SYMMETRY PROBLEM

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Abstract. A novel approach to an old symmetry problem is developed. A new proof is given for the following symmetry problem, studied earlier: if \( \Delta u = 1 \) in \( D \subset \mathbb{R}^3 \), \( u = 0 \) on \( S \), the boundary of \( D \), and \( u_N = \text{const} \) on \( S \), then \( S \) is a sphere. It is assumed that \( S \) is a Lipschitz surface homeomorphic to a sphere. This result has been proved in different ways by various authors. Our proof is based on a simple new idea.

1. Introduction

Symmetry problems are of interest both theoretically and in applications. A well-known, and still unsolved, symmetry problem is the Pompeiu problem (see [9], [10], and the references therein). In modern formulation this problem consists of proving the following conjecture:

If \( D \subset \mathbb{R}^n, n \geq 2, \) is a domain homeomorphic to a ball, and the boundary \( S \) of \( D \) is smooth (\( S \in C^{1,\lambda}, \lambda > 0, \) is sufficient), and if the problem

\[
(\nabla^2 + k^2)u = 0 \quad \text{in} \quad D, \quad u|_S = c \neq 0, \quad u_N|_S = 0, \quad k^2 = \text{const} > 0,
\]

where \( c \) is a constant, has a solution, then \( S \) is a sphere.

A similar problem (Schiffer’s conjecture) is also unsolved (see also [4]):

If the problem

\[
(\nabla^2 + k^2)u = 0 \quad \text{in} \quad D, \quad u|_S = 0, \quad u_N|_S = c \neq 0, \quad k^2 = \text{const} > 0
\]

has a solution, then \( S \) is a sphere.

Standing assumptions. In this paper we assume that \( D \subset \mathbb{R}^3 \) is a bounded domain homeomorphic to a ball, with a sufficiently smooth boundary \( S \) (\( S \) is Lipschitz suffices).

We use the following notation: \( D' = \mathbb{R}^3 \setminus D \), \( B_R = \{ x : |x| \leq R \} \), \( B_R \supset D, \mathcal{H} \) is the set of all harmonic functions in \( B_R \), \( R > 0 \) is an arbitrary large number, such that the ball \( B_R \) contains \( D \), \( |D| \) and \( |S| \) are the volume of \( D \) and the surface area of \( S \), respectively.

In [12] it is proved that if

\[
\int_D \frac{dy}{4\pi|x - y|} = \frac{c}{|x|}, \quad \forall x \in B'_R, \quad c = \text{const} > 0,
\]

then \( D \) is a ball. The proof in [12] is based on an idea similar to the one we are using in this paper.

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In [13] a symmetry problem of interest in elasticity theory is studied by A.D. Alexandrov’s method of a moving plane ([1]), used also in [14]. The result in [14], which is formulated below in Theorem 1, was proved in [15] by a method, different from the one given in [14], and discussed also in [2]. The argument in [2] remained unclear to the author.

In [5] another symmetry problem of potential theory was studied.

Our goal is to give a new proof of Theorem 1. The result of Theorem 1 was obtained in [14] for \( R^n, n \geq 2 \).

**Theorem 1.** If \( D \supset R^3 \) is a bounded domain, homeomorphic to a ball, \( S \) is its Lipschitz boundary, and the problem

\[
\Delta u = 1 \quad \text{in} \ D, \quad u \big|_S = 0, \quad u_N \big|_S = c := \frac{|D|}{|S|} > 0
\]

has a solution, then \( S \) is a sphere.

This result is equivalent to the following result:

If

\[
\int_D h(x)dx = c \int_S h(s)ds, \quad \forall h \in \mathcal{H}, \quad c := \frac{|D|}{|S|},
\]

then \( S \) is a sphere.

The equivalence of (4) and (5) can be proved as follows.

Suppose (4) holds. Multiply (4) by an arbitrary \( h \in \mathcal{H} \), integrate by parts and get

\[
\int_D h(x)dx = c \int_S h(s)ds.
\]

If \( h = 1 \) in (6), then one gets \( c = \frac{|D|}{|S|} \), so (3) is identical to (5).

