Wave scattering by many small particles and creating materials with a desired refraction coefficient

A.G. Ramm

Mathematics Department,
Kansas State University,
Manhattan, KS66506, USA
ramm@math.ksu.edu
www.math.ksu.edu/~ramm
Abstract

The novel points in this work are:
1) Asymptotic and numerical methods for solving wave scattering problem by many small bodies embedded in an inhomogeneous medium.
2) Derivation of the equation for the field in the limit \( a \to 0 \), where \( a \) is the characteristic size of the bodies (particles), and their number \( M = M(a) \) tends to infinity at a suitable rate. Multiple scattering is taken into account.
3) A recipe for creating materials with a desired refraction coefficient by embedding many small particles in a given material.
4) Discussion of possible applications:
   a) creating materials with negative refraction,
   b) creating materials with a desired radiation pattern.
5) A novel approach to homogenization theory.
Recipe for creating materials with a desired refraction coefficient:

**Step 1.** Given the original refraction coefficient $n_0^2(x)$ and the desired refraction coefficient $n^2(x)$, calculate $p(x)$ by formula

$$p(x) = k^2[n_0^2(x) - n^2(x)].$$

*This step is trivial.*
Step 2. Given \( p(x) = 4\pi h(x)N(x) \), calculate the functions \( h(x) \) and \( N(x) \). These functions satisfy the following restrictions:

\[
\text{Im} \ h(x) \leq 0, \quad N(x) \geq 0.
\]

This step is also trivial, and it has many solutions. For example, one can fix an arbitrary \( N(x) > 0 \), and then find \( h(x) = h_1(x) + ih_2(x) \), where \( h_1 = \text{Re} \ h, \ h_2 = \text{Im} \ h \), by the formulas

\[
h_1(x) = \frac{p_1(x)}{4\pi N(x)}, \quad h_2(x) = \frac{p_2(x)}{4\pi N(x)},
\]

where \( p_1 = \text{Re} \ p, \ p_2 = \text{Im} \ p \). The condition \( \text{Im} \ h \leq 0 \) holds if \( \text{Im} \ p \leq 0 \), i.e., \( \text{Im} \ [n_0^2(x) - n^2(x)] \leq 0. \)
Abstract

Scattering problem

Many-body scattering problem

Step 3. Prepare \( M = \frac{1}{a^{2-\kappa}} \int_D N(x)dx[1 + o(1)] \) small balls \( B_m(x_m, a) \) with the boundary impedances \( \zeta_m = \frac{h(x_m)}{a^\kappa}, \)

\( 0 \leq \kappa < 1, \) where the points \( x_m, 1 \leq m \leq M, \) are distributed in \( D \) according to formula \( N(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \)

\( \Delta \subset D \) is an arbitrary open subset. Embed in \( D \) \( M \) balls \( B_m(x_m, a) \) with boundary impedance \( \zeta_m, \) \( d = O(a^{(2-\kappa)/3}). \)

The material, obtained after the embedding of these \( M \) small balls will have the desired refraction coefficient \( n^2(x) \) with an error that tends to zero as \( a \to 0. \)

Step 3 is the only non-trivial step in this recipe from the practical point of view.
Technological problems

The first technological problem is:
How can one embed many, namely \( M = M(a) \), small balls in a given material so that the centers of the balls are points \( x_m \) distributed as desired?
The stereolitography process can be used.

The second technological problem is:
How does one prepare a ball \( B_m \) of small radius \( a \) with boundary impedance \( \zeta_m = \frac{h(x_m)}{a^\kappa} \), \( 0 \leq \kappa < 1 \) which has a desired frequency dependence?

