Research announcement

Uniqueness of the solution to inverse obstacle scattering with non-over-determined data

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Abstract

It is proved that the scattering amplitude $A(\beta, \alpha_0, k_0)$, known for all $\beta \in S^2$, where $S^2$ is the unit sphere in $\mathbb{R}^3$, $\alpha_0 \in S^2$ is fixed, $k_0 > 0$ is fixed, determines the surface $S$ of the obstacle and the boundary condition on $S$ uniquely. The boundary condition on $S$ is either the Dirichlet, or Neumann, or the impedance one. The uniqueness theorems for the solution of inverse scattering problems with non-over-determined data were not known for many decades. Such a theorem is proved in this paper for inverse scattering by obstacles for the first time.

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1. Introduction

The uniqueness theorems for the solution of inverse scattering problems with non-over-determined data were not known for many decades. Such a theorem is proved in this paper for inverse scattering by obstacles for the first time. In [1–3] such theorems are proved for the first time for inverse scattering by potentials.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected $C^2$—smooth boundary $S$, $D' := \mathbb{R}^3 \setminus D$ be the unbounded exterior domain and $S^2$ be the unit sphere in $\mathbb{R}^3$.

Consider the scattering problem:

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \quad D', \quad \Gamma_j u|_S = 0, \quad u = e^{ik\alpha \cdot x} + v,$$

where $k > 0$ is a constant, $\alpha \in S^2$ is a unit vector in the direction of the propagation of the incident plane wave $e^{ik\alpha \cdot x}$, $S^2$ is the unit sphere in $\mathbb{R}^3$, $\Gamma_1 u := u$, $\Gamma_2 u := u_N$, $N$ is the unit normal to $S$ pointing out of $D$, $u_N$ is the normal derivative of $u$, $\Gamma_3 u := u_N + hu$, $h = const$, $\text{Im}h \geq 0$, $h$ is the boundary impedance, and the scattered field $v$ satisfies the radiation condition

$$v_r - iku = o\left(\frac{1}{r}\right), \quad r := |x| \to \infty.$$
The scattering amplitude $A(β, α, k)$ is defined by the following formula:

$$v = A(β, α, k) e^{ikr} \frac{1}{r} + o \left( \frac{1}{r} \right), \quad r := |x| \to \infty, \quad \frac{x}{r} = β,$$

(3)

where $α, β \in S^2$, β is the direction of the scattered wave, $α$ is the direction of the incident wave. For a bounded domain $o(\frac{1}{r}) = O(\frac{1}{r^2})$ in formula (3). The function $A(β, α, k)$, the scattering amplitude, can be measured experimentally. Let us call it the scattering data. It is known (see [4], p. 25) that the solution to the scattering problem (1)–(3) does exist and is unique.

The inverse scattering problem (IP) consists of finding $S$ and the boundary condition on $S$ from the scattering data. It was first proved by M. Schiffer in the sixties of the last century that if the boundary condition is the Dirichlet one, then the surface $S$ is uniquely determined by the scattering data $A(β, α_0, k)$ known for a fixed $α = α_0$ all $β \in S^2$ and all $k ∈ (a, b)$, $0 ≤ a < b$. M. Schiffer’s proof was not published by him. This proof can be found in [4], p. 85, and the acknowledgement of M. Schiffer’s contribution is on p. 399 in [4].

A.G. Ramm was the first to prove that the scattering data $A(β, α, k_0)$, known for all $β$ in a solid angle, all $α$ in a solid angle and a fixed $k = k_0 > 0$, determine uniquely the boundary $S$ and the boundary condition. This condition was assumed of one of the three types $I_j$, $j = 1, 2$ or 3, (see [4], Chapter 2, for the proof of these results). By subindex zero fixed values of the parameters are denoted, for example, $k_0$, $α_0$. By solid angle in this paper an open subset of $S^2$ is understood.

In [4], p. 62, it is proved that for smooth bounded obstacles the scattering amplitude $A(β, α, k)$ is an analytic function of $β$ and $α$ on the analytic variety $M := \{z|z ∈ C^3, z \cdot z = 1\}$, where $z \cdot z := \sum_{m=1}^{3} z^2_m$. The unit sphere $S^2$ is a subset of $M$. If $A(β, α, k)$ as a function of $β$ is known on an open subset of $S^2$, it is uniquely extended to all of $S^2$ (and to all of $M$) by analyticity. The same is true if $A(β, α, k)$ as a function of $α$ is known on an open subset of $S^2$.

