SOME RESULTS ON INVERSE SCATTERING

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Revised

A review of some of the author’s results in the area of inverse scattering is given. The following topics are discussed: 1) Property C and applications, 2) Stable inversion of fixed-energy 3D scattering data and its error estimate, 3) Inverse scattering with "incomplete" data, 4) Inverse scattering for inhomogeneous Schrödinger equation, 5) Krein’s inverse scattering method, 6) Invertibility of the steps in Gel’fand-Levitan, Marchenko, and Krein inversion methods, 7) The Newton-Sabatier and Cox-Thompson procedures are not inversion methods, 8) Resonances: existence, location, perturbation theory, 9) Born inversion as an ill-posed problem, 10) Inverse obstacle scattering with fixed-frequency data, 11) Inverse scattering with data at a fixed energy and a fixed incident direction, 12) Creating materials with a desired refraction coefficient and wave-focusing properties.

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1. Introduction

This paper contains a brief description of some of the author’s results in the area of inverse scattering. The proofs are omitted if the results were published. In this case references are given. For some new results proofs are given. The following problems are discussed:

1) Property C and applications,
2) Stable inversion of fixed-energy 3D scattering data and its error estimate,
3) Inverse scattering with "incomplete" data,
4) Inverse scattering for inhomogeneous Schrödinger equation,
5) Krein’s inverse scattering method,
6) Invertibility of the steps in Gel’fand-Levitan, Marchenko, and Krein inversion methods,
7) The Newton-Sabatier and Cox-Thompson procedures are not inversion methods,
8) Resonances: existence, location, perturbation theory,

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9) Born inversion as an ill-posed problem,
10) Inverse obstacle scattering with fixed-frequency data,
11) Inverse scattering with data at a fixed energy and a fixed incident direction,
12) Creating materials with a desired refraction coefficient and wave-focusing properties.

2. Property C and applications

Property C, completeness of the set of products of solutions to homogeneous equations, was introduced first in 26 and then applied to many inverse problems: 3D inverse scattering with fixed-energy data, inverse boundary-value problem, inverse problems for the heat and wave equations, impedance tomography problem, etc. (see 26–49, 54, 70 and references therein).

Definition 1. If \( L_1 \) and \( L_2 \) are two linear partial differential expressions (PDE), \( D \subset \mathbb{R}^n, n \geq 2, N_j = \{ u : L_j u = 0 \text{ in } D \}, j = 1, 2, \) then the pair \( \{ L_1, L_2 \} \) has Property C if the set of products \( \{ u_1 u_2 \}_{u_1 \in N_1} \) is total in \( L^2(D) \).

Necessary and sufficient condition for a pair \( \{ L_1, L_2 \} \) of PDE with constant coefficients to have Property C is given in 53, (see also 54, 70). This condition is easy to verify. If \( L_j := \{ z : z \in \mathbb{C}^n, L_j(z) = 0 \} \) is the algebraic variety, corresponding to the PDE \( L_j \) with constant coefficients and characteristic polynomial \( L_j(z) \), then the necessary and sufficient condition for a pair \( \{ L_1, L_2 \} \) of PDE with constant coefficients to have Property C can be stated as follows: the union of the algebraic varieties \( L_1 \) and \( L_2 \) is not a union of parallel hyperplanes in \( \mathbb{C}^n \). If the pair \( \{ L, L \} \) has property C, then we say that the operator \( L \) has this property. Classical operators \( \nabla^2, \partial_t - \nabla^2, \partial^2_t - \nabla^2, i \partial_t - \nabla^2 \) all have property C, Schrödinger pair \( \{ L_1, L_2 \} \) has property C, where \( L_j = \nabla^2 + k^2 - q_j(x), k = \text{const} \geq 0, j = 1, 2, q_j \in Q_a := \{ q_j(x) \in L^2(B_a), q_j \neq \emptyset, B_a := \{ x : |x| \leq a, x \in \mathbb{R}^2 \}, q_i = 0 \text{ if } |x| > a \} (54, 70).

Example of application of property C to inverse scattering. Let \( A_q(\alpha', \alpha, k) := A(\alpha', \alpha), k = \text{const} > 0 \) is fixed, be the scattering amplitude corresponding to \( q \in Q_a \). The inverse scattering problem with 3D fixed energy data consists of finding \( q \) given \( A(\alpha', \alpha) \) for all \( \alpha', \alpha \in S^2, S^2 \) is the unit sphere in \( \mathbb{R}^3 \). This problem has been open for several decades (from 1942). Below we write \( \beta \) in place of \( \alpha' \) sometimes. In 1987 the author proved that \( q \in Q_a \) is uniquely determined by \( A(\alpha', \alpha) \) (27, 49). The idea of the proof is simple. One uses the formula (see 54, p. 67):

\[
-4\pi A(\beta, \alpha) = \int_{\mathbb{R}^3} p(x)u_1(x, \alpha)u_2(x, \beta)dx,
\]

where \( A := A_1 - A_2, p(x) := q_1 - q_2, u_j \) is the scattering solution, corresponding to \( q_j, j = 1, 2, \) and \( A_j \) is the corresponding scattering amplitude. This formula is derived by using the formula

\[
G(x, y) = e^{ik|y|} \frac{1}{4\pi|y|} u(x, \alpha) + o(\frac{1}{|y|}), \quad |y| \to \infty, \quad \frac{y}{|y|} = -\alpha,
\]
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This is an orthogonality relation:

\[ p(x)u_1(x, \alpha)u_2(x, \beta)dx = 0 \quad \forall \alpha, \beta \in S^2. \]

This is an orthogonality relation: \( p(x) \) is orthogonal to the set \( \{u_1u_2\}_{\alpha, \beta \in S^2} \). The set \( \{u_j(x, \alpha)\}_{\alpha \in S^2} \) is total in the set \( N_j := \{u : l_ju = [\nabla^2 + k^2 - q_j(x)]u = 0 \} \) in \( B_a \). The pair \( \{L_1, L_2\} \) has property \( C \). Thus, \( p(x) = 0 \), and the uniqueness theorem for inverse scattering with fixed-energy data is proved.