Remark: It is not necessary to have large boundary impedance: if \( \kappa = 0 \), or \( \kappa = O(\frac{1}{|\ln a|}) \), then \( \zeta_m \) is bounded. However, if \( \kappa = 0 \), then \( M = O(a^{-2}) \), so more particles have to be embedded.
Scattering problem in the absence of embedded particles

\[ L_0u_0 := [\nabla^2 + k^2 n_0^2(x)]u_0 := [\nabla^2 + k^2 - q_0(x)]u_0 = 0 \text{ in } \mathbb{R}^3, \]
\[ u_0 = e^{ik\alpha \cdot x} + v_0, \quad \lim_{r \to \infty} r(u_r -iku) = 0. \]
\[ \text{Im } n_0^2(x) \geq 0, \quad \alpha \in S^2, \quad k = \text{const} > 0. \]
\[ L_0G = -\delta(x - y) \text{ in } \mathbb{R}^3. \]
\[ n_0^2(x) = 1 - k^{-2}q_0(x), \quad q_0(x) = k^2 - k^2 n_0^2(x), \quad \text{Im } q_0(x) \leq 0, \]
\[ n_0^2(x) = 1 \text{ in } D' := \mathbb{R}^3 \setminus D, \quad q_0(x) = 0 \text{ in } D'. \]
Many-body scattering problem

\[
\begin{aligned}
L_0 u_M &= 0 \text{ in } \Omega' := \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m; \\
\frac{\partial u_M}{\partial N} &= \zeta_m u_M \text{ on } S_m := \partial D_m, \\
u_M &= u_0 + v_M,
\end{aligned}
\]

where $N$ is the outer unit normal to $S_m$, and $h(x) \in C(D)$ is an arbitrary function, $h = h_1 + ih_2$, $h_2 \leq 0$, $\zeta_m$ is impedance, $d := \min_{m \neq j} \text{dist} (x_m, x_j)$. 

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Wave scattering
Basic assumptions

Let $n_0^2 := \max_{x \in \mathbb{R}^3} |n_0^2(x)|$. We assume that:

$$kn_0a \ll 1, \quad d \gg a,$$

$$N(\Delta) := \sum_{x_m \in \Delta} 1 = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(x) dx [1 + o(1)], \quad a \to 0. \quad (*)$$

Here $N(x)a^{-(2-\kappa)} \geq 0$ is the density of the distribution of particles, $d$ is the minimal distance between neighboring particles,

$$d = O(a^{(2-\kappa)/3}). \quad (**)$$

$$M = M(a) \sim O(a^{-(2-\kappa)}), \quad 0 \leq \kappa \leq 1.$$ 

Since $d^{-3} = O(M)$, relation (**) follows from (*).
Representation of the solution

\[ u_M(x) = u_0(x) + \sum_{m=1}^{M} \int_{S_m} G(x, t) \sigma_m(t) dt = \]

\[ = u_0(x) + \sum_{m=1}^{M} G(x, x_m) Q_m + \sum_{m=1}^{M} J_m. \]

\[ Q_m := \int_{S_m} \sigma_m(t) dt, \quad J_m := \int_{S_m} [G(x, t) - G(x, x_m)] \sigma_m(t) dt, \]

\[ I_m := |G(x, x_m) Q_m|. \]

Basic result:

\[ |J_m| \ll I_m, \quad a \to 0. \]
Impedance bc (boundary condition)

For the *impedance boundary condition* the limiting field $u$ solves the equation:

$$u(x) = u_0(x) - \int_D G(x, y)p(y)u(y)dy,$$

where

$$p(y) = 4\pi N(y)h(y),$$

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \quad 0 \leq \kappa < 1.$$

If the small bodies $D_m$ are of arbitrary shape, such that $|S_m| = ca^2$, then the factor $4\pi$ is replaced by the factor $c$. This factor may depend on $m$ if the small bodies are not identical.
Effective field 1

If \( |J_m| \ll I_m \), then, as \( a \to 0 \), one has

\[
u_M(x) = u_0(x) + \sum_{m=1}^{M} G(x, x_m) Q_m, \quad |x - x_m| \geq a.
\]

Define effective field acting on \( m \)-th particle:

\[
u_e := u_e^{(m)} := u_M(x) - \int_{S_m} G(x, t) \sigma_m(t) dt,
\]

If \( |x - x_m| \gg a \), then \( u_e \sim u_M \) as \( a \to 0 \). We prove below that

\[
Q_m \sim -4\pi u_e(x_m) h(x_m) a^{2-\kappa}.
\]
The equation for the effective field $u_e(x)$, as $a \to 0$, is

$$u_e(x) = u_0(x) - 4\pi \sum_{m=1}^{M} G(x, x_m) u_e(x_m) h_m a^{2-\kappa},$$

where $x \in \Omega', \quad \Omega' := \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m, \quad h_m := h(x_m)$.