In papers [5] and [6] a new approach to a proof of the uniqueness theorems for inverse obstacle scattering problem (IP) was given. This approach is used in our paper.

Since the sixties of the last century the uniqueness theorem for IP with non-over-determined data $A(β) := A(β, α_0, k_0)$ was not known. In paper [7] the uniqueness theorem for IP with such data was proved for strictly convex smooth obstacles. The proof in [7] was based on the results concerning location of resonances for a pair of such obstacles. These results are technically difficult to obtain and they hold for two strictly convex obstacles with a positive distance between them.

The purpose of this paper is to prove the uniqueness theorem for IP without any convexity assumption about $S$. By the boundary condition any of the three conditions $I_j$ are understood below.

**Theorem 1.** The surface $S$ and the boundary condition on $S$ are uniquely determined by the data $A(β)$ known in a solid angle.

In Section 2 some auxiliary material is formulated in five lemmas and Theorem 1 is proved.

2. Proof of Theorem 1

First we give an auxiliary material. It consists of five lemmas which are proved by the author, except for Lemma 3, which was known (it was proved first by V. Kupradze in 1934, then by H. Freudenthal in 1938, then by I. Vekua in 1943 and by F. Rellich in 1943, see references in the monograph [4], p. 397). This lemma is often referred to as Rellich’s lemma. A proof of it, based on a new idea, is given in paper [8].

Denote by $G(x, y, k)$ the Green’s function corresponding to the scattering problem (1)–(3). The parameter $k > 0$ is assumed fixed in what follows. For definiteness we assume below the Dirichlet boundary condition,
but our proof is valid for the Neumann and impedance boundary conditions as well. If there are two surfaces $S_m$, $m = 1, 2$, we denote by $G_m$ the corresponding Green’s functions.

**Lemma 1 ([4], p. 46).** One has:

$$
G(x, y, k) = g(|y|)u(x, \alpha, k) + \mathcal{O}\left(\frac{1}{|y|^2}\right), \quad |y| \to \infty, \quad y = -\alpha.
$$  

(4)

Here $g(|y|) \equiv \frac{e^{ik|y|}}{4\pi|y|}$, $u(x, \alpha, k)$ is the scattering solution, i.e., the solution to problem (1)–(3), and the notation $\gamma(r) := 4\pi g(r) = \frac{e^{ik|r|}}{|r|}$ will be used below.

**Remark 1.** The solutions to Eq. (1) have unique continuation property:

If $u$ solves (1) and vanishes on a set $\tilde{D} \subset D'$ of positive Lebesgue measure, then $u$ vanishes everywhere in $D'$.

**Remark 2.** In [4] the remainder in (4) was $o\left(\frac{1}{|y|}\right)$, but for bounded domains $D$ the proof given in [4] yields formula (4). Moreover, this proof shows that formula (4) holds if $\tau > 0$ is a scalar and $\eta$ is an arbitrary fixed vector orthogonal to $\alpha \in S^2$, that is, $\eta \cdot \alpha = 0$. If $\alpha \cdot \eta = 0$ and $y = -\tau \alpha + \eta$, then $|y| = 1 + O\left(\frac{1}{|\tau|^2}\right)$ as $\tau \to \infty$. The relation $|y| \to \infty$ is equivalent to the relation $\tau \to \infty$, and $g(|y|) = g(\tau)(1 + O\left(\frac{1}{\tau}\right))$.

**Lemma 2 ([6], formula (4); [9], formula (5.1.30), global perturbation lemma).** One has:

$$
4\pi[A_1(\beta, \alpha, k) - A_2(\beta, \alpha, k)] = \int_{S_12} [u_1(s, \alpha, k)u_{2N}(s, -\beta, k) - u_{1N}(s, \alpha, k)u_2(s, -\beta, k)]ds,
$$  

(5)

where $A_m(\beta, \alpha, k)$ corresponds to obstacle $S_m$, $m = 1, 2$.

Denote by $D_{12} := D_1 \cup D_2$, $D'_{12} := \mathbb{R}^3 \setminus D_{12}$, $S_{12} := \partial D_{12}$, $\tilde{S}_1 := S_{12} \setminus S_2$, $\tilde{S}_2 \notin D_2$, $B'_R := \mathbb{R}^3 \setminus B_R$, $B_R := \{x : |x| \leq R\}$. The number $R$ is sufficiently large, so that $D \subset B_R$.