Let us give an example of applications of property \( C \) to inverse boundary-value problem. Let

\[ Lu := [\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in} \quad D \subset \mathbb{R}^3, \]

where \( D \) is a bounded domain with a smooth boundary \( S, u \big|_S = f \). Assume that zero is not a Dirichlet eigenvalue of \( L \) in \( D \). Then \( f \) defines \( u = u(x; f) \) uniquely, and the map \( \Lambda : f \rightarrow h := u_N \), where \( u_N \) is the normal derivative of \( u \) on \( S \), is well defined. The inverse problem is:

**Given the set \( \{f,h\}_{f \in H^{3/2}(S)} \) can one determine \( q \) uniquely?**

The answer is yes. Indeed, if \( q_1 \) and \( q_2 \) generate the same set \( \{f,h\}_{f \in H^{3/2}(S)} \), i.e., the same \( \Lambda \), then one derives, as above, the orthogonality relation

\[ \int_D p(x)u(x; f)v(x)dx = 0 \quad \forall f \in H^{3/2}(S), \quad \forall v \in N_2, \]

where \( p := q_1 - q_2 \). The set \( \{uv\} \) is total in \( L^2(D) \) by property \( C \) for the pair \( \{L_1, L_2\} \). Thus, \( p = 0 \), and the uniqueness is proved. Many other examples one finds in \( 54, 70 \).

### 3. Stable solution of 3D inverse scattering problem

If \( q \in Q := Q_0 \cap L^\infty(B_a) \) and \( A(\beta, \alpha) \) is the exact scattering amplitude at a fixed \( k > 0 \), then it is proved in \( 25 \) that \( A \) admits an analytic continuation from \( S^2 \times S^2 \) to the algebraic variety \( M_k := \{\Theta : \Theta \in \mathbb{C}^3, \Theta \cdot \Theta = k^2\} \), where \( \Theta \cdot \Theta = \sum_{j=1}^3 \Theta_j^2 \).

This implies that the knowledge of \( A(\beta, \alpha) \) on \( S^2 \times S^2 \), where \( S^2 \times S^2 \), \( j = 1, 2 \), are arbitrary small open subsets of \( S^2 \), determines \( A(\beta, \alpha) \) uniquely on \( S^2 \times S^2 \).

For any \( \xi \in \mathbb{R}^3 \) there exist (many) \( \Theta', \Theta \in M_k \) such that \( \Theta' - \Theta = \xi, |\Theta| \rightarrow \infty \). We take \( k = 1 \) in this Section without loss of generality. Let \( M := M_1, \tilde{q}(\xi) := \int_{B_a} e^{-ik \cdot x} q(x)dx, Y_{\ell}(\Theta) = Y_{\ell m}(\Theta), -\ell \leq m \leq \ell \), are orthonormal spherical harmonics, \( A_\ell(\alpha) := \int_{S^2} A(\beta, \alpha)Y_\ell(\beta)d\beta \), the overbar stands for complex conjugate, \( h_{\ell}(r) = e^{i\frac{\pi}{2}(\ell+1)} \sqrt{\frac{\ell + \frac{1}{2}}{2\pi}} H_\ell^{(1)}(r) \) is the Hankel function, \( \sum_{\ell=0}^\infty := \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell b \) and \( a_1 \) are arbitrary numbers satisfying the inequality \( b > a_1 > a \).

Let \( \|p\| := \|p\|_{L^2(B_a \setminus B_{a_1})}, \rho := e^{-i\Theta x} \int_{S^2} u(x, \alpha)\nu(\alpha)d\alpha - 1, \nu \in L^2(S^2), \Theta \in M \). Let \( d(\Theta) = \inf_{\nu \in L^2(S^2)} \|\rho(\nu)\| \). One has

\[ d(\Theta) \leq c|\Theta|^{-1}, \quad \Theta \in M, \quad |\Theta| \rightarrow \infty, \]
where \( c = \text{const} > 0 \) is independent of \( \Theta \), see \(^{49}\). Let \( \Theta' \in M, \Theta' - \Theta = \xi \), where \( \xi \in \mathbb{R}^3 \) is an arbitrary vector. If \( \|\rho(\nu)\| < 2d(\Theta) \), \( \nu = \nu(\alpha, \Theta) \), and
\[
\tilde{q} := -4\pi \int_{S^2} A(\Theta', \alpha) \nu(\alpha, \Theta) d\alpha,
\]
then
\[
|\tilde{q}(\xi) - \tilde{q}| \leq \frac{c}{|\Theta|}, \quad |\Theta| \to \infty,
\]
where \( c > 0 \) stands for various constants. Thus, \(^{70}\):
\[
-4\pi \lim_{\theta' - \theta = \xi, \|\nu(\alpha, \Theta)\| < 2d(\Theta)} \int_{S^2} A(\Theta', \alpha) \nu(\alpha, \Theta) d\alpha = \tilde{q}(\xi).
\]
If \( \max_{\beta, \alpha \in S^2} |A_1(\beta, \alpha) - A_2(\beta, \alpha)| < \delta \), then
\[
\max_{\xi \in \mathbb{R}^3} |\tilde{q}_1(\xi) - \tilde{q}(\xi)| < c \left| \frac{\ln |\ln \delta|}{|\ln \delta|} \right|,
\]
as was proved in \(^{55}\).

Suppose the "noisy data" \( A_\delta \) are given, \( \sup_{\beta, \alpha \in S^2} |A_\delta(\beta, \alpha) - A(\beta, \alpha)| < \delta \), where \( A_\delta \) is not necessarily a scattering amplitude, \( A(\beta, \alpha) \) is the scattering amplitude corresponding to \( q \in Q \). The author’s method for calculating a stable estimate of \( \tilde{q}(\xi) \) is as follows.