Here $h_m$ are known, but $u_m := u_e(x_m)$ are unknown.

To calculate $u_m$ one can use a linear algebraic system (LAS):

$$u_j = u_{0j} - 4\pi a^{2-\kappa} \sum_{m=1, m \neq j}^{M} G(x_j, x_m) h_m u_m, \quad 1 \leq j \leq M.$$

**The order of this system is substantially reduced: see next slide.**
Reduction of the order of LAS.

To reduce the order $M$ of this system, consider a partition of $D$ into a union of small cubes $\Delta_p, 1 \leq p \leq P, P \ll M, y_p \in \Delta_p, \text{diam} \Delta_p \gg d$. Then a linear algebraic system (LAS) for $u_p$ is

\[
  u_q = u_{0q} - 4\pi \sum_{p \neq q}^P G(y_q, y_p) h(y_p) u_p N(y_p) |\Delta_p|,
\]

where $1 \leq q \leq P, P \ll M, u_q = u(y_q), u_{0q} = u_0(y_q)$. The LAS (*) is used for efficient numerical solution of many-body scattering problems when the scatterers are small.
How efficient can this reduction be?

Let the small particles be distributed in a cube with side $L = 10^{-1}\text{m}$, $a = 10^{-8}\text{m}$, $d = 10^{-6}\text{m}$. Then $M = \left(\frac{L}{d}\right)^3 = 10^{15}$.

Let the side $b$ of the partition cubes $\Delta_p$ be $b = a^{1/6} = 10^{-\frac{4}{3}}\text{m}$. Then $P = \left(\frac{L}{b}\right)^3 = 10$. The reduction of the order $M$ of the LAS in this example is from $10^{15}$ to $10$.

If $b = a^{1/4} = 10^{-2}\text{m}$, then $P = \left(\frac{L}{b}\right)^3 = 10^3$. In this case the reduction of the order $M$ of the LAS is from $10^{15}$ to $10^3$. 
Asymptotic formula for $Q_m$

\[ u_{eN} - \zeta_m u_e + \frac{A_m \sigma_m - \sigma_m}{2} - \zeta_m T_m \sigma_m = 0 \quad \text{on } S_m. \]

\[ A_m \sigma_m := 2 \int_{S_m} \frac{\partial G(s, t)}{\partial N_s} \sigma_m(t) dt, \quad T_m \sigma_m := \int_{S_m} G(s, t) \sigma_m(t) dt. \]

\[ G(x, y) = \frac{1}{4\pi |x - y|} [1 + O(|x - y|)], \quad |x - y| \to 0. \]

\[ \frac{4}{3} \pi a^3 \Delta u_e(x_m) - \zeta_m 4\pi a^2 u_e(x_m) = Q_m + \zeta_m \int_{S_m} ds \int_{S_m} \frac{\sigma_m(t) dt}{4\pi |s - t|}, \]

\[ \int_{S_m} A \sigma_m dt = - \int_{S_m} \sigma_m dt, \]
Abstract

Scattering problem

Many-body scattering problem

Derivation of the formula for \( Q_m \)

Formula for \( \sigma_m \)

\[
\int_{S_m} \frac{ds}{4\pi|s-t|} = a
\]

\[
\frac{4}{3} \pi a^3 \Delta u_e(x_m) - 4\pi \zeta_m u_e(x_m) a^2 = Q_m (1 + \zeta_m a).
\]

\[
Q_m = \frac{a^3 \left[ \frac{4\pi}{3} \Delta u_e(x_m) - 4\pi u_e(x_m) \zeta_m a^{-1} \right]}{1 + \zeta_m a}.
\]