**Lemma 3 ([4], p. 25).** If $v \in L^2(B'_R)$ or $\lim_{r \to \infty} \int_{|x|=r} |v|^2ds = 0$, $k > 0$, and $v$ satisfies Eq. (1), then $v = 0$ in $B'_R$.

**Lemma 4 (lifting lemma).** If $A_1(\beta, \alpha, k) = A_2(\beta, \alpha, k)$ for all $\beta, \alpha \in S^2$, then $G_1(x, y, k) = G_2(x, y, k)$ for all $x, y \in D'_{12}$ if $A_1(\beta, \alpha_0, k) = A_2(\beta, \alpha_0, k)$ for all $\beta \in S^2$ and a fixed $\alpha = \alpha_0$, then $G_1(x, y_0, k) = G_2(x, y_0, k)$ for all $x \in D'_{12}$ and all $y_0 = -\alpha_0 \tau + \eta$, where $\tau > 0$ is arbitrary and $\eta$ is an arbitrary fixed vector orthogonal to $\alpha_0$, that is, $\alpha_0 \cdot \eta = 0$.

**Proof of Lemma 4.** The function $w := G_1(x, y, k) - G_2(x, y, k)$ satisfies Eq. (1) in $D'_{12}$ as a function of $y$ and also as a function of $x$, and the radiation condition as a function of $y$ and also as a function of $x$.

By Lemma 1 one has:

$$
w = g(|y|)[u_1(x, \alpha, k) - u_2(x, \alpha, k)] + \mathcal{O}\left(\frac{1}{|y|^2}\right), \quad |y| \to \infty, \quad \alpha = -\frac{y}{|y|}.
$$  

(6)

If $A_1(\beta, \alpha, k) = A_2(\beta, \alpha, k)$ for all $\beta \in S^2$ and fixed $\alpha$ and $k$, then

$$
u_1(x, \alpha, k) - u_2(x, \alpha, k) = \gamma(|x|)[A_1(\beta, \alpha, k) - A_2(\beta, \alpha, k)] + \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \quad \beta = \frac{x}{|x|},
$$  

(7)

because, for $m = 1, 2$, one has:

$$u_m(x, \alpha, k) = e^{ik|x|} + A_m(\beta, \alpha, k)\gamma(|x|) + \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \quad \beta = \frac{x}{|x|}.
$$  

(8)
If $A_1(\beta, \alpha, k) = A_2(\beta, \alpha, k)$, then
\[ u_1(x, \alpha, k) - u_2(x, \alpha, k) = O\left(\frac{1}{|x|^2}\right). \] (9)

Since $u_1 - u_2$ solves Eq. (1) in $D'_1$ and relation (9) holds, it follows from Lemma 3 that $u_1 = u_2$ in $B'_R$. By Remark 1, $u_1 = u_2$ everywhere in $D'_1$. Since $u_1 = u_2$, from formula (6) it follows that,
\[ w = O\left(\frac{1}{|y|^2}\right) \leq O\left(\frac{1}{|x|^2}\right), \quad |y| > |x| \geq R. \] (10)

Therefore, by Lemma 3, it follows that $w = w(x, y) = 0$ in $B'_R$ and, by Remark 1, $w = 0$ everywhere in $D'_1$. Thus, the first part of Lemma 4 is proved. Its second part is a particular case of the first when $\alpha = \alpha_0$.

Let us point out the following implications:
\[ G(x, y, k) \rightarrow u(x, \alpha, k) \rightarrow A(\beta, \alpha, k), \] (11)
which hold by Lemma 1 and formula (8). The first arrow means, for example, that the knowledge of $G(x, y, k)$ determines uniquely $u(x, \alpha, k)$ for all $\alpha \in S^2$.

The reversed implications also hold:
\[ A(\beta, \alpha, k) \rightarrow u(x, \alpha, k) \rightarrow G(x, y, k). \] (12)

These implications follow from Lemmas 1 and 3 and formula (8).

