Asymptotic of some integral

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Summary: Consider an integral \( I(s) := \int_0^b e^{-s(x^2-icx)}x^2 \, dx \), where \( c > 0 \) and \( b > 0 \) are arbitrary positive constants. It is proved that \( I(s) \sim \frac{-2i}{s^3 c^3} \) as \( s \to +\infty \). Possible applications of this result to the Pompeiu problem are outlined.

1 Formulation of the result

Let \( c > 0 \) be a constant and \( s > 0 \) be a large parameter. Consider the integral

\[
I(s) := \int_0^b e^{-s(x^2-icx)}x^2 \, dx,
\]

where \( b > 0 \) is an arbitrary fixed constant. The choice of \( b > 0 \) does not influence the asymptotic as \( s \to \infty \). This is proved in Remark 2.2 below. So, one can take \( b = \infty \) and use some analytical results from [1] and [6].

The aim of this paper is to derive an asymptotic formula for \( I(s) \) as \( s \to +\infty \) and to describe the idea of its application to the Pompeiu problem. In what follows we write \( \infty \) for \( +\infty \). The result is quite simple.

**Theorem 1.1** One has

\[
I(s) \sim \frac{-2i}{s^3 c^3} \quad \text{as} \quad s \to +\infty.
\]

In Section 2 a proof of Theorem 1.1 is given. In Section 3 possible applications to the Pompeiu problem are discussed.

In the large literature on the stationary phase method and the steepest descent method the integrals of the type \( \int_D e^{i\lambda S(x)} f(x) \, dx \) when \( \lambda \to \infty \) were studied, and it is usually assumed that the phase \( S(x) \) is a real-valued or a purely imaginary function, see [2], [4]. There are some works on steepest descent when the phase function is assumed analytic and one chooses a steepest descent contour on which the phase function \( S(x) \) is either a real-valued or a purely imaginary function. The novelty in this paper is in the study of the example when the phase function is complex-valued. The special choice of \( f(x) \) is motivated by a study of the Pompeiu problem, see [8], [11]. Our derivation of asymptotic

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formula differs from the usual derivations in [2] and [4]. Namely, we use some formulas from the theory of special functions, [1], [6].

2 Proof of Theorem 1.1

Let us start with two formulas from [1], formulas 1.4.11 and 2.4.18:

\[
\int_0^\infty e^{-ax^2} \cos(xy)dx = \frac{\pi^{1/2}}{2a^{1/2}} e^{-\frac{y^2}{4a}},
\]
(2.1)

\[
\int_0^\infty e^{-ax^2} \sin(xy)dx = \frac{ye^{-\frac{y^2}{4a}}}{2a} F\left(\frac{1}{2};\frac{3}{2};\frac{y^2}{4a}\right),
\]
(2.2)

where \(F(y;c;x)\) is the degenerate hypergeometric function, defined in [6], Chapter 9, by the formula

\[
F(y;c;x) = \sum_{k=0}^{\infty} \frac{(y)_k x^k}{(c)_k k!},
\]

where \((y)_k := \frac{\Gamma(y+k)}{\Gamma(y)}\), \((y)_0 := 1\). In [1] the function \(F(y;c;x)\) is denoted sometimes by \(1_F(y;c;x)\).

One has ([6], formula (9.12.8)):

\[
F(y;c;x) \sim \frac{\Gamma(c)}{\Gamma(y)} e^x x^{c-y} \left[1 + O\left(\frac{1}{x}\right)\right], \quad x \to \infty.
\]
(2.3)

It follows from (2.1)–(2.3) that

\[
J := \int_0^\infty e^{-sx^2} e^{icsx} dx \sim \frac{i}{cs}, \quad s \to \infty.
\]
(2.4)

One can calculate \(I(s)\) using the formula:

\[
I(s) = s^{-2} (-i d/dc)^2 J \sim -2is^{-3} c^{-3}, \quad s \to \infty.
\]
(2.5)

where the differentiation with respect to parameter \(c\) is justified. The choice \(b = \infty\) does not restrict the generality, as was mentioned above, see Remark 2.2. Theorem 1.1 is proved.

Remark 2.1 The result (2.4) can be formally obtained if one neglects the term \(-sx^2\) in the phase, uses a standard asymptotic of the integral \(\int_0^a e^{icsx} f(x)dx\) as \(s \to \infty\) and assumes that \(f \in C^1([0,a])\), \(f = 1\) in a neighborhood of the origin \(x = 0\) and \(f = 0\) for \(x \geq a\). This is just a formal argument, and its rigorous justification is given in the proof of Theorem 1.1.