If \( \zeta_m = \frac{h(x_m)}{a^\kappa} \), \( \kappa < 1 \), then

\[
Q_m \sim -4\pi u_e(x_m) h(x_m) a^{2-\kappa}.
\]
Asymptotic formula for $\sigma_m$

\[ u_M = u_e + \sigma_m \int_{S_m} \frac{dt}{4\pi|x-t|} = u_e + \frac{\sigma_m a^2}{|x|}, \quad |x - x_m| = O(a). \]

\[ u_e N - h(x_m) \frac{1}{a^\kappa} u_e - \sigma_m - \frac{h(x_m)}{a^\kappa} \sigma_m a = 0 \]

\[ \sigma_m = \frac{u_e N - h(x_m) u_e(x_m) a^{-\kappa}}{1 + h(x_m) a^{1-\kappa}} \]

If $\kappa < 1$, $a \to 0$, then $\sigma_m \sim -h(x_m) u_e(x_m) a^{-\kappa}$. 

A.G. Ramm

Wave scattering
Why is $I_m \gg |J_m|$?

\[
|G(x, x_m)Q_m| = I_m \sim \frac{a^{2-\kappa}}{d}, \quad d \sim a^\theta, \quad \theta \in (0, 1).
\]

\[
J_m \sim \max \left\{ \frac{a}{d} + ka \right\} \frac{a^{2-\kappa}}{d}, \quad ka \ll 1, \quad \frac{a}{d} \ll 1.
\]

$I_m \gg J_m$ if $0 < \theta < 1$.

Formula for calculating the field $u_M(x)$ is:

\[
u_M(x) = u_0(x) - 4\pi \sum_{m=1}^{M} G(x, x_m) h(x_m) u_e(x_m) a^{2-\kappa},
\]

where

\[
|x - x_m| \geq a^\theta, \quad d \geq O(a^\theta), \quad 0 < \theta < 1.
\]
Limiting procedure as $a \to 0$

\[
4\pi \sum_{m=1}^{M} G(x, x_m) h(x_m) u_e(x_m) a^{2-\kappa} = 4\pi \sum_{p=1}^{P} G(x, y^{(p)}) h(y^{(p)}) u_e(y^{(p)}).
\]

\[
a^{2-\kappa} \sum_{x_m \in \Delta_p} 1 = 4\pi \sum_{p=1}^{P} G(x, y^{(p)}) h(y^{(p)}) u_e(y^{(p)}) N(y^{(p)}) |\Delta_p|(1+\varepsilon_p)
\]

\[
\to \int_D G(x, y)p(y)u(y)dy, \quad p(y) := 4\pi h(y)N(y).
\]

\[
\mathcal{N}(\Delta_p) = a^{-(2-\kappa)} \int_{\Delta_p} N(x)dx[1 + o(1)], \quad a \to 0.
\]

\[
u(x) = u_0(x) - \int_D G(x, y)p(y)u(y)dy.
\]
An auxiliary lemma

**Lemma.** If \( f \in C(D) \) and \( x_m \) are distributed in \( D \) so that

\[
N(\triangle) = \frac{1}{\varphi(a)} \int_{\triangle} N(x) \, dx \left[1 + o(1)\right], \quad a \to 0,
\]

for any subdomain \( \triangle \subset D \), where \( \varphi(a) \geq 0 \) is a continuous, monotone, strictly growing function, \( \varphi(0) = 0 \), then

\[
\lim_{a \to 0} \sum_{x_m \in D} f(x_m) \varphi(a) = \int_D f(x) N(x) \, dx.
\]

**Remark:** *This lemma holds for bounded \( f \) with the set of discontinuities of Lebesgue’s measure zero. It can be generalized to a class of unbounded \( f \).*
Proof of the Lemma

