1. Problems in Elementary Mathematics.

Below TF means: true or false?

1. There exist infinitely many natural numbers $n$ which cannot be written as $n = n_1^3 + n_2^3 + n_3^3$ where $n_1, n_2$, and $n_3$ are natural numbers. TF

2. No prime number can be written as a sum of two squares of positive integers in two distinct ways. TF

3. The remainder when a prime $p$ is divided by 30 is also a prime (or 1). TF

4. The equation $15x^2 - 7y^2 = 9$ is not solvable in integers. TF

5. Any natural number $n$ can be written as $n = \sum_{i=1}^{5} n_i^3$ where the $n_i$ are integers. TF

2. Problems in Elementary Physics.

1. A ball of radius $r$ is falling from the edge of a table of height $h$ where $r << h$. At what distance $x$ from the table does the ball hit the ground?

2. A point weight $m$ is falling from a horizontal position. The weight is attached by a string to a fixed point 0. At what angle $\alpha$ is the vertical component $v$ of the velocity of $m$ maximal?

3. Why do frankfurters split longitudinally (and not transversally) while boiling?

4. A cobra of length $l$ and mass $m$ is rising vertically with velocity $v$ getting ready to strike. What is the pressure the cobra applies to the earth?

5. You can open a door by applying pressure with your finger, but you cannot open it by shooting a bullet through it. Why?

6. Two cars have power $P_1$ and $P_2$ and velocities $v_1$ and $v_2$, respectively, obtained by applying these powers. What will their velocity be if you join the cars by a cable?
7. The point of suspension of a hard pendulum of length \( l \) moves horizontally so that its displacement is given by \( x = a \cos(\omega t) \). For small oscillations, find the amplitude and phase of these oscillations.

3. Problems in Undergraduate Mathematics.

1. A car starts at a point \( A \) and stops at a point \( B \). Let \( d = \text{dist}(A, B) \) and let \( T \) be the time for the trip. Is it true that at some point of time the acceleration \( a \) of the car satisfies \(|a| \geq \frac{4d}{T^2}\)?

2. What is the maximal number of zeros that \( f(x) = \sum_{k=1}^{n} a_k e^{b_k x} \) can have on \((-\infty, \infty)\)? Here \( a_k \) and \( b_k \) are real numbers and \( b_k \neq b_j \) if \( k \neq j \).

3. Compute \( \lim_{x \to +\infty} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^x} \).

4. What is the maximal number of real roots of the equation \( x^5 + ax^4 + bx^3 + c = 0 \), where \( c \neq 0 \) and \( a, b, \) and \( c \), are real numbers.

5. Let \( A \) be an \( n \times n \) matrix, \( I \) be the unit matrix, and suppose that \((I + A)^m = 0\) for some \( m \). Then \( A \) is invertible. TF

6. Is there a polynomial \( p(x) \) such that \( p(\sin x) = \cos x \) for \( a \leq x \leq b \), where \( b > a \)?

7. Let \( f(x) = \sum_{i=1}^{n} a_i x^i \) for \( x > 0 \) where \( a_i \) and \( b_i \) are real numbers satisfying \( b_i \neq b_j \) if \( i \neq j \), \( a_i > 0 \) for \( 1 \leq i \leq n - 1 \), and \( a_n < 0 \). What is the maximal number of positive roots of \( f(x) \)?

8. Let \( f(x, y) \) and \( \frac{\partial f(x, y)}{\partial y} \) be continuous on \( R^2 \), \( f_y' > 0 \) on \( R^2 \), and \( f(x + T, y) = f(x, y) \) on \( R^2 \). What is the maximal number of \( T \)-periodic solutions to the equation \( y' = f(x, y) \)?

9. If \( f(a) = f(b) = 0 \) and \( f' \) is continuous on \([a, b]\), then \( \max_{a \leq x \leq b} |f'| \geq \frac{4}{(b-a)^2} \int_{a}^{b} |f|dx \). TF

10. Let \( n \) be a natural number and let \( \int_{a}^{b} x^m f(x)dx = 0 \) where \( 0 \leq m \leq n \). If \( f \) is continuous, then \( f(x) \) has at least \( N \) zeros on \([a, b]\). Find \( N \).

11. Let \( S_N = \sum_{n=1}^{N} \frac{\sin n}{n} = S_N^+ + S_N^- \), where \( S_N^+(S_N^-) \) is the sum of the positive (negative) terms. Find \( \lim_{N \to \infty} S_N^+ \).

12. Let \( f \in C^{\infty}(-1, 1) \). Suppose that \( f^{(n)}(0) \neq 0 \) \( \forall n \), and \( f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n)}(\theta)}{n!} x^n \) for \( 0 < \theta < 1 \). Find \( \lim_{x \to 0} \theta \).