Remark 2.2 If \(b > 0\) is an arbitrary fixed number, and \(\epsilon > 0\) is an arbitrary small fixed number, then

\[
\int_0^b e^{-sx^2} e^{icsx} dx \sim -2is^{-3} c^{-3}, \quad s \to \infty.
\]
(2.6)
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This follows from the estimate

\[ \left| \int_{\epsilon}^{b} e^{-sx^2} e^{ixs} \, dx \right| \leq O(e^{-s\epsilon^2}) = o(s^{-3}), \quad s \to \infty. \tag{2.7} \]

**Remark 2.3** One can prove the following result: if \( J(s) := \int_{0}^{b} e^{s(x^2 + ix)} \, dx \), where \( c > 0 \) is a constant, then \( J(s) \sim \frac{e^{s(x^2 + ixc)}}{s(2b+iC)} \) as \( s \to \infty \).

3 An idea of an application of Theorem 1.1 to the Pompeiu problem

Let us outline a possible application of the above results to the Pompeiu problem. This problem has been open since 1929 (see [7]). Some new results and historical remarks about this problem can be found in [8], [10] and in [12]. The problem in the modern formulation can be stated as follows.

Let \( f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}' \), where \( \mathcal{S}' \) is the Schwartz class of distributions, and

\[ \int_{\sigma(D)} f(x) \, dx = 0 \quad \forall \sigma \in G, \tag{3.1} \]

where \( G \) is the group of all rigid motions of \( \mathbb{R}^n \), \( n \geq 2 \), consisting of all translations and rotations, and \( D \subset \mathbb{R}^n \) is a bounded domain, the closure \( \tilde{D} \) of which is diffeomorphic to a closed ball. Under these assumptions the complement of \( \tilde{D} \) in \( \mathbb{R}^n \) is connected and path connected by the isotopy extension theorem, see [5]. In [7] the following question was raised:

**Does (3.1) imply that \( f = 0 \)?**

If yes, then we say that \( D \) has \( P \)-property (Pompeiu’s property), and write \( D \in P \). Otherwise, we say that \( D \) fails to have \( P \)-property, and write \( D \in \bar{P} \). Pompeiu claimed in 1929 that every plane bounded domain has \( P \)-property. This claim turned out to be false: a counterexample was given 15 years later in [3]. The counterexample is a domain \( D \) which is a disc, or a ball in \( \mathbb{R}^n \) for \( n > 2 \). If \( D \) is a ball, then there are \( f \neq 0 \) for which equation (1) holds. The set of all \( f \neq 0 \), for which equation (1) holds, was constructed in [9]. There are infinitely many (a continuum) such \( f \). A set of counterexamples can also be found in [8]). In [11] some recent conjectures and results on the Pompeiu problem are given.

It is known (see, for example, [8]) that if the condition

\[ \int_{D} e^{ik\alpha \cdot x} \, dx = 0 \quad \forall \alpha \in S^2, \tag{3.2} \]

and for some \( k > 0 \), implies that \( D \) is a ball, then the Pompeiu problem in its modern formulation is solved in the following sense: every domain \( D \) which fails to have \( P \)-property is a ball.

The idea of a possible application of Theorem 1.1 to the Pompeiu problem can now be described. It is easy to check that (3.2) holds for every \( \alpha \in M \) if it holds for every...
\( \alpha \in S^2 \), where \( M := \{ z : z \in \mathbb{C}^3, z \cdot z = 1 \} \), and \( z \cdot z := \sum_{j=1}^{3} z_j^2 = 1 \). Here \( z_j \in \mathbb{C} \). The algebraic variety \( M \) is a non-compact variety in \( \mathbb{C}^3 \).

Consider the following element of \( M \):

\[
\alpha = is e_3 + (s^2 + 1)^{1/2}(e_1 \cos \theta + e_2 \sin \theta),
\]

where \( \{e_j\}_{j=1}^{3} \) is an orthonormal basis of \( \mathbb{R}^3 \), \( s \in \mathbb{R} \) is a real-valued parameter that we take to \( 1 \), and \( \theta \in [0, 2\pi) \). The integral (3.2) can be written as

\[
I_1 := \int_{0}^{B} \int_{D_z} e^{ik(s^2 + 1)^{1/2}(x_1 \cos \theta + x_2 \sin \theta)} dx_1 dx_2,
\]

where \( D_z \) is the cross-section of \( D \) by the plane perpendicular to some chosen direction \( e_3 \), while \( z \) is the coordinate along this direction. The value \( z = 0 \) corresponds to the tangent plane to \( S \), the boundary of \( D \), perpendicular to \( e_3 \), and \( B \) corresponds to the value of \( z \) describing the tangent plane \( z = B \) to \( S \) above the plane \( z = 0 \). Using the Radon transform and denoting \( p := x_1 \cos \theta + x_2 \sin \theta \), one writes \( I_1 \) as

\[
I_1 = \int_{0}^{B} \int_{b_j}^{b_2} dpe^{ik(s^2 + 1)^{1/2} p} L(p),
\]

where \( L(p) \) is the length of the intersection of \( D_z \) with the straight line orthogonal to the line whose direction is given by the vector with the components \( (\cos \theta, \sin \theta) \), \( b_j = b_j(z, \theta) \), \( j = 1, 2 \). The asymptotic of the integral \( \int_{b_1}^{b_2} dpe^{ik(s^2 + 1)^{1/2} p} L(p) \) as \( s \to \infty \) can be calculated by the standard methods since the phase function \( k(s^2 + 1)^{1/2} p \) is real-valued.