Suppose (5) holds. Then (3) holds. Let \( v \) solve the problem \( \Delta v = 1 \) in \( D \), \( v \big|_S = 0 \). This \( v \) exists and is unique. Using (3), the equation \( \Delta h = 0 \) in \( D \), and the Green’s formula, one gets

\[
c \int_S h(s)ds = \int_D h(x)dx = \int_D h(x)\Delta vdx = \int_S h(s)v_N ds.
\]

Thus,

\[
\int_S h(s)[v_N - c]ds = 0, \quad \forall h \in \mathcal{H}.
\]

We will need the following lemma:

**Lemma A.** The set of restrictions on \( S \) of all harmonic functions in \( D \) is dense in \( L^2(S) \).

Proof of Lemma A. We give a proof for the convenience of the reader. The proof is borrowed from [12]. Suppose that \( g \in L^2(S) \), and \( \int_S g(s)h(s)ds = 0 \) \( \forall h \in \mathcal{H} \). Since \((4\pi|x-y|)^{-1}\) is in \( \mathcal{H} \) if \( y \in D' \), one gets

\[
w(y) := \int_S g(s)(4\pi|s-y|)^{-1}ds = 0 \quad \forall y \in D'.
\]
Thus, a single layer potential \( w \), with \( L^2 \) density \( g \), vanishes in \( D' \), and, by continuity, on \( S \). Since \( w \) is a harmonic function in \( D \) vanishing on \( S \), it follows that \( w = 0 \) in \( D \). By the jump formula for the normal derivative of the single-layer potential across a Lipschitz boundary, one gets \( g = 0 \).

Thus, (3) implies \( u_N \big|_S = c \). Therefore, (4) holds.

A result, related to equation (3), was studied in [7] for a two-dimensional problem. The arguments in [7] were not quite clear to the author.

Our main result is a new proof of Theorem 1. The proof is simple, and the method of the proof is new. This method can be used in other problems (see [5], [10], [12], [11]).

2. Proofs

Proof of Theorem 1. We denote by \( D' \) the complement of \( D \) in \( \mathbb{R}^3 \), by \( S^2 \) the unit sphere, by \( [a,b] \) the cross product of two vectors, by \( g = g(\phi) \) the rotation about an axis, directed along a vector \( \alpha \in S^2 \), by the angle \( \phi \), and note that if \( h(x) \) is a harmonic function in any ball \( B_{R'} \), containing \( D \), then \( h(g(x)) \) is also a harmonic function in \( B_{R'} \).

Take \( h = h(g(\phi)x) \) in (3), differentiate with respect to \( \phi \) and then set \( \phi = 0 \). This yields:

\[
\int_D \nabla h(x) \cdot [\alpha, x] \, dx = c \int_S \nabla h(s) \cdot [\alpha, s] \, ds.
\]

Using the divergence theorem, one rewrites this as

\[
\alpha \cdot \int_S [s, N] h(s) \, ds = \alpha \cdot \int_S [s, c \nabla h(s)] \, ds.
\]

Since \( \alpha \in S^2 \) is arbitrary, one gets

\[
(9) \quad \int_S [s, N] h(s) \, ds = \int_S [s, c \nabla h(s)] \, ds, \quad \forall h \in \mathcal{H},
\]

where \( N = N_s \) is a unit normal to \( S \) at the point \( s \in S \), pointing into \( D' \).

Let \( y \in B_{R'}^c \) be an arbitrary point, and \( h(x) = \frac{1}{|x-y|} \in \mathcal{H} \), where \( x \in B_R \). Then equation (9) implies that

\[
(10) \quad v(y) := \int_S [s, N] \, ds \bigg/ |s-y| = c \int_S \, ds \bigg/ |s-y|, \quad \forall y \in B_{R'}^c,
\]

because

\[
(11) \quad c \int_S [s, \nabla_s \frac{1}{|s-y|}] \, ds = c \int_S [\frac{s}{|s-y|^3}, y] \, ds = c \big[ \nabla_y \int_S \frac{ds}{|s-y|} \big],
\]