Let \( N(\delta) := \left\lfloor \frac{\ln \delta}{\ln x} \right\rfloor \), \( x \) is the integer closest to \( x > 0 \),
\[
\hat{A}_\delta(\Theta', \alpha) := \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(\Theta'), \quad u_\delta(x, \alpha) := e^{i\alpha \cdot x} + \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(x^0) h_\ell(r),
\]
where \( r := |x|, x^0 := \frac{x}{r}, \Theta', \Theta \in M, \Theta' - \Theta = \xi \)
\[
\rho_\delta := e^{-i\Theta' \cdot x} \int_{S^2} u_\delta(x, \alpha) \nu(\alpha) d\alpha - 1, \quad \nu \in L^2(S^2),
\]
\[
\mu(\delta) := e^{-\gamma\alpha \cdot x}, \quad \gamma := \ln \frac{\alpha}{\kappa} > 0, \quad a(\nu) := \|\nu\|_{L^2(S^2)}, \quad \kappa = |\text{Im } \Theta|,
\]
\[
F(\nu, \Theta) := \|\rho_\delta(\nu)\| + a(\nu) e^{\alpha \cdot \mu(\delta)} = \inf_{\nu \in L^2(S^2), \Theta = \Theta_\delta} := t(\delta),
\]
\[
\tau(\delta) := \left( \frac{\ln |\ln \delta|}{|\ln \delta|} \right)^2.
\]
We prove that \( t(\delta) = O(\tau(\delta)) \) as \( \delta \to 0 \), and \( t(\delta) \) is independent of \( \xi \). If \( \nu = \nu_\delta(\alpha) \in L^2(S^2) \) and \( \Theta = \Theta_\delta \) are such that \( F(\nu_\delta, \Theta_\delta) < 2t(\delta) \), and
\[
\tilde{q}_\delta := -4\pi \int_{S^2} \hat{A}_\delta(\Theta'_\delta, \alpha) \nu_\delta(\alpha) d\alpha,
\]
then
\[
\sup_{\xi \in \mathbb{R}^3} |\tilde{q}_\delta - \tilde{q}(\xi)| \leq c \tau(\delta), \quad 0 < \delta \ll 1,
\]
where \( c = \text{const} > 0 \) depends only on a norm of \( q \) (see \(^{65}\) and \(^{79}\)).
4. Inverse scattering with "incomplete" data

4.1. Spherically symmetric potentials

Let \( q \in Q \) be spherically symmetric. It was proved in \(^{50,52}\) that a necessary and sufficient condition for \( q \in Q \) to be spherically symmetric is \( A(\beta, \alpha) = A(\beta \cdot \alpha) \). It was known for decades that if \( q(x) = q(|x|) \) then \( A(\beta, \alpha) = A(\beta \cdot \alpha) \). This follows easily from the separation of variables. The converse is a non-trivial fact, which follows from the author’s uniqueness theorem for inverse scattering problem. If \( q = q(r) \), \( r = |x| \), and \( q \in Q \), then the knowledge of the scattering amplitude \( A(\beta \cdot \alpha) \) is equivalent to the knowledge of all fixed-energy phase shifts \( \delta \ell, A^\ell(\alpha) = A^\ell(Y^\ell(\alpha)), A^\ell = 4\pi e^{i\delta \ell} \sin \delta \ell, k = 1 \). One can find the radius \( a \) of the ball \( B_a \), out of which \( q = 0 \), by the formula (70, p. 173):

\[
a = 2e^{-1} \lim_{\ell \to \infty} (\ell |\delta \ell|^1)^{1/2},
\]

and the author has proved that if \( L \) is any subset of positive integers such that

\[
\sum_{\ell \in L} \ell^{-1} = \infty,
\]

then the set \( \{\delta \ell\}_{\ell \in L} \) determines \( q \in Q \) uniquely (\(^{62}\)). He conjectured that (4.1) is necessary for the uniqueness of the recovery of \( q \). This conjecture was proved in \(^4\). Examples of two quite different, piecewise-constant potentials \( q_j(r) \), generating practically the same sets of fixed-energy phase shifts, are given in \(^{64,61}\). "Practically the same" means here that \( \max_{\ell \geq 0} |\delta^{(1)}_\ell - \delta^{(2)}_\ell| < 10^{-5} \).

4.2. Rapidly decaying potentials

To find \( q(x) \in L_{1,1} := \{q : q = q, \int_0^\infty x|q(x)|dx < \infty\} \) in the 1D inverse scattering problem on a half-line one needs the following scattering data

\[
S := \{S(k) := \left. \frac{f(-k)}{f(k)} \right|_{vk \geq 0}, k_j, s_j, 1 \leq j \leq J\},
\]

where \( f(k) \) is the Jost function, \( k_j > 0 \) are constants, \(-k_j^2\) are the bound states, \( f(ik_j) = 0, 1 \leq j \leq J, s_j > 0 \) are the norming constants,

\[
s_j = -\frac{2ik_j}{f(ik_j)f'(0,ik_j)},
\]

where

\[
f(x,k) = e^{ikx} + \int_x^\infty A(x,y)e^{iky}dy
\]
is the Jost solution, \( A(x,y) \) is the transformation kernel, \( f(k) := f(0,k), \dot{f}(k) := \frac{df}{dk}, f'(0,k) := \left. \frac{df(x,k)}{dx} \right|_{x=0} \).

