A problem in analysis

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Received: November 17, 2011

Summary: Assume that $D \subset \mathbb{R}^2$ is a bounded domain, diffeomorphic to a disc, star-shaped, with a $C^{1,\lambda}$ boundary $C$, $\lambda > 0$, which can be represented in polar coordinates as $r = f(\phi)$, where $f > 0$ is a smooth $2\pi$-periodic function. Let $\psi_{\pm n} := \psi_{\pm n}(\phi) := e^{\pm in\phi} f^n(\phi)$.

Theorem. Assume that

$$\int_0^{2\pi} \psi_{\pm n} f^2(\phi) d\phi = 0 \quad n = 1, 2, \ldots$$

Then $f = \text{const.}$

1 Formulation of the result

Assume that $D \subset \mathbb{R}^2$ is a bounded domain, diffeomorphic to a disc, star-shaped, with a $C^{1,\lambda}$ boundary $C$, $\lambda > 0$, which can be represented in polar coordinates as $r = f(\phi)$, where $f > 0$ is a smooth $2\pi$-periodic function. Let $\psi_{\pm n} := \psi_{\pm n}(\phi) := e^{\pm in\phi} f^n(\phi)$.

Theorem 1.1 Assume that

$$\int_0^{2\pi} \psi_{\pm n} f^2(\phi) d\phi = 0 \quad n = 1, 2, \ldots \quad (1.1)$$

Then $f = \text{const.}$

Remark 1.2 A similar result is true for $D \subset \mathbb{R}^m$, $m > 2$. Its proof is essentially the same.

Remark 1.3 The author raised the question, answered in Theorem 1.1, while thinking about the Pompeiu problem, see Chapter 11 in [1]. This question is of interest regardless of its relation to the Pompeiu problem since it gives an unusual result concerning completeness of a set of functions.

In Section 2 a proof is given.

AMS 2010 subject classification: 42C30

Key words and phrases: Completeness of a set of functions
2 Proof

Assumption (1.1) implies that

$$\int_{D} h_n dx = 0 \quad n = 1, 2, \ldots, \quad (2.1)$$

where $h_n := r^{|n|} e^{\pm in \phi}$ are harmonic functions regular at the origin, $x \in \mathbb{R}^2$, $x = (r, \phi)$, where $(r, \phi)$ are polar coordinates. To see that (1.1) is equivalent to (2.1), write the left-hand side of (2.1) in polar coordinates, integrate over $r$ from 0 to $f(\phi)$, and get (1.1).

Let $y \in \mathbb{R}^2$, $B_R$ be a ball (disc), centered at the origin and containing $D$ inside, $B'_R$ be its complement in $\mathbb{R}^2$, and $G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}$ be the fundamental solution of the Laplace equation in $\mathbb{R}^2$. Let

$$r := |x|, \quad r' := |y|, \quad x \cdot y = rr' \cos \theta.$$ 

Then, for $r > r'$, one has

$$2\pi G(x, y) = - \left[ \ln r + \frac{1}{2} \left( \ln \left(1 - \frac{r'}{r} e^{i\theta}\right) + \ln \left(1 - \frac{r'}{r} e^{-i\theta}\right) \right) \right], \quad r > r'. \quad (2.2)$$

Expanding $\ln(1 - \frac{r'}{r} e^{i\theta})$ in Taylor series, which is possible since $\frac{r'}{r} < 1$, one gets

$$\ln \left(1 - \frac{r'}{r} e^{i\theta}\right) = - \sum_{n=1}^{\infty} \frac{h_n}{n r^{n+1}}, \quad r > r', \quad h_n = (r')^n e^{\pm in \theta}. \quad (2.3)$$

We conclude from the assumption (2.1) and from (2.2)–(2.3) that

$$\int_{D} G(x, y) dy = - \frac{1}{2\pi} |D| \ln r, \quad r > R, \quad (2.4)$$

where $|D|$ denotes area of $D$.

Using the method from [2] (see also [3]) we derive from (2.4) that $D$ is a disc.

It follows from (2.4) that the harmonic in $D' = \mathbb{R}^2 \setminus D$ function

$$u(x) := \int_{D} G(x, y) dy = - \frac{1}{2\pi} |D| \ln r, \quad r > R, \quad (2.5)$$

solves the equation

$$\Delta u(x) = - \eta |D|, \quad (2.6)$$

where $\eta$ is the characteristic function of $D$, that is, $\eta = 1$ in $D$, and $\eta = 0$ in $D'$. Let $C_R$ be the boundary of $B_R$. A harmonic in $B_R$ function $h$ satisfies the conditions

$$\int_{C_R} h_N ds = 0, \quad \int_{C_R} h ds = 2\pi h(0). \quad (2.7)$$
It follows from (2.5) that the functions $u(x)$ and $u_N(x)$ are constant on $C_R$, since the normal $N$ on $C_R$ is directed along the radius. Multiply (2.6) by an arbitrary regular at the origin harmonic function $h = h_n$, integrate over a disc $B_R$, and use (2.7) to get

$$
\int_D h dx = \int_{C_R} (uh_N - u_N h) ds = ch(0), \quad c = const. \tag{2.8}
$$

If $h$ is harmonic in $B_R$, then so is $h(gx)$, where $g$ is a rotation by an arbitrary angle $\alpha$ around $z$-axis, the axis perpendicular to $D$. Since $h(g0) = h(0)$, one can replace $h(x)$ by $h(gx)$ in (2.8), differentiate with respect to $\alpha$ and then set $\alpha = 0$. This yields

$$
\int_D \nabla h(x) \cdot [e_3, x] dx = 0, \tag{2.9}
$$

where $e_3$ is a unit vector along $z$-axis, $\cdot$ stands for the scalar product, $[e_3, x]$ is the vector product in $\mathbb{R}^3$, and $h$ is an arbitrary harmonic function in $B_R$, regular at the origin. One has

$$
\nabla h(x) \cdot [e_3, x] = \nabla \cdot (h[e_3, x]), \tag{2.10}
$$

because $\nabla \cdot [e_3, x] = 0$. Thus, integrating by parts in (2.9), one gets

$$
\int_C (-N_1 s_2 + N_2 s_1) h ds = 0, \tag{2.11}
$$

where $N_j, j = 1, 2$, are the components of the outer unit normal $N$ to $C$. It is proved in [2] that the set of restrictions of all harmonic functions in $B_R$, regular at the origin, onto a closed curve $C \subset B_R$, diffeomorphic to a circle, is dense in $L^2(C)$. Therefore, (2.11) implies

$$
-N_1 s_2 + N_2 s_1 = 0 \quad \forall s \in C. \tag{2.12}
$$

Let us derive from equation (2.12) that $C$ is a circle. Geometrically equation (2.12) means that the radius-vector $r := s_1 e_1 + s_2 e_2$ of the boundary $C$ is parallel to the normal $N$ to $C$, namely, $[r, N] = 0$. The unit tangential vector to $C$ is $t = dr/ds$, where $s$ is the arclength of $C$, and the normal $N$ is directed along $dt/ds$.

Since the normal $N$ is orthogonal to $t$, and $N$ is parallel to $r$ according to (2.12), it follows that $t \cdot r = 0$. Thus,

$$
dr/ds \cdot r = 0 \quad \forall s \in C. \tag{2.13}
$$

Consequently,

$$
r \cdot r = const \quad \forall s \in C. \tag{2.14}
$$

Therefore, $C$ is a circle, and $D$ is a disc.

Theorem 1.1 is proved.
References


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