Relation (11) actually holds for all \( y \in D' \), because of the analyticity of its left and right sides in \( D' \). Let

\[
w(y) := \int_S |s-y|^{-1} \, ds.
\]

Denote \( y^0 := y/|y| \). It is known (see, e.g., [3]) that

\[
(12) \quad |y-s|^{-1} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{4\pi}{2n+1} Y_{nm}(y^0) \overline{Y_{nm}(s^0)} |s|^{n} |y|^{-(n+1)}, \quad |y| > |s|,
\]

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where the overline stands for the complex conjugate, \( y^0 \) is the unit vector characterized by the angles \( \theta, \phi \) in spherical coordinates, \( Y_{nm} \) are normalized spherical harmonics:

\[
Y_{nm}(y^0) = Y_{nm}(\theta, \phi) = \gamma_{nm} P_{n|m|}(\cos \theta)e^{i\phi}, \quad -n \leq m \leq n,
\]

\[
\gamma_{nm} = \left( \frac{(2n+1)(n-m)!}{4\pi(n+m)!} \right)^{1/2}
\]

are normalizing constants:

\[
(Y_{nm}(y^0), Y_{pq}(y^0))_{L^2(S^2)} = \delta_{np}\delta_{mq},
\]

and

\[
P_{n|m|}(\cos \theta) = (\sin \theta)^{|m|} (\frac{d}{d\cos \theta})^{|m|} P_n(\cos \theta)
\]

are the associated Legendre functions, where \( P_n(\cos \theta) \) are the Legendre polynomials.

If \( z = \cos \theta \), then

\[
P_{n,m}(z) = (z^2 - 1)^{m/2}(\frac{d}{dz})^m P_n(z), \quad m = 1, 2, ..., \]

\[
P_n(z) = (2^n n!)^{-1}(\frac{d}{dz})^n(z^2 - 1)^n, \quad P_0(z) = 1
\]

(see [3]). The definitions of \( P_{n,m}(z) \) in various books can differ by a factor \((-1)^m\).

Using formula (12), one obtains

\[
w(y) = \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_{nm}(y^0)|y|^{-(n+1)} c_{nm}, \quad c_{nm} := \int_S |s|^n Y_{nm}(s^0) ds.
\]

Substitute this in (10), equate the terms in front of \(|y|^{-(n+1)} \), and define vectors

\[
a_{nm} := \int_S [s,N]|s|^n Y_{nm}(s^0) ds
\]

to obtain

\[
\sum_{m=-n}^{n} Y_{nm}(y^0)a_{nm} = \sum_{m=-n}^{n} c_{nm} [e_\theta e_\phi Y_{nm}(y^0) + e_\phi (\sin \theta)^{-1}\partial_\phi Y_{nm}(y^0), e_r],
\]

where \( e_\theta, e_\phi, \) and \( e_r \) are orthogonal unit vectors of the spherical coordinate system, \([e_\phi, e_r] \) is the cross product, \([e_\phi, e_r] = e_\theta, [e_\theta, e_r] = -e_\phi, y = ry^0, r = |y|, y^0 = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), \( \partial_\theta = \frac{\partial}{\partial \theta} \).

Therefore, formula (15) can be rewritten as

\[
\sum_{m=-n}^{n} Y_{nm}(y^0)a_{nm} = \sum_{m=-n}^{n} c_{nm} \left( -e_\phi \partial_\theta Y_{nm}(y^0) + \partial_\phi Y_{nm}(y^0) \right).
\]

From (16) we want to derive that

\[
a_{nm} = 0, \quad n \geq 0, -n \leq m \leq n.
\]

If (17) is established, then it follows from (14) and from the completeness in \( L^2(S) \) of the system \( \{ |s|^n Y_{nm}(s^0) \}_{n \geq 0, -n \leq m \leq n} \) that \( [s,N] = 0 \) on \( S \), and this implies that \( S \) is a sphere, as follows from Lemma 1 formulated and proved below. Consequently, Theorem 1 is proved as soon as relations (17) are established. The completeness of the system \( \{ |s|^n Y_{nm}(s^0) \}_{n \geq 0, -n \leq m \leq n} \) in \( L^2(S) \) follows from Lemma B:

The functions \( |x|^n Y_{nm}(x^0), n \geq 0, -n \leq m \leq n \), are harmonic in any ball, centered at the origin.
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Lemma B. The set of restrictions of the above functions to any Lipschitz surface homeomorphic to a sphere is complete in $L^2(S)$.