Given \( S \), one solves the Marchenko (M) equation for \( A(x,y) \):

\[
A(x,y) + \int_y^\infty A(x,s)F(s+y)dt + F(x+y) = 0, \quad 0 \leq x \leq y < \infty,
\]
where the function $F$ is expressed via the scattering data as follows:

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 - S(k)]e^{ikx}dk + \sum_{j=1}^{J} s_{j}e^{-k_{j}x},$$

and then the potential is found by the formula

$$q(x) = -2 \frac{dA(x,x)}{dx}.$$  

If one knows that $q = 0$ for $x > a$, then any of the data $\{S(k)\}$, $\{f(k)\}$, $\delta(k)$, $f'(k)$, $\forall k \geq 0$, where $\delta(k)$ is the phase shift, determine $q$ uniquely (70, p. 180). The phase shift $\delta(k)$ is defined by the relation $S(k) = e^{2i\delta(k)}$.

In fact, a weaker assumption was made in 70: it was assumed that $|q(x)| \leq c_{1}e^{-c_{2}|x|}$, where $\gamma > 1$ and $c_{1}, c_{2}$ are positive constants. This assumptions implies that $f(k)$ is an entire function, $S(k)$ is meromorphic, and the only poles of $S(k)$ in $\mathbb{C}_{+} = \{k : \text{Im} k > 0\}$ are $ik_{j}$, $1 \leq j \leq J$. Thus, $S(k)$ determines $J$ and $k_{j}$ uniquely, and $s_{j} = i\text{Res}_{k=ik_{j}}S(k)$ are also uniquely determined. Therefore, $q$ is uniquely determined by $\{S(k)\}_{k \geq 0}$. We refer the reader to 70 for the uniqueness proof in the cases of other data.

4.3. Potentials vanishing on half-line

If the inverse scattering on the full line is considered, then the scattering data are

$$\left\{r(k) \mid \forall k \geq 0, k_{j}, s_{j}, 1 \leq j \leq J \right\},$$

where $r(k)$ is the reflection coefficient, $-k_{j}^{2}$ are the bound states, and $s_{j} > 0$ are norming constants (see 8 and 54, p. 284). These data determine $q \in L_{1,1}(\mathbb{R})$ uniquely. However, if one knows that $q(x) = 0$ for $x < 0$, then the data $\{r(k)\}_{k \geq 0}$ alone determine $q$ on $\mathbb{R}_{+} = [0, \infty)$ uniquely 70, p. 181. Indeed, if $q = 0$ on $\mathbb{R}_{-}$, then the scattering solution on $\mathbb{R}$ is $u = e^{ikx} + r(k)e^{-kx}$ for $x < 0$ and $u = t(k)f(x,k)$ for $x > 0$, where $t(k)$ is the transmission coefficient, which is unknown, and $f(x,k)$ is the Jost solution. Thus

$$\frac{ik[1 - r(k)]}{1 + r(k)} = \frac{u'(-0,k)}{u(0,k)} = \frac{u'(0,k)}{u(0,k)} = \frac{f'(0,k)}{f(0,k)} := I(k).$$  \hspace{1cm} (4.3)

Therefore, $r(k)$ determines uniquely the $I$-function $I(k)$. This function determines $q$ uniquely 70, p. 108. The proof in 70 is based on $C_{+}$ property of the pair $\{\ell_{1}, \ell_{2}\}$, $\ell_{j} := -\frac{d^{2}}{dx^{2}} + q_{j}(x) - k^{2}$. This property holds for $q_{j} \in L_{1,1}$ and says that the set $\{f_{1}(x,k), f_{2}(x,k)\}_{k \geq 0}$ is complete (total) in $L^{1}(\mathbb{R}_{+})$ 70, p. 104.

4.4. Potentials known on a part of the interval

Consider the equation $\ell u := -u'' + q(x)u = k^{2}u$, $x \in [0,1]$, $u(0) = u(1) = 0$, $q = \varphi \in L^{1}([0,1])$. Fix $0 < b \leq 1$. Assume $q$ on $[b,1]$ known and the subset
\[ \{ \lambda_{m(n)} \}_{n=1,2,3...} \] of the eigenvalues \( \lambda_n = k_n^2 \) of \( \ell \) is known, where \( m(n)/n = 1/\sigma (1+\varepsilon_n) \), \( \sigma = \text{const} > 0, |\varepsilon_n| < 1, \sum_{n=1}^{\infty} |\varepsilon_n| < \infty. \]

**Theorem 1.** (59, 69, 70, p. 176) If \( \sigma \geq 2b \), then the above data determine \( q \) on \([0,b]\) uniquely.

For example, if \( \sigma = 1 \) and \( b = \frac{1}{2} \), then the theorem says that \( q \) is uniquely determined by one spectrum.

5. Inverse scattering for inhomogeneous Schrödinger equation

Let \( \ell u - k^2 u := -u'' + q(x)u - k^2 u = \delta(x), x \in \mathbb{R}^1, \lim_{x \to \infty} \left( \frac{\partial u}{\partial x} - iku \right) = 0 \). Assume \( q = \bar{q}, q = 0 \) for \( |x| > 1 \), \( q \in L^\infty[-1,1] \). Suppose the data \( \{u(-1,k), u(1,k)\}_{k \geq 0} \) are given. It is proved in 60 that these data determine \( q(x) \) uniquely (see also 70, p. 204).