Proof. Let \( D = \bigcup_p \Delta_p \) be a partition of \( D \) into a union of small cubes \( \Delta_p \), having no common interior points. Let \( |\Delta_p| \) denote the volume of \( \Delta_p \), \( \delta := \max_p \text{diam} \Delta_p \), and \( y^{(p)} \) be the center of the cube \( \Delta_p \). One has

\[
\lim_{a \to 0} \sum_{x_m \in D} f(x_m) \varphi(a) = \lim_{a \to 0} \sum_{y^{(p)} \in \Delta_p} f(y^{(p)}) \varphi(a) \sum_{x_m \in \Delta_p} 1
\]

\[
= \lim_{a \to 0} \sum f(y^{(p)}) N(y^{(p)}) |\Delta_p|[1 + o(1)] = \int_D f(x) N(x) dx.
\]

The last equality holds since the preceding sum is a Riemannian sum for the continuous function \( f(x) N(x) \) in the bounded domain \( D \). Thus, the Lemma is proved. \( \square \)
New equation for the limiting effective field

\[ u(x) = u_0(x) - \int_D G(x, y)p(y)u(y)dy, \quad p(x) = 4\pi h(x)N(x). \]

\[ Lu := [\nabla^2 + k^2 - q(x)]u = 0, \quad k^2 - q(x) := k^2n^2(x). \]

\[ L = L_0 - p(x) := \nabla^2 + k^2 - q_0(x) - 4\pi h(x)N(x). \]

New refraction coefficient \( n^2(x) \) and new potential \( q(x) \) are in 1-to-1 correspondence:

\[ n^2(x) = 1 - k^{-2}q(x); \quad q(x) = q_0(x) + p(x). \]

\[ k^2[n_0^2(x) - n^2(x)] = p(x). \]
Creating new materials

Step 1.

\[ \{n^2(x), n_0^2(x)\} \Rightarrow p(x) = k^2(n_0^2 - n^2). \]

Step 2.

Given \( p(x) = p_1 + ip_2 \), find \( \{h(x), N(x)\} \).

Here \( h(x) = h_1(x) + ih_2(x) \), \( N(x) \geq 0 \), \( h_2(x) \leq 0 \).

We have \( p(x) = 4\pi N(x)h(x) \). Thus,

\[
  h_1(x) = \frac{p_1(x)}{4\pi N(x)}, \quad h_2(x) = \frac{p_2(x)}{4\pi N(x)}.
\]

There are many solutions, because \( N(x) \geq 0 \) can be arbitrary.
Step 3. 
Embed $\mathcal{N}(\Delta_p) = \frac{1}{a^{2-\kappa}} \int_{\Delta_p} N(x) dx$ small particles in $\Delta_p$, where $\bigcup_p \Delta_p = D$.

Physical properties of these particles are given by their boundary impedances $\zeta_m = \frac{h(y^{(p)})}{a^\kappa}$ for all $x_m \in \Delta_p$.

The distance between neighboring particles is $d = O(a^{\frac{2-\kappa}{3}})$.

**Theorem.** The resulting new material has the function $n^2(x)$ as its refraction coefficient with the error which tends to 0 as $a \to 0$.

**Remark.** The total volume $V_p$ of the embedded particles in the limit $a \to 0$ is zero.

**Proof.** Since $\kappa \geq 0$, one has:

$$V_p = \lim_{a \to 0} O(a^3/a^{2-\kappa}) = \lim_{a \to 0} O(a^{1+\kappa}) = 0.$$
Technological problems

The first technological problem is:
How can one embed many, namely $M = M(a)$, small balls in a given material so that the centers of the balls are points $x_m$ distributed as desired?
The stereolithography process and chemical methods for growing small particles can be used.

The second technological problem is:
How does one prepare a ball $B_m$ of small radius $a$ with boundary impedance $\zeta_m = \frac{h(x_m)}{a^\kappa}$, $0 \leq \kappa < 1$, which has a desired frequency dependence?