Let us explain why the knowledge of $u(x, \alpha, k)$ determines uniquely $G(x, y, k)$. If there are two $G_m$, $m = 1, 2$, to which the same $u(x, \alpha, k)$ corresponds, then $w := G_1 - G_2$ solves Eq. (1) in $D'_1$ and, by Lemma 1, $w = O\left(\frac{1}{|x|^2}\right)$. Thus, by Lemma 3, $w = 0$, so $G_1 = G_2$ in $D'_1$. This implies, by the argument given in the proof of Theorem 1, that $D_1 = D_2 := D$.

In particular, if $y = y_0 = -\tau\alpha_0 + \eta$, $\tau > 0$ is arbitrary, $\alpha_0$ is a fixed unit vector, and $\eta$ is an arbitrary fixed vector orthogonal to $\alpha_0$, then
\[ G(x, y_0, k) \rightarrow u(x, \alpha_0, k) \rightarrow A(\beta, \alpha_0, k), \] (13)
where $\alpha_0$ is a free unit vector, that is, a vector whose initial point is arbitrary. The reversed implications also hold:
\[ A(\beta, \alpha_0, k) \rightarrow u(x, \alpha_0, k) \rightarrow G(x, y_0, k). \] (14)

The first of these implications follows from Lemma 3 and the asymptotic of the scattering solution, while the second follows from Lemma 1. \qed

Remark 3. There is another way to prove that the knowledge of $u(x, \alpha, k)$ for all $\alpha \in S^2$ determines uniquely $G(x, y, k)$. Namely, the function $u(x, \alpha)$ determines uniquely $A(\beta, \alpha, k) \forall \beta, \alpha \in S^2$, by formula (8). The scattering amplitude $A(\beta, \alpha, k)$, known for all $\beta, \alpha \in S^2$, determines uniquely $S$ and the boundary condition on $S$ ([4], p. 87, Theorem 1). Consequently, the Green’s function $G(x, y, k)$ is uniquely determined. The proof of Theorem 1, given below, yields an alternative proof of Theorem 1 on p. 87 of [4].

Lemma 5. One has
\[ \lim_{x \to t} G_{2N}(x, s, k) = \delta(s - t), \quad t \in S_2, \]
where $\delta(s - t)$ denotes the delta-function on $S_2$ and $x \to t$ denotes a limit along any straight line non-tangential to $S_2$. 
**Proof of Lemma 5.** Let \( f \in C(S_2) \) be arbitrary. Consider the following problem: \( W \) solves Eq. (1) in \( D'_2 \), \( W \) satisfies the boundary condition \( W = f \) on \( S_2 \), and \( W \) satisfies the radiation condition. The unique solution to this problem is given by Green’s formula:

\[
W(x) = \int_{S_2} G_{2N}(x, s)f(s)ds.
\]

Since \( \lim_{x \to t \in S_2} W(x) = f(t) \) and \( f \in C(S_2) \) is arbitrary, the conclusion of Lemma 5 follows. \( \square \)

**Proof of Theorem 1.** If \( A_1(\beta) = A_2(\beta) \) for all \( \beta \) in a solid angle, then the same is true for all \( \beta \in S^2 \) as was proved in [4], p. 62. So, we can assume that \( A_1(\beta) = A_2(\beta) \) for all \( \beta \in S^2 \). If \( A_1(\beta) = A_2(\beta) \) for all \( \beta \) but \( S_1 \neq S_2 \), then Lemma 2 yields the following conclusion:

\[
0 = \int_{S_{12}} [u_1(s, \alpha_0, k)u_{2N}(s, -\beta, k) - u_{1N}(s, \alpha_0, k)u_2(s, -\beta, k)]ds, \quad \forall \beta \in S^2,
\]

where \( k > 0 \) and \( \alpha_0 \in S^2 \) is fixed. By the lifting lemma, that is, by Lemma 4, Eq. (15) implies

\[
0 = \int_{S_{12}} [G_1(s, y_0, k)G_{2N}(s, x, k) - G_{1N}(s, y_0, k)G_2(s, x, k)]ds, \quad \forall x \in D_{12}',\]

where \( y_0 = -\alpha_0 \tau, \tau \geq 0 \). Here vector \( -\alpha \) is a free vector in the sense that its origin can be placed at any point \( \eta \) such that \( \eta \cdot \alpha_0 = 0 \) in the chosen coordinate system. Indeed, the incident plane wave \( e^{ik_0 \cdot x} \) is not changed when \( x \) is replaced by \( x + \eta \), provided that \( \eta \cdot \alpha_0 = 0 \). The scattered field \( v \), satisfying the radiation condition, will satisfy the radiation condition when the origin of the coordinate system is moved to a point \( \eta \) provided that \( \eta \cdot \alpha_0 = 0 \), and there is always a coordinate origin such that the vector \( -\alpha_0 \tau \) intersects the point \( t \in S_2 \). In Remark 2 it is stated that formula (4) is valid with the same \( u(x, \alpha, k) \) if \( y = -\tau \alpha + \eta \), for any fixed vector \( \eta \) orthogonal to \( \alpha \).