6. Krein’s inversion method

In 70, p. 186, apparently for the first time, Krein’s inversion method (see 5, 6) was presented with detailed proofs, and it was proved additionally that this method yields the unique potential which reproduces the original scattering data \( S \). For simplicity, let us describe Krein’s method assuming that there are no bound states, so that the scattering data are \( \{S(k)\}_{k \geq 0} \). Then Krein’s method can be described as follows:

Given \( S(k) \) with \( \text{ind}_R S(k) = 0 \), where \( \text{ind}_R S(k) \) is the index of \( S(k) \), one finds \( f(k) \) by the formula

\[ f(k) = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln S(-y) \frac{dy}{y-k} \right\}, \quad \text{Im} \ k > 0, \]

then one calculates

\[ H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikt} \left( \frac{1}{|f(k)|^2} - 1 \right) dk, \]

then one finds \( \Gamma_x(t,s) \) by solving the equations:

\[ \Gamma_x(t,s) + \int_0^x H(t-u) \Gamma_x(u,s) du = H(t-s), \quad 0 \leq t, s \leq x. \quad (6.1) \]

This equation is uniquely solvable if

\[ S(k) = S(-k) = S^{-1}(k), \quad k \in \mathbb{R}; \quad \text{ind}_R S(k) = 0, \quad (6.2) \]

and

\[ \| F(x) \|_{L^\infty(\mathbb{R}^+)} + \| F(x) \|_{L^1(\mathbb{R}^+)} + \| xF'(x) \|_{L^1(\mathbb{R}^+)} < \infty, \quad (6.3) \]

where \( F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 - S(k)] e^{ikx} dk \). From \( \Gamma_x(t,s) \) one calculates \( 2\Gamma_{2x}(x,0) := a(x) \) and, finally, \( q(x) = a^2(x) + a'(x) \). In 70, p. 197 it is proved that the steps
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of the above inversion procedure are invertible, and the constructed potential generates the original data $S(k)$ if conditions (6.2)-(6.3) hold, the case when bound states are present is considered, and the advantages of Krein’s method for numerical implementation are discussed.

7. Invertibility of the steps in Gelfand-Levitan, Marchenko, and Krein’s methods

The Gel’fand-Levitan (GL) method for finding $q(x)$ given the corresponding spectral function $\rho(\lambda)$, consists of the following steps:

$$\rho \Rightarrow L \Rightarrow K \Rightarrow q,$$

(7.1)

where

$$L(x,y) = \int_{-\infty}^{\infty} \varphi_0(x,\lambda) \varphi_0(y,\lambda) d\sigma(\lambda), \quad d\sigma = d(\rho - \rho_0),$$

$$K(x,y)$$ is found from the GL equation

$$K(x,y) + \int_{0}^{x} K(x,s) L(s,y)ds + L(x,y) = 0, \quad 0 \leq y \leq x,$$

(7.2)

and the potential is found by the formula

$$q(x) = 2 \frac{dK(x,x)}{dx}.$$

Let us assume that the spectral functions $\rho(\lambda)$ satisfy two assumptions:

$A_1$) If $h \in L_0^2(\mathbb{R}^+) \text{ is arbitrary, } H(\lambda) := \int_{0}^{\infty} h(x) \varphi_0(x,\lambda) dx$, and $\int_{-\infty}^{\infty} H^2(\lambda) d\rho(\lambda) = 0$, then $h = 0$. Here $\varphi_0(x,\lambda) = \frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}$.

$A_2$) If $\int_{-\infty}^{\infty} |H(\lambda)|^2 d(\rho_1 - \rho_2) = 0 \forall h \in L_0^2(\mathbb{R}^+)$, then $\rho_1 = \rho_2$.

These assumptions are satisfied, for example, for the operators $l_j = -\frac{d^2}{dx^2} + q_j(x)$, which are in the limit-point at infinity. Under these assumptions it is proved in $70$, p. 128, 69, that each step in (7.1) is invertible:

$$\rho \Leftrightarrow L \Leftrightarrow K \Leftrightarrow q$$

Methods for calculating $d\rho$ from $S$, the scattering data, and $S$ from $d\rho$, are given in $70$, p. 131, and the set of spectral functions, corresponding to $q \in H_m^{m+1}(\mathbb{R}^+)$ is characterized: these are the $\rho$’s satisfying assumptions $A_1$) and $A_3$), where $A_1$) is stated above and $A_3$) is the following assumption: the function $L(x) \in H_m^{m+1}(\mathbb{R}^+)$, where

$$L(x) := \int_{-\infty}^{\infty} \frac{1 - \cos(x\sqrt{\lambda})}{2\lambda} d(\rho - \rho_0),$$

so that $L(x) = L(\frac{\pi}{2}, \frac{\pi}{2})$, where $L(x,y)$ is defined below (7.1).
The Marchenko (M) inversion consists of the following steps:

\[ S \Rightarrow F \Rightarrow A \Rightarrow q, \quad (7.3) \]

where \( S := \{ S(k), k_j, s_j, 1 \leq j \leq J \} \) are the scattering data,

\[
F(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 - S(k)]e^{ikx}dk + \sum_{j=1}^{J} s_j e^{-k_j x},
\]

\( A = A(x, y) \) is the (unique) solution of the equation (4.2), Section 4.2, and

\[
q(x) = -2 \frac{dA(x, x)}{dx}.
\]

It is proved in \(70\), p. 143, that each step in (7.3) is invertible if \( q \in L_{1,1} \). The scattering data, corresponding to \( q \in L_{1,1} \), are characterized by the following conditions:

a) \( \text{ind}_{R} S(k) = \begin{cases} -2J & \text{if } f(0) \neq 0 \\ -2J - 1 & \text{if } f(0) = 0, \end{cases} \)

b) \( k_j > 0, s_j > 0, 1 \leq j \leq J; S(k) = S(-k) = S^{-1}(k), k \geq 0, S(\infty) = 1, \)

c) Inequality (6.3) holds.

Let \( A(y) := A(0, y) \), where \( A(x, y) \) is the transformation kernel defined in Section 4.2. It is proved in \(70\), p. 147, that \( A(y) \) solves the equation:

\[
F(y) + A(y) + \int_{-\infty}^{\infty} A(t)F(t + y)dt = A(-y), \quad -\infty < y < \infty. \quad (7.4)
\]

If conditions a) - c) hold and \( k(\|f(k)\|^2 - 1) \in L^2(\mathbb{R}_+) \), then \( q \in L^2(\mathbb{R}_+) \).

We have mentioned invertibility of the steps in Krein’s inversion method in Section 5. The proof of these results is given in \(70\).