The resulting integral is a one-dimensional integral of the form investigated in Theorem 1.1. The idea is to show that if \( I_1 = 0 \) as \( s \to \infty \) for all \( \theta \) and all the directions of the \( z \)-axis, then \( D \) is a ball.

Let us give more details. One uses the known formula (see, for example, [4]):

\[
\int_{0}^{a} e^{ixx} f(x)x^{\beta - 1} dx \sim f(0)\Gamma(\beta)e^{i\pi \beta /2}s^{-\beta} + o(s^{-\beta}), \quad s \to \infty,
\]

where \( \Gamma(z) \) is the Gamma function, \( f \) is a smooth function, \( f = 0 \) for \( x \geq a \), and the number \( \beta \) is positive. The asymptotic of \( L(p) \) as \( p \to b_j \) can be calculated as follows. Introduce the local coordinate system on the plane \( D_z \) by choosing the origin at the point \( b_j \). Then the equation of the boundary of \( D_z \) is \( b_j - p = \frac{x^2}{2\rho(b_j)} \) up to the higher order terms as \( p \to b_j \) and \( j = 2 \), and \( \rho(b_j) \) is the radius of curvature of the boundary of \( D_z \) at the point \( b_j \). At the point \( b_1 \) the role of \( b_2 - p \) is played by \( p - b_1 \). The quantity \( L(p) \) can be calculated by the formula:

\[
L(p) = 2[2\rho(b_2)(b_2 - p)]^{1/2}.
\]
By formula (3.5) with $\beta = 3/2$ one gets

$$
\int_{b_1}^{b_2} dpe^{ik(s^2+1)^{1/2}}p L(p) \sim \left[ e^{ik(s^2+1)^{1/2}b_2} \rho(b_2)^{1/2}2^{1/2}\Gamma(3/2)e^{-i\pi/4}(k(s^2+1)^{1/2})^{-3/2}
+ e^{ik(s^2+1)^{1/2}b_1} \rho(b_1)^{1/2}2^{1/2}\Gamma(3/2)e^{i\pi/4}(k(s^2+1)^{1/2})^{-3/2} \right].
$$

One has $\Gamma(3/2) = \sqrt{\pi}/2$.

Inserting (3.6) into $I_1$ and taking into account the input, similar to (3.6), from the crossection $D_z$ in $I_1$ near the point $z = 0$, one gets

$$
I_1 = \int_0^B dz e^{-skz} \left[ e^{ik(s^2+1)^{1/2}b_2} \rho(b_2)^{1/2}2^{1/2}\Gamma(3/2)e^{-i\pi/4}(k(s^2+1)^{1/2})^{-3/2}
+ e^{ik(s^2+1)^{1/2}b_1} \rho(b_1)^{1/2}2^{1/2}\Gamma(3/2)e^{i\pi/4}(k(s^2+1)^{1/2})^{-3/2} + I_2 \right].
$$

Here

$$
I_2 := e^{ik(s^2+1)^{1/2}b_2'} \rho(b_2')^{1/2}2^{1/2}\Gamma(3/2)e^{-i\pi/4}(k(s^2+1)^{1/2})^{-3/2}
+ e^{ik(s^2+1)^{1/2}b_1'} \rho(b_1')^{1/2}2^{1/2}\Gamma(3/2)e^{i\pi/4}(k(s^2+1)^{1/2})^{-3/2},
$$

where the $b_j'$ are the points analogous to $b_j$ on the boundary of the cross-section $D_z'$ by the plane $z = \text{const}$ when $z \to 0$.

The question is:

Does the relation $I_1 = 0$ for all $\theta \in [0,2\pi)$ and all directions of the z-axis imply that $D$ is a ball?

One has

$$
e^{ik(s^2+1)^{1/2}b_2} \sim e^{ikscz^{1/2}}, \quad s \to \infty,
$$

where $c(\theta)$ is a constant in the formula $b_2 = c(\theta)\sqrt{z}$. After introducing a new variable $z = x^2$ integral $I_1$ takes the form (1.1). Its asymptotic as $s \to \infty$ can be calculated using Theorem 1.1. Note also that $\rho(b_j)^{1/2} = O(\sqrt{z})$ if $z \to 0$.

References


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