Since \( G_m = 0 \) on \( S_m, m = 1, 2 \), one can write (16) as

\[
\int_{S_2} G_1(s, y_0, k)G_{2N}(s, x, k) = \int_{S_1} G_{1N}(s, y_0, k)G_2(s, x, k)ds, \quad \forall x \in D_{12}',
\]

where \( S_1 \) was defined before Lemma 3: it is the part of \( S_{12} \) which does not belong to \( S_2 \), and \( S_2 \) is defined similarly.

Take point \( x \) in (17) to \( t \in S_2 \) and use Lemma 5 and the boundary condition \( G_2(s, t) = 0 \) for \( t \in S_2 \) to get

\[
G_1(t, y_0, k) = 0, \quad \forall t \in S_2, \quad \forall y_0 = -\alpha_0 \tau.
\]

Let \( y_0 \to t \) along vector \( -\alpha_0 \). This is possible if \( -\alpha_0 \) is a free vector in the following sense: the free vector \( -\alpha_0 \) can be written as \( -\alpha_0 + \eta \), where \( \eta \) is a vector orthogonal to \( \alpha_0 \). Then, on the one hand, \( G_1(t, y_0, k) \to \infty \), because \( G_1(t, y_0, k) \) has a singularity as \( y_0 \to t \):

\[
|G_1(t, y_0, k)| = O\left(\frac{1}{|t - y_0|}\right).
\]

On the other hand, \( G_1(t, y_0, k) = 0 \) for any \( y_0 = -\alpha_0 \tau + \eta, \tau > 0, \eta \cdot \alpha_0 = 0 \), by formula (18) since \( t \in S_2 \). This contradiction, which is due to the assumption \( S_1 \neq S_2 \), proves that \( S_1 = S_2 \).

If \( S_1 = S_2 := S \), then \( D_1 = D_2 := D \) and \( u_1(x, \alpha_0, k) = u_2(x, \alpha_0, k) \) for \( x \in D' \), and, consequently, the boundary condition on \( S \) is determined.

Theorem 1 is proved.
The above proof is given under the assumption that the boundary condition on $S_2$ is the Dirichlet one, but it remains valid for other boundary conditions $\Gamma_j$, $j = 2, 3$.

**Remark 4.** Using the results from [4] one can derive the following formula:

$$G(x, y, \zeta) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{u(x, \kappa)u(y, \kappa)}{|\kappa|^2 - \zeta^2} d\kappa, \quad x, y \in \mathbb{R}^3,$$

where $\kappa$ is a vector, $\kappa := |\kappa|\omega \in \mathbb{R}^3$, $\omega \in S^2$, $S^2$ is the unit sphere in $\mathbb{R}^3$, $\text{Im}\zeta > 0$, $u(x, \kappa)$ is the scattering solution in $D'$ and $G(x, y, \zeta)$ is the Green's function of the Dirichlet Laplacian in $D'$. A similar formula holds in $\mathbb{R}^2$. By the limiting absorption principle ([4], p. 43) the Green's function has a limit as $\text{Im}\zeta \to +0$ when $x \neq y$. Thus, the knowledge of $u(x, \kappa)$ determines uniquely $G(x, y, \zeta)$. In (20) the scattering solutions are functions of vector $\kappa$, so they are assumed known for all $\kappa = |\kappa|\omega$, that is, for all $|\kappa| \geq 0$ and for all $\omega \in S^2$.

To verify formula (20), apply the operator $-(\nabla^2 + \zeta^2)$ to (20), take into account that $-(\nabla^2 + \zeta^2)u(x, \kappa) = (|\kappa|^2 - \zeta^2)u(x, \kappa)$, and use the formula

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} u(x, \kappa)u(y, \kappa) d\kappa = \delta(x - y),$$

proved in [4], p. 48.

**References**