8. The Newton-Sabatier and Cox-Thompson procedures are not inversion methods.

The Newton-Sabatier and Cox-Thompson procedures \((0, 1, 2)\) for finding \( q \) from the set of fixed-energy phase shifts are not inversion methods: it is not possible, in general, to carry these procedures through, and it is not proved that if these procedures can be carried through, then the obtained potential reproduces the original phase shifts. These procedures are fundamentally wrong because their basic assumptions are wrong: the integral equation, used in these procedures, in general, is not uniquely solvable for some \( r > 0 \), and then the procedures break down. A detailed analysis of the Newton-Sabatier procedure is given in \(66, 68, 70\), p. 166, and a counterexample to the uniqueness claim in \(2\) is given in \(67\).
9. Resonances

If \( q \in Q_a \), then the numbers \( k \in \mathbb{C} \) \( = \{ z : \text{Im} z \leq 0 \} \), for which the equation \( u + T(k)u = 0 \) has non-trivial solutions, are called resonances. Here

\[
T(k)u = \int_{B_a} g(x, y, k)q(y)u(y)dy, \quad g(x, y, k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}.
\]

In the one-dimensional case resonances are zeros of the Jost function \( f(k) \) in \( \mathbb{C} \). It was proved in \(^{11}\) that if \( q \in Q_a \cap C^1(B_a) \) then there are no resonances in the region \( \text{Im} k > c - b \ln |k| \) for large \( |k| \), where \( b > 0 \) and \( c \) are constants. In \(^{12},^{13},^{14},^{15},^{16},^{17},^{18},^{19},^{23},^{24},^{25}\) methods for calculating resonances, perturbation theory for resonances, variational principles for resonances, asymptotics of resonant states, and the relation to eigenmode and singularity expansion methods are given. In \(^{20}\) existence of infinitely many purely imaginary resonances is proved for \( q \in Q_a \). In \(^{54}\), pp. 278–283, the following results are proved:

1. If \( q \in L_{1,1} \) then \( q = 0 \) for \( x > 2a \) if and only if the corresponding Jost function \( f(k) \) is entire, of exponential type \( \leq 2a \), bounded in \( \mathbb{C}_+ \) and \( \lim_{|k| \to \infty, k \in \mathbb{C}_+} f(k) = 1 \);
2. Let \( Q_n := \int_0^{\infty} x^n |q(x)|dx \). If \( Q_n = O(n^b) \), \( 0 \leq b < 1 \), and \( q \neq 0 \), then \( q \) generates infinitely many resonances;
3. Let \( a > 0 \) be arbitrarily small fixed number. There exists \( q = \tilde{q} \in C_0^\infty(0,a) \), which generates infinitely many purely imaginary resonances.

Note that any \( q \in C_0^\infty(0,\infty), q = \tilde{q}, \text{supp} \ q \subset [0,a], \text{which does not change sign in an arbitrary small left neighborhood of} \ a, \text{i.e.}, \ (a - \delta, a), \text{where} \ \delta > 0 \text{ is arbitrarily small, cannot produce infinitely many purely imaginary resonances.}

Example of application of the result (2): if \( |q(x)| \leq c_1 e^{-\gamma x^\gamma}, \ x \geq 0, \ \gamma > 1, \) then \( q \) generates infinitely many resonances. Indeed, if \( t^\gamma = y \) and \( c := c_2, \) then

\[
Q_n = c_1 \int_0^{\infty} x^n e^{-\gamma x^\gamma} dx \leq c_1 \frac{1}{e^{\gamma n}} \int_0^{\infty} (cx)^n e^{-(cx)^\gamma} dx = c_1 \frac{1}{e^{\gamma n}} \int_0^{\infty} t^n e^{-t^\gamma} dt
\]

\[
= \frac{c_2}{\gamma n} \int_0^{\infty} y^{\frac{n+1}{\gamma}} e^{-y} dy = c_2 \frac{\Gamma(\frac{n+1}{\gamma})}{\gamma n} \leq c_3 n^{\beta n}, \ b = \frac{1}{\gamma} > 1.
\]

Thus, if \( q \in Q_a \), then it generates infinitely many resonances.

10. Born inversion is always an ill-posed problem

Born inversion has been quite popular among engineers and physicists. The exact scattering amplitude is:

\[
-4\pi A(\beta, \alpha, k) = \int_D e^{-ik\beta x} u(x, \alpha, k)q(x)dx,
\]

\[(10.1)\]
where $u(x, \alpha, k)$ is the scattering solution. In the Born approximation one replaces $u(x, \alpha, k)$ in (10.1) by the incident field $u_0 = e^{ik \alpha \cdot x}$ and gets:

$$-4\pi A(\beta, \alpha, k) \approx \int_D e^{-i k (\beta - \alpha) \cdot x} q(x) \, dx,$$

(10.2)

where the error of this formula is small if $q$ is small in some sense, specified in 51. If $A(\beta, \alpha, k)$ is known for all $k > 0, \beta, \alpha \in S^2$, then, for any $\xi \in \mathbb{R}^3$, one can find (non-uniquely) $k, \beta, \alpha$ so that $\xi = k(\alpha - \beta)$, and (10.2) gives the equation $\tilde{q}(\xi) = \int_D e^{-i \xi \cdot x} q(x) \, dx$. In 51 it was pointed out that although the direct scattering problem can be solved in the Born approximation if $q$ is small quite accurately, and the error estimate for this solution can be derived easily, the inverse scattering problem in the Born approximation is always an ill-posed problem, no matter how small $q$ is. In 51 and in 70, p. 307, this statement is explained and a stable inversion scheme in the Born approximation is given with an error estimate, provided that $q$ is small. It is also explained that collecting very accurate scattering data is not a good idea if the inversion is done in the Born approximation: in this approximation even the exact data have to be considered as noisy data.

Let us state and prove just one Theorem that makes the above easier to understand.