Proof of Lemma B. The proof is given for completeness. It is similar to the proof of Lemma A. Suppose that $g \in L^2(S)$ and

$$\int_S g(s)|s|^nY_{nm}(s^0)ds = 0, \quad \forall n \geq 0, |m| \leq n.$$ 

This and (12) imply that

$$\int_S g(s)(4\pi|s-y|)^{-1}ds = 0 \quad \forall y \in D',$$

and the argument, given in the proof of Lemma A, yields the desired conclusion $g = 0$. □

Vector $a_{nm}$ is written in the Cartesian basis $\{e_j\}_{1 \leq j \leq 3}$ as

$$a_{nm} = \sum_{j=1}^3 a_{nm,j}e_j.$$

The relation between the components $F_1, F_2, F_3$ of a vector $F$ in Cartesian coordinates and its components $F_r, F_\theta, F_\phi$ in spherical coordinates can be found, e.g., in [6], Section 6.5:

$$F_1 = F_r \sin \theta \cos \phi + F_\theta \cos \theta \cos \phi - F_\phi \sin \phi,$$

$$F_2 = F_r \sin \theta \sin \phi + F_\theta \cos \theta \sin \phi + F_\phi \cos \phi,$$

$$F_3 = F_r \cos \theta - F_\theta \sin \theta.$$

Using these relations one derives from (16) the following formulas:

(18) $$\sum_{m=-n}^n a_{nm,1}Y_{nm}(y^0) = \sum_{m=-n}^n cc_{nm} \left( \partial_\phi Y_{nm}(y^0) \sin \phi + \partial_\theta Y_{nm}(y^0) \cot \theta \cos \phi \right),$$

(19) $$\sum_{m=-n}^n a_{nm,2}Y_{nm}(y^0) = \sum_{m=-n}^n cc_{nm} \left( -\partial_\theta Y_{nm}(y^0) \cos \phi + \partial_\phi Y_{nm}(y^0) \cot \theta \sin \phi \right),$$

(20) $$\sum_{m=-n}^n a_{nm,3}Y_{nm}(y^0) = - \sum_{m=-n}^n cc_{nm} \partial_\phi Y_{nm}(y^0).$$

From formulas (18)-(20) one derives (17).

If $n = 0$, then $a_{00} = 0$, as the following calculation shows:

$$a_{00} = \frac{1}{(4\pi)^{1/2}} \int_S [s, N]ds = -\frac{1}{(4\pi)^{1/2}} \int_D [
abla, x]dx = 0.$$

If $n > 0$, then multiply equation (20) by $e^{-im\phi}$, integrate with respect to $\phi$ over $[0, 2\pi]$, write $P_{n,m}$ for $P_{n,m}(\cos \theta)$, and obtain

(21) $$a_{nm,3}P_{n,m} = -cc_{nm}imP_{n,m}, \quad c_{nm} := c_{nm}.$$

One concludes that $a_{n0,3} = 0$ and $a_{nm,3} = -imcc_{nm,m}$. If one derives from (18)-(19) that $c_{n,m} = 0$, then equation (17) follows, and Theorem 1 is proved.
The quantities \( a_{nm} \) of their asymptotics as in independence of the system of functions and Theorem 1 is proved.

From (18) and (19) one derives analogs of (21):

\[
2ia_{nm,1}\gamma_{nm} P_{n,m} = cc_{n,m-1}\gamma_{n,m-1} (\partial_{\theta} P_{n,m-1} - (m - 1) \cot \theta P_{n,m-1})
\]

\[
2ia_{nm,2}\gamma_{nm} P_{n,m} = cc_{n,m-1}\gamma_{n,m-1} (-\partial_{\theta} P_{n,m-1} + (m - 1) \cot \theta P_{n,m-1})
\]