Remark: It is not necessary to have large boundary impedance: if $\kappa = 0$, or $\kappa = O\left(\frac{1}{\ln a}\right)$, then $\zeta_m$ is bounded. However, if $\kappa = 0$, then $M = O(a^{-2})$, so more particles have to be embedded.
Playing with numbers

\[ \mathcal{N} \sim 10^6; \quad \mathcal{N} \sim \frac{1}{a^{2-\kappa}}; \quad d \sim a^{(2-\kappa)/3} \]

\[ \mathcal{N} = 10^6; \quad \kappa = 1, \quad a = 10^{-6}; \quad d = 10^{-2}. \]

\[ \mathcal{N} = 10^6; \quad \kappa = 1/2, \quad a = 10^{-4}; \quad d = 10^{-2}. \]

The difference between the solution of the limiting integral equation for the effective field and the solution to the linear algebraic system for \( u_e(x_m) \) is \( O(1/n) \), where \( 1/n \) is the side of a partition cube.
Spatial dispersion. Negative refraction

\[ u = \sum_k a(k) e^{i[k \cdot r - \omega(k)t]}, \quad |k - \overline{k}| + |\omega(k) - \omega(\overline{k})| < \delta \]

\[ v_g = \nabla_k \omega(k), \quad v_p = \frac{\omega}{|k|} k^0. \]

\[ \nabla_k |k| = k^0 := \frac{k}{|k|}; \quad \frac{\omega^2 n^2}{c^2} = k^2, \quad \frac{\omega n}{c} = |k|. \]

\[ \left[ \frac{n}{c} + \frac{\omega}{c} \frac{\partial n}{\partial \omega} \right] \nabla_k \omega = k^0. \]

\[ \{ v_g = -\text{const} \cdot v_p, \quad \text{const} > 0 \} \iff \text{negative refraction}. \]

\[ n + \omega \frac{\partial n}{\partial \omega} < 0 \]
If $\omega > 0$, $\omega = \omega(k)$, $k := |k|$, then $v_p \cdot v_g = \omega'(k) \frac{\omega}{k} < 0$, provided that

$\omega'(|k|) < 0$.

Indeed,

$$v_g := \nabla_k \omega(k) = \omega'(k) k^0,$$

$$v_p := \frac{\omega}{k} k^0$$

$$\nabla_k \omega(k) \cdot v_p = \omega'(k) \frac{\omega}{k},$$

$k^0 := k/k$.

**Terminology in optics:**

*Negative refraction means* $v_g$ *is directed opposite to* $v_p$;  
*Negative index means that* $\epsilon < 0$ *and* $\mu < 0$. 
Inverse scattering with data at fixed energy and fixed incident direction

\[
\left[ \nabla^2 + k^2 - q(x) \right] u = 0 \text{ in } \mathbb{R}^3, \quad u = e^{ik\alpha \cdot x} + v := u_0 + v,
\]

\[
v = A(\beta) \frac{e^{ikr}}{r} + o\left( \frac{1}{r} \right), \quad r = |x| \to \infty, \quad \frac{x}{r} := \beta,
\]

\[
A(\beta) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx, \quad h(x) := q(x)u(x, \alpha).
\]

Here \( \alpha \) is a unit vector in the direction of propagation of the incident wave, \( A(\beta) \) is the scattering amplitude. **We assume that \( \alpha \) and \( k > 0 \) are fixed.**
Inverse Scattering Problem with fixed energy and fixed incident direction

**IP** (inverse problem): Given \( f(\beta) \in L^2(S^2) \), \( \alpha \in S^2 \), \( k > 0 \), and \( \epsilon > 0 \), \( D \subset \mathbb{R}^3 \) (a bounded domain), find \( q \in L^2(D) \) such that

\[
\| f(\beta) - A(\beta) \|_{L^2(S^2)} < \epsilon. \tag{2}
\]

A priori it is not clear that this problem has a solution. We prove that it has a solution. If this **IP** has a solution, then it has infinitely many solutions because small variations of \( q \) lead to small variations of \( A(\beta) \).
Claim 1. The set \( \{ \int_D e^{-ik\beta \cdot x} h(x) dx \} \forall h \in L^2(D) \) is dense in \( L^2(S^2) \).

Corollary 1. Given \( f \in L^2(S^2) \) and \( \epsilon > 0 \), one can find \( h \in L^2(D) \) such that
\[
\| f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx \| < \epsilon.
\]

Claim 2. The set \( \{ q(x) u(x, \alpha) \} \forall q \in L^2(D) \) is dense in \( L^2(D) \).