**Theorem 2.** If $q = \bar{q}$ and $|q(x)| \leq c(1 + |x|)^{-b}, b > 3$, then the equation

$$-4\pi A(\beta, \alpha, k) = \int_D e^{-i k (\beta - \alpha) \cdot x} q(x) \, dx$$

(10.3)

implies $q = 0$.

**Proof.** The exact scattering amplitude satisfies the relation (optical theorem):

$$4\pi \text{Im} A(\beta, \beta, k) = k \int_{S^2} |A(\beta, \alpha, k)|^2 \, d\alpha.$$

(10.4)

From (10.3) and the assumption $q = \bar{q}$ it follows that $\text{Im} A(\beta, \beta, k) = 0$, so (10.4) implies

$$A(\beta, \alpha, k) = 0.$$  

(10.5)

If (10.5) holds for all $\beta, \alpha \in S^2$ and $k > 0$, then $q = 0$. If $q \in Q_0$ and (10.5) holds for a fixed $k > 0$ and all $\beta, \alpha \in S^2$, then $q = 0$ by the Ramm’s uniqueness theorem from Section 3.

11. Inverse obstacle scattering with fixed-frequency data.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary $S$ (or less smooth boundary, a boundary with finite perimeter, see 70, pp. 227-234). Let $A(\beta, \alpha)$ be the corresponding scattering amplitude at a fixed $k > 0$. The boundary condition on $S$, which we denote $\Gamma$, is homogeneous, one of the three types: the Dirichlet ($D$): $u|_S = 0$, the Neumann ($N$): $u_N|_S = 0$, the Robin ($R$): $(u_N + h(s)u)|_S = 0$. 

Here $N$ is the outer unit normal to $S$, $h(s)$ is a piecewise-continuous bounded function, $\text{Im} \ h \geq 0$. The last inequality guarantees the uniqueness of the solution to the scattering problem. This solution for nonsmooth $S$ is understood in the weak sense. The inverse scattering problem consists of finding $S$ and the boundary condition $(D)$, $(N)$ or $(R)$ type, and the function $h$ if condition $(R)$ holds. The uniqueness of the solution to this problem was first proved by the author (25) for Lyapunov boundaries and then in 57 for boundaries with finite perimeter. For such boundaries the normal is understood in the sense of Federer 3, 70, p. 227. A domain $D \subset \mathbb{R}^3$ has finite perimeter if $S := \partial D$ has finite Hausdorff $H_2(S)$ measure, i.e., $H_2(S) = \lim_{\varepsilon \to 0} \inf_{\bar{S}} \sum_j r_j^2 < \infty$, where the infimum is taken over all coverings of $S$ by open two-dimensional balls of radii $r_i < \varepsilon$. For domains with finite perimeter Green’s formula holds:

$$\int_D \nabla u \, dx = \int_{\partial D} N(s) \, u(s) H_2(ds),$$

where $\partial D$ is the reduced boundary of $D$, i.e. the subset of points of $\partial D$ at which the normal in the sense of Federer exists.

It is proved in 57 that if $D_1$ and $D_2$ are two bounded domains with finite perimeter, $A_j$ are the corresponding scattering amplitudes, and $u_j(x, \alpha)$ are the corresponding scattering solutions, then

$$4\pi [A_1(\beta, \alpha) - A_2(\beta, \alpha)] = \int_{S_{12}} [u_1(s, -\beta)u_{2N}(s, \alpha) - u_1N(s, -\beta)u_2(s, \alpha)]ds,$$  \hspace{1cm} (11.1)

where $S_{12} := \partial D_{12}$, $D_{12} := D_1 \cup D_2$, and we assume $S_j$ Lipschitz for simplicity of the formulation of the result. If $A_1 = A_2 \ \forall \alpha, \beta \in S^2$, then (11.1) yields

$$0 = \int_{S_{12}} [u_1(s, -\beta)u_{2N}(s, \alpha) - u_1N(s, -\beta)u_2(s, \alpha)]ds \ \forall \alpha, \beta \in S^2.$$  \hspace{1cm} (11.2)

From (11.2) by the "lifting" method 70, p. 236, one derives that

$$G_j(x, y) = G_2(x, y) \ \forall x, y \in D'_2 := \mathbb{R}^3 \setminus D_{12},$$  \hspace{1cm} (11.3)

where $G_j$, $j = 1, 2$ are Green’s functions for the operator $\nabla^2 + k^2$ in domain $D_j$, corresponding to the boundary condition $\Gamma_j$ ($\Gamma_j = (D), (N)$ or $(R)$). If $S_1 \neq S_2$ then (11.3) leads to a contradiction, which proves that $S_1 = S_2$. Indeed, one takes a point $x \in S_1 \cap D'_2$, and let $y \to x$. Since $x \notin D_2$ one has

$$|\Gamma_1 G_2(x, y)| \to \infty \ \text{and} \ \Gamma_1 G_1(x, y) = 0 \ \forall y \in D'_1.$$ 

This contradiction proves that $S_1 = S_2 = S$, so $D_1 = D'_2 = D$, $\Gamma_1 = \Gamma_2 := \Gamma$.

If $\Gamma_1(x, y) = 0$, $x \in S$, $y \in D'$, then $\Gamma = (D)$.

If $G_N(x, y) = 0$, $x \in S$, $y \in D'$, then $\Gamma = (N)$.

If $\frac{\partial u}{\partial n} := -h$ on $S$, then $\Gamma = (R)$ and $h$ on $S$ is recovered.

