}s_i = s_j\) for some open cover \(X = \bigcup U_i\) with trivializations over each \(U_i\) and transition functions \(g_{ij}\). We can also define sections of a vector bundle over each open subset \(U \subset X\). The set of all regular sections of \(E\) over \(U\) is denoted by \(\Gamma(U, E)\). The set of global sections of \(E\) is also denoted simply by \(\Gamma(E) = \Gamma(X, E)\).

We will often omit the word "regular" when talking about regular sections.

**Example 3.** The line bundle \(\mathcal{O}_X\), or simply \(\mathcal{O}\), over \(X\) is the trivial line bundle \(X \times \mathbb{k}\). Its sections over \(U \subset X\) are the regular functions over \(U\). They form the ring \(\mathbb{k}[U]\).

Note that the set \(\Gamma(U, E)\) is in bijection with the set of morphisms \(\mathcal{O}_U \to E|_U\): for every \(s \in \Gamma(U, E)\) we can map \((x, \lambda) \mapsto \lambda s(x) \in f^{-1}(x)\), where \(x \in U\), \(\lambda \in \mathbb{k}\).

Similarly, for every \(\Phi : \mathcal{O}_U \to E|_U\) we can define \(s(x) = \Phi(x, 1)\) and get a section of \(E\).

**Example 4.** Let \(X = \mathbb{P}^2\) with coordinates \((x : y)\), and let \(U_x = \{x \neq 0\}\) and \(U_y = \{y \neq 0\}\). The coordinate on \(U_x\) is \(y/x\), and the coordinate on \(U_y\) is \(x/y\). The line bundle \(\mathcal{O}(1)\) has transition functions \(g_{xy} = x/y\) and \(g_{yx} = y/x = (g_{xy})^{-1}\). Then a section \(s\) is a pair of functions \(s_x\) on \(U_x\) and \(s_y\) on \(U_y\) such that \(g_{xy}s_x = s_y\). One example of such a section will be \(s_x = 1, s_y = x/y\).

6. **The Definition of a Sheaf**

Let \(X\) be a topological space. A presheaf \(\mathcal{F}\) of abelian groups on \(X\) is the data of an abelian group \(\mathcal{F}(U)\) for every open subset \(U \subset X\), and a morphism \(\mathcal{F}(U) \to \mathcal{F}(V)\) whenever \(V \subset U\), called the restriction morphism. Elements of the group \(\mathcal{F}(U)\) are called sections of \(\mathcal{F}\) over \(U\). A presheaf \(\mathcal{F}\) is a sheaf, if it satisfies the following property:

- For any collection \((U_i, f_i \in \mathcal{F}(U_i))\) such that for any \(i, j\) the sections \(f_i\) and \(f_j\) coincide when restricted to \(U_i \cap U_j\), there is a unique section \(f \in \mathcal{F}(\bigcup U_i)\) such that \(f_i = f|_{U_i}\).

One can also consider a presheaf, or a sheaf, of sets, rings, vector spaces, modules (over a ring, or a sheaf of rings or algebras) etc. We can also denote the set \(\mathcal{F}(U)\) by \(\Gamma(U, \mathcal{F})\), similar to the sections of a vector bundle.

Examples:

1. The constant sheaf \(\mathbb{k}_X\) has \(\mathbb{k}\) for sections over every open subset, with identity restriction maps. One can use any group, or object, instead of \(\mathbb{k}\).
2. The structure sheaf \(\mathcal{O}_X\) is a sheaf of functions on \(X\). For an algebraic \(X\), we take the sheaf of regular functions: for open \(U \subset X\), set \(\mathcal{O}_X(U) = \mathbb{k}[U]\). It is a sheaf of algebras.
3. Sections of any vector bundle \(E\) over \(X\) form a sheaf of modules over \(\mathcal{O}_X\): for every open \(U \subset X\) the set \(\Gamma(U, E)\) is a module over \(\mathcal{O}_X(U)\). This sheaf is locally free: every \(x \in X\) has a neighborhood \(U\) such that \(\Gamma(U, E)\) is a free \(\mathcal{O}_X(U)\)-module (a neighborhood over which \(E\) is trivial will work). The reverse is true: a locally free sheaf of modules is a sheaf of sections for some vector bundle.
4. The skyscraper sheaf of a point \(p \in X\) is the sheaf \(\mathcal{F}\) such that \(\mathcal{F}(U) = \mathbb{k}\) when \(x \in U\) and \(\mathcal{F}(U) = 0\) when \(x \notin U\).
7. LINE BUNDLES ON $\mathbb{P}^n$

For vector bundles over topological spaces with continuous transition functions, it is relatively easy to prove that any vector bundle over a contractible space such as $\mathbb{R}^n$ is trivial. In the algebraic setting however this question stood for twenty years, being known as the Serre’s Conjecture, before finally being proven by Quillen and by Suslin in the 70’s:

**Theorem 1.** Any vector bundle over the affine space $k^n$ is trivial.

Let $X$ be $\mathbb{P}^n$ with coordinates $(x_0 : \ldots : x_n)$, covered by affine charts $U_i \simeq k^n$. Let $L$ be a line bundle over $X$ that is trivial over each affine chart. The transition functions $g_{ij}$ must be rational functions that have neither zeroes nor poles over $U_i \cap U_j$, which is the set where $x_i \neq 0$ and $x_j \neq 0$. Therefore, $g_{ij}$ must be quotient of two monomials in $x_i, x_j$ of the same degree, i.e. a power of $x_i/x_j$. Then since $g_{ij}g_{jk} = g_{ik}$, the degree must be the same for all transition functions. Recall that we defined $O(k)$ on $\mathbb{P}^n$ as the line bundle with transition functions $g_{ij} = (x_i/x_j)^k$ for some $k \in \mathbb{Z}$.

**Proposition 1.** Every line bundle on $\mathbb{P}^n$ is isomorphic to $O(k)$ for some $k$.

To describe all (global) regular sections of $O(k)$, observe that if a section is given by a polynomial $P \left( \frac{z_1}{x_0}, \ldots, \frac{z_n}{x_0} \right)$ in the chart $U_0$ where $x_0 \neq 0$, then it is given by $\left( \frac{z_0}{x_0} \right)^k P \left( \frac{z_1}{x_0}, \ldots, \frac{z_n}{x_0} \right)$ in the chart $U_i$. In order for that to be a regular function of $\frac{z_0}{x_0}, \ldots, \frac{z_n}{x_0}$ we need $P$ to be a homogeneous polynomial of degree $k$. This means that for $k < 0$ there are no global regular sections, and for $k \geq 0$ the space of global regular sections is isomorphic to the space of all homogeneous polynomials of degree $k$. In particular, for $k = 0$ the space of all regular sections of the structure sheaf $O$ is one-dimensional and consists only of constant functions. For each homogeneous polynomial $P(x_0, \ldots, x_n)$ of degree $k$ the corresponding section of $O(k)$ is $\frac{P}{x_i^k}$ on $U_i$. 