Corollary 2. Given \( h \in L^2(D) \) and \( \epsilon > 0 \), one can find \( q \in L^2(D) \) such that
\[
\| h(x) - q(x) u(x, \alpha) \|_{L^2(D)} < \epsilon.
\]

Since the scattering amplitude
\[
A(\beta) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx
\]
depends continuously on \( h \), the inverse problem IP is solved by Claims 1,2.
Proof of Claim 1

Assume the contrary. Then $\exists \psi \in L^2(S^2)$ such that

$$0 = \int_{S^2} d\beta \psi(\beta) \int_D e^{-ik\beta \cdot x} h(x) dx \quad \forall h \in L^2(D).$$

Thus,

$$\int_{S^2} d\beta \psi(\beta) e^{-ik\beta \cdot x} = 0 \quad \forall x \in \mathbb{R}^3.$$

Therefore,

$$\int_0^\infty d\lambda \lambda^2 \int_{S^2} d\beta e^{-i\lambda \beta \cdot x} \psi(\beta) \frac{\delta(\lambda - k)}{k^2} = 0 \quad \forall x \in \mathbb{R}^3.$$

By the injectivity of the Fourier transform, one gets

$$\psi(\beta) \frac{\delta(\lambda - k)}{k^2} = 0.$$

Therefore, $\psi(\beta) = 0$. Claim 1 is proved.
Proof of Claim 2

Given $h \in L^2(D)$, define

$$
u := u_0 - \int_D g(x, y)h(y)dy, \quad g := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad (3)$$

$$
q(x) := \frac{h(x)}{u(x)}. \quad (4)
$$

If $q \in L^2(D)$, then this $q$ solves the problem, and $u$, defined in (3), is the scattering solution:

$$
u = u_0 - \int_D g(x, y)q(y)u(y)dy, \quad (5)$$

and

$$A(\beta) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot y} h(y)dy.$$
If \( q \) is not in \( L^2(D) \), then the null set 
\( N := \{ x : x \in D, \, u(x) = 0 \} \) is non-void. Let 
\[
N_\delta := \{ x : |u(x)| < \delta, x \in D \}, \quad D_\delta := D \setminus N_\delta.
\]

**Claim 3.** \( \exists h_\delta = \begin{cases} h, &\text{ in } D_\delta, \\ 0, &\text{ in } N_\delta, \end{cases} \) such that \( \| h_\delta - h \|_{L^2(D)} < c \epsilon \),
\[
q_\delta := \begin{cases} \frac{h_\delta}{u_\delta}, &\text{ in } D_\delta, \\ 0, &\text{ in } N_\delta, \end{cases} \quad q_\delta \in L^\infty(D), \quad u_\delta := u_0 - \int_D g h_\delta dy.
\]

**Proof of Claim 3.** The set \( N \) is, generically, a line 
\( l = \{ x : u_1(x) = 0, \, u_2(x) = 0 \} \), where \( u_1 = \Re u \) and \( u_2 = \Im u \).

Consider a tubular neighborhood of this line, \( \rho(x, l) \leq \delta \). Let the origin \( O \) be chosen on \( l \), \( s_3 \) be the Cartesian coordinate along the tangent to \( l \), and \( s_1 = u_1 \), \( s_2 = u_2 \) are coordinates in the plane orthogonal to \( l \), \( s_j \)-axis is directed along \( \nabla u_j \big|_l \), \( j = 1, 2 \).
The Jacobian $\mathcal{J}$ of the transformation $(x_1, x_2, x_3) \mapsto (s_1, s_2, s_3)$ is nonsingular, $|\mathcal{J}| + |\mathcal{J}^{-1}| \leq c$, because $\nabla u_1$ and $\nabla u_2$ are linearly independent. Define

$$
\begin{align*}
  h_\delta &:= \begin{cases} 
  h, & \text{in } D_\delta, \\
  0, & \text{in } N_\delta,
  \end{cases} \\
  u_\delta &:= u_0 - \int_D g(x, y)h_\delta(y)dy,
\end{align*}
$$

$$
\begin{align*}
  q_\delta &:= \begin{cases} 
  \frac{h_\delta}{u_\delta}, & \text{in } D_\delta, \\
  0, & \text{in } N_\delta.
  \end{cases}
\end{align*}
$$