13. If \( f \in C^1(R^n) \cap L^1(R^n) \), then \( \sup_{x \in R^n} |f| \leq c_n \sup_{x \in R^n} |\nabla f|^{\frac{n}{n+1}} (\int_{R^n} |f|dx)^{\frac{1}{n+1}} \). TF
If true, estimate \( c_n \).
14. Calculate $\int_0^1 \ln(1+x) \frac{1}{1+x^2} \, dx$.
15. Let $a_n > 0$, $a_n \geq a_{n+1}$, and $\sum_{n=1}^{\infty} a_n < \infty$.
   TF: Then $\lim_{n \to \infty} na_n = 0$.
16. Let $a_n > 0$, $\sum_{n=1}^{\infty} a_n < \infty$.
   TF: Then there exists a $b_n > 0$, such that $b_n \leq b_{n+1}$, $\lim_{n \to \infty} b_n = \infty$, and $\sum_{n=1}^{\infty} a_n b_n < \infty$.
17. If $f$ is continuous on $[0, \infty)$ and $\int_0^\infty \frac{f(x)}{x} \, dx < \infty$, then $\int_0^\infty \frac{f(ax)-f(bx)}{x} \, dx = f(0) \ln \frac{b}{a}$, $a > 0, b > 0$.
18. Let $a > b > 1$. Which number is greater $a^b$ or $b^a$?
19. Let $f(x, y)$ be continuously differentiable function on the plane, and $f_x + f_y = 0$.
   Then $f = \text{const}$. TF


1. If $\sum_{n=1}^{\infty} a_n < \infty$, then what can be said about the convergence of $\sum_{n=1}^{\infty} a_n^{2k+1}$, $k = 1, 2, \ldots$?
2. $\int_0^1 x^{-x} \, dx = \sum_{n=1}^{\infty} n^{-n}$. TF
3. $\sum_{n=1}^{\infty} \frac{\cos n}{1+n^2} =$?
4. $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n!} =$?
5. Compute the flux of the vector field $\nabla \phi \times \nabla \psi$ through the sphere $r = \sqrt{x_1^2 + x_2^2 + x_3^2} = 1$, where, $\phi$ and $\psi$ are $C^2$ functions in the region $|x| \leq 1$.
6. Let $a > 1$ and $P_{n+1} \geq P_n > 0$ for each $n$. Does $\sum_{n=1}^{\infty} \frac{P_{n+1}-P_n}{P_{n+1}P_n}$ converge? What can be said about convergence of this series if $0 < a \leq 1$?
7. Let $f \in C[0, 1]$ and $f > 0$. Find $\lim_{n \to +\infty} \left( \int_0^1 \frac{f(x)}{1+x^2} \, dx \right)^n$.
8. Let $J = \int_\infty^{-\infty} f(x) \, dx < \infty$. Find $\int_{-\infty}^{\infty} f \left( x - \frac{1}{x} \right) \, dx$.
9. If $f(x)$ is uniformly continuous on $(0, \infty)$, then $\lim_{x \to 0} f(x)$ and $\lim_{x \to +\infty} f(x)$ exist. TF
10. Let $f \in C^1([0, 1])$ where $f(0) = 0$. Suppose that there is a positive constant $c$ such that $|f'(x)| \leq c|f(x)|$ for each $x$. Then $f = 0$. TF
11. Calculate $\frac{d^{100}}{dx^{100}} \left( \frac{1}{x^2+3x^2+2} \right) |_{x=0}$. 

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12. Let \( P(x, y) > 0 \) be a polynomial, \( x, y \in \mathbb{R}^1 \). Is its infimum necessarily attained at a point of \( \mathbb{R}^2 \)?

13. Does the integral \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx dy}{1+x^2+y^2} \) converge?

14. Calculate \( \int_{-\infty}^{\infty} \frac{dx}{(x^4+4x+4)^{1000}} \) with a relative error 10% analytically.

15. Solve:
   a) \( \ddot{x} = x^5 + x^2 \dot{x} \).
   b) \( y' = (2x - y^2)^{-1} \).
   c) \( \ddot{y} - ty - y = 0 \).
   d) Does the problem \( y' = y^2 + x, y(0) = 0 \) have the solution \( y \in C^1 \) on the interval \([0, 3]\)?
   e) Is every solution to \( y' = (1 + x^2 + y^2)^{-1} \) bounded on the whole \( x \)-axis?
   f) Can the function \( y = t^2 \sin(t) \) solve the equation \( \ddot{y} + a(t) \dot{y} + b(t) = 0 \) on the interval \((-1, 1)\), where \( a \) and \( b \) are continuous functions on \((-1, 1)\)?
   g) Find a continuous \( f \) from the equation \( f(x) = f(2x) \).
   h) \( [\sin(x)/x]^3 \geq \cos(x), 0 < x < \pi/2 \).