Let us take \( \theta \to 0 \) in the above equations. It is known (see [3], Section 3.9.2, formula (4)) that

\[
P_{n,m}(z) \sim b(n,m)(z - 1)^m/2, \quad z \to 1, \quad b(n,m) := \frac{(n + m)!}{2^m/m!(n - m)!}.
\]

Equation (22) can be considered as a linear combination

\[
\sum_{j=1}^3 A_j f_j(z) = 0,
\]

where the \( A_j \) are constants:

\[A_1 = 2ia_{nm,1}\gamma_{nm}, \quad A_2 = -cc_{n,m-1}\gamma_{n,m-1}, \quad A_3 = cc_{n,m+1}\gamma_{n,m+1},\]

and

\[f_1(z) = P_{n,m}(z),\]
\[f_2(z) = -(1 - z^2)^{1/2}P_{n,m-1}'(z) - (m - 1) \frac{z}{(1 - z^2)^{1/2}} P_{n,m-1}(z),\]
\[f_3(z) = -(1 - z^2)^{1/2}P_{n,m+1}'(z) - (m + 1) \frac{z}{(1 - z^2)^{1/2}} P_{n,m+1}(z), \quad z = \cos \theta.
\]

If the system of functions \( \{ f_j(z) \}^3_{j=1} \) is linearly independent on the interval \([-1, 1]\), then all \( A_j = 0 \) in (25), that is, \( A_1 = 0, A_2 = 0, \) and \( A_3 = 0. \) This implies that

\[a_{nm,1} = c_{n,m}, \quad -n \leq m \leq n.
\]

The quantities \( a_{nm,2} \) and \( a_{nm,3} \) are proportional to \( c_{n,m}. \) Since \( c_{n,m} = 0, \) it follows that

\[a_{nm,2} = a_{nm,3} = 0, \quad -n \leq m \leq n,
\]

and Theorem 1 is proved.

Thus, to complete the proof of Theorem 1 it is sufficient to verify the linear independence of the system of functions \( \{ f_j(z) \}^3_{j=1} \) on the interval \([-1, 1].\)

From formula (24) it follows that these functions have the following main terms of their asymptotics as \( z \to 1:\)

\[f_1(z) \sim B_1(z - 1)^m/2, \quad f_2(z) \sim B_2 \frac{(z - 1)^{(m+1)/2}}{(1 - z^2)^{1/2}}, \quad f_3(z) \sim B_3 \frac{(z - 1)^{(m+3)/2}}{(1 - z^2)^{1/2}}.
\]

where the constants \( B_j \neq 0, 1 \leq j \leq 3, \) depend on \( n, m. \) The linear independence of the system \( \{ f_j(z) \}^3_{j=1} \) holds because the system

\[\{(z - 1)^m/2, \quad \frac{(z - 1)^{(m+1)/2}}{(1 - z^2)^{1/2}}, \quad \frac{(z - 1)^{(m+3)/2}}{(1 - z^2)^{1/2}}\}
\]

is linearly independent. The linear independence of this system holds if the system

\[\{1, \quad (1 + z)^{-0.5}, \quad (z - 1)(1 + z)^{-0.5}\}
\]
is linearly independent on the interval \([-1, 1]\). The linear independence of this system on the interval \([-1, 1]\) is obvious.

Theorem 1 is proved. \[\square\]

**Lemma 1.** If \(S\) is a \(C^2\) smooth closed surface and \([s, N_s] = 0\) on \(S\), then \(S\) is a sphere.

**Proof of Lemma 1.** Let \(s = r(u, v)\) be a parametric equation of \(S\). Then the vectors \(r_u\) and \(r_v\) are linearly independent and \(N_s\) is directed along the vector \([r_u, r_v]\). Thus, the assumption \([s, N_s] = 0\) on \(S\) implies that

\[ [r, [r_u, r_v]] = r_u(r, r_v) - r_v(r, r_u) = 0. \]

Since the vectors \(r_u\) and \(r_v\) are linearly independent, it follows that \((r, r_v) = (r, r_u) = 0\). Thus, \((r, r) = R^2\), where \(R^2\) is a constant. This means that \(S\) is a sphere. Lemma 1 is proved. \[\square\]

**References**


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