One has $u_\delta = u_0 - \int_D ghdy + \int_D g(x, y)(h - h_\delta)dy$,

$$
|u_\delta(x)| \geq |u(x)| - c \int_{N_\delta} \frac{dy}{4\pi|x - y|} \geq \delta - I(\delta), \quad x \in D_\delta, \quad c = \max_{x \in N_\delta} |h(x)|.
$$

If one proves, that $I(\delta) = o(\delta)$, $\delta \to 0$, $\forall x \in D_\delta$ then $q_\delta \in L^\infty(D)$, and Claim 3 is proved.
Claim 4:

\[ I(\delta) = \mathcal{O}(\delta^2 |\ln(\delta)|), \quad \delta \to 0. \]

Proof of Claim 4.

\[
\int_{N_\delta} \frac{dy}{|x - y|} \leq \int_{N_\delta} \frac{dy}{|y|} = c_1 \int_0^{c_2 \delta} \rho \int_0^1 \frac{ds_3}{\sqrt{\rho^2 + s_3^2}} d\rho \\
= c_1 \int_0^{c_2 \delta} d\rho \rho \ln(s_3 + \sqrt{\rho^2 + s_3^2})\bigg|_0^1 \leq c_3 \int_0^{c_2 \delta} \rho \ln \left(\frac{1}{\rho}\right) d\rho \\
\leq \mathcal{O}(\delta^2 |\ln(\delta)|). 
\]
The condition $|\nabla u_j|_l \geq c > 0, j = 1, 2$, implies that a tubular neighborhood of the line $l$, $N_\delta = \{x : \sqrt{|u_1|^2 + |u_2|^2} \leq \delta\}$, is included in a region $\{x : |x| \leq c_2 \delta\}$ and includes a region $\{x : |x| \leq c'_2 \delta\}$. This follows from the estimates

$$c'_2 \rho \leq |u(x)| = |\nabla u(\xi) \cdot (x - \xi)| \leq c_2 \rho.$$ 

Here $\xi \in l$, $x$ is a point on a plane passing through $\xi$ and orthogonal to $l$, $\rho = |x - \xi|$, and $\delta > 0$ is sufficiently small, so that the terms of order $\rho^2$ are negligible, $c_2 = \max_{\xi \in l} |\nabla u(\xi)|$, $c'_2 = \min_{\xi \in l} |\nabla u(\xi)|$. Claim 4, and, therefore, Claim 2 are proved.
Calculation of $h$ given $f(\beta)$ and $\epsilon > 0$

1. Let $\{\phi_j\}$ be a basis in $L^2(D)$,

$$h_n = \sum_{j=1}^{n} c^{(n)}_j \phi_j,$$

$$\psi_j(\beta) := -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} \phi_j(x) \, dx.$$

Consider the problem:

$$\|f(\beta) - \sum_{j=1}^{n} c^{(n)}_j \psi_j(\beta)\| = \min.$$  \hfill (6)

A necessary condition for (6) is a linear system for $c^{(n)}_j$. 
An analytical solution.

2. Let

\[ D = \{ x : |x| \leq 1 \} := B, \text{ or } B \subset D, \text{ } h = 0 \text{ in } D \setminus B. \]

One has:

\[
h_{lm} = \begin{cases} 
(-1)^{l+1} \frac{f_{l,m}}{\sqrt{\frac{\pi}{2k} g_{1,l+\frac{1}{2}}(k)}}, & l \leq L, \\
0, & l > L,
\end{cases}
\]

where \( g_{\mu,\nu}(k) = \int_{0}^{1} x^{\mu + \frac{1}{2}} J_{\nu}(kx) dx \) (Bateman-Erdelyi book, formula (8.5.8))

and \( L \) is chosen so that

\[
\sum_{l>L} |f_{l,m}|^2 < \epsilon^2.
\]
References I


References II


References III


References IV


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References V