16. Calculate the first term of the asymptotics of \( \int_{-\infty}^{\infty} \frac{e^{ikx} dx}{\sqrt{1+x^2}} \) as \( k \to +\infty \).

17. \[
\begin{cases}
  u_{xx} + \lambda u = \sin x, & u = u(x), \quad 0 \leq x \leq \pi, \quad \lambda = \text{const} > 0, \\
  u(0) = u(\pi) = 0
\end{cases}
\]
How many solutions does this problem have?

18. If \( a_n > 0 \) for all \( n \), then \( \lim_{n \to \infty} \left( \frac{a_{1+n} + a_n}{a_n} \right)^n \geq e \). TF

19. If \( f \geq 0 \), then \( \int_0^{\infty} f dt \leq \sqrt{\pi} \left( \int_0^{\infty} f^2 dt \right)^{1/4} \left( \int_0^{\infty} t^2 f^2 dt \right)^{1/4} \). TF

20. Let \( f(x) \in C^1(0, \infty) \). Suppose that:
   i) there is a positive constant \( c \) such that \( |f'| \leq c(1 + x)^{-1} \) for \( x > 0 \), and
   ii) \( R^{-1} \int_0^R |f| dx \to 0 \) as \( R \to \infty \).
   Then \( f(x) \to 0 \) as \( x \to \infty \).
   TF
   Will the answer change if the assumption ii) is replaced by
   iii) \( R^{-1} \int_0^R f dx \to 0 \) as \( R \to \infty \)?

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21. $\|u\|_{L^2(R^3)} \leq c\|\text{grad} u\|_{L^2(R^3)}$ for any $u \in C^\infty_0(R^3)$ with $c > 0$ independent of $u$. TF. If true, give a possible value of $c$.

22. Calculate $\sum_{-\infty}^{\infty} (1 + j^2)^{-1}$.

23. Calculate $\int_{-\infty}^{\infty} e^{ikx}(e^x + e^{-x})^{-2} dx$.

24. Assume that the linear span $L$ of an orthonormal set $\{f_j\}_{j=1}^{\infty}$ in an infinite-dimensional Hilbert space $H$ is total in a linear subset $M$ of $H$, and $M$ is dense in $H$. Does it imply that $L$ is total in $H$? A set $\{f_j\}_{j=1}^{\infty}$ is total in $H$ if $(g, f_j) = 0$ for all $j$ implies $g = 0$, where $g \in H$ is an arbitrary element.

Consider the case $H = L^2(0, 1)$, $L = C(0, 1)$, where $C(0, 1)$ is the space of continuous functions on $[0, 1]$.

25. Assume that $A$ is a linear bounded operator in a Hilbert space $H$, defined on all of $H$, and $(Au, u) \geq 0$ for all $u \in H$. Is it true that $A$ is selfadjoint? Is it true that $||A|| = \sup_{||u||=1} (Au, u)$?

Suppose $A$ is a linear operator densely defined in $H$, and $(Au, u) \geq 0$ for all $u \in D(A)$, where $D(A)$ is the domain of definition of $A$. Assume that $\sup_{||u||=1, u \in D(A)} (Au, u) < \infty$.

Does it follow that $||A|| < \infty$?

26. Let $a > 0$, $b > 2$, and $az^3 - z + b = e^{-z}(z + 2)$. Does this equation have roots in the half-plane $\text{Re} z \geq 0$?

27. Assume that $f$ is a monotone continuous real-valued function on the interval $[0, 1]$. Can the function $F(z) := \int_0^1 f(x) \cos(xz) dx$ have complex (that is, not purely real-valued) roots?

28. Find asymptotics of the integral $\int_0^\pi e^{-t \sin x} \sin(2mx) dx$ as $t \to +\infty$. Here $m > 0$ is a given number.

29. Suppose that $A$ is a bounded linear operator defined on all of $H$, where $H$ is a Hilbert space, and the operator $A^*A$ is compact with eigenvalues $s_j^2$. What are the eigenvalues of $AA^*$? How are the corresponding eigenvectors of $A^*A$ and $AA^*$ related?

30. Let $A$ be a linear densely defined operator in $H$, where $H$ is a Hilbert space. Is it true that

a) $R(A)^\perp = N(A^*)$,
b) $N(A)^\perp = R(A^*)^\perp$?

Here $R(A)$ is the range of $A$, and $N(A)$ is the null-space of $A$. 

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