In 70, p. 237, a more complicated problem is treated: let

$$Lu := \nabla \cdot (a(x)\nabla u) + q(x)u = 0 \ \text{in} \ \mathbb{R}^3,$$  \hspace{1cm} (11.4)
where

\[ a(x) = a^+ \text{ in } D, \ a(x) = a^- := a_0 \text{ in } D' := \mathbb{R}^3 \setminus D, \ \ q(x) = k^2 a^+ := q^+ \text{ in } D, \]

\[ q(x) = k_0^2 a_0 := q^- \text{ in } D', \quad a^\pm, q^\pm, k_0, k > 0 \text{ are constants.} \quad (11.5) \]

\[ a^+ u_N^+ = a_0 u_N^0 \text{ on } S, \quad a^+ \neq a^-, \]

\[ u = u_0 + A(\beta, \alpha, k_0) e^{ik_0 r} + o(x), \quad r := |x| \to \infty, \ u_0 := e^{ik_0 x}. \quad (11.8) \]

Assume \( k_0 > 0 \) is fixed, denote \( A(\beta, \alpha, k_0) := A(\beta, \alpha) \).

Theorem 3. (70, p. 237) The data \( A(\beta, \alpha) \ \forall \beta, \alpha \in S^2, \ a_0 \text{ and } k_0 \) determine \( S, a^+, \) and \( k \) uniquely.

The proof is based on the result in 58 on the behavior of fundamental solutions to elliptic equations with discontinuous senior coefficients. In 56 stability estimates are obtained for the recovery of \( S \) from the scattering data and an inversion formula is given. Let us formulate these results. Consider two star-shaped obstacles \( D_j \) with boundaries \( S_j \) which are described by the equations

\[ r = f_j(\alpha), \quad r = |x|, \quad \alpha = \frac{x}{|x|}, \quad j = 1, 2. \]

Assume that \( 0 < c \leq f_j(\alpha) \leq C, \forall \alpha \in S^2, \) and \( S_j \in C^{2, \lambda}, 0 < \lambda \leq 1, \) with \( C^{2, \lambda} \)-norm of the functions, representing the boundaries \( S_j \) bounded by a fixed constant. Let

\[ \rho := \max \{ \sup_{x \in S_1} \inf_{y \in S_2} |x - y|, \sup_{y \in S_1} \inf_{x \in S_2} |x - y| \} \]

be the Hausdorff distance between the obstacles, and

\[ \sup_{\beta, \alpha} |A_1(\beta, \alpha) - A_2(\beta, \alpha)| < \delta, \]

\( k > 0 \) is fixed, and the Dirichlet boundary condition holds on \( S_1 \) and \( S_2. \)

In 56, 70, p. 240, the following results are proved:

Theorem 4. Under the above assumptions one has

\[ \rho \leq c_1 \left( \frac{\log | \log \delta |}{| \log \delta |} \log \delta \right)^{c_2}, \]

where \( c_1 \) and \( c_2 \) are positive constants independent of \( \delta. \)

Theorem 5. There exists a function \( \nu_\eta(\alpha, \theta) \) such that

\[ -4\pi \lim_{\eta \to 0} \int_{S_2} A(\theta', \alpha) \nu_\eta(\alpha, \theta) d\alpha = -\frac{|\xi|^2 \tilde{\chi}_D(\xi)}{2}, \]

where

\[ \tilde{\chi}_D(\xi) := \int_D e^{-i \xi \cdot x} dx, \quad \theta, \theta' \in M_k, \ k(\theta' - \theta) = \xi \in \mathbb{R}^3, \]

and

\[ \tilde{\chi}_D(\xi) := \int_D e^{-i \xi \cdot x} dx, \quad \theta, \theta' \in M_k, \ k(\theta' - \theta) = \xi \in \mathbb{R}^3. \]
Some results on inverse scattering

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$M_k := \{ \theta : \theta \in \mathbb{C}^3, \sum_{j=1}^3 \theta_j^2 = k^2 \}, \ k > 0$ is fixed, and the equation for the scattering solution is

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \ D' := \mathbb{R}^3 \setminus D, u|_S = 0,$$  \hspace{1cm} (11.10)

$$u = e^{ik\alpha \cdot x} + A(\beta, \alpha) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \ \beta = \frac{x}{r}, \ \alpha \in S^2.$$  \hspace{1cm} (11.11)

It is an open problem to find an algorithm for calculating $\nu(q, \theta)$ given $A(\beta, \alpha)$. In the literature there is a large number of papers containing various methods for finding $S$ given $A(\beta, \alpha)$. In 70, pp. 245-253 there are some comments on these methods, The basic points of these comments are: most of these methods are not satisfactory, there are no error estimates for these methods, no guaranteed accuracy for the solution of the inverse obstacle scattering problem are currently available. The published numerical results with good agreement between the original obstacle and its reconstruction are obtained because the original obstacle was known a priori. Thus, it is still an open problem to develop a stable inversion method for finding $S$ from $A(\beta, \alpha)$ and to give an error estimate for such a method. In 25, p. 94 (see also 21, 22, 54, pp. 126-130), for strictly convex obstacles an analytic formula for calculating $S$ from high-frequency data $A(\beta, \alpha, k)|_{k \to \infty}$ is given, and the error estimate for the $S$ recovered from this formula when the noisy data $A_k(\beta, \alpha, k)$ are used is also given.

12. Inverse scattering with data at a fixed-energy and a fixed incident direction

Let us pose the following inverse scattering problem:

ISP: Given an arbitrary function $f(\beta) \in L^2(S^2)$, an arbitrary small number $\varepsilon > 0$, a fixed $k > 0$, and a fixed incident direction $\alpha \in S^2$, can one find a potential $q \in L^2(D)$ such that the corresponding scattering amplitude $A_q(\beta, \alpha, k) := A(\beta)$ satisfies the inequality

$$\| f(\beta) - A(\beta) \| < \varepsilon,$$  \hspace{1cm} (12.1)

where $\| \cdot \| = \| \cdot \|_{L^2(S^2)}$, $D \subset \mathbb{R}^3$ is a bounded domain, $D \subset B_a := \{ x : |x| \leq a \}$?

Let us prove that the answer is yes, that there are infinitely many such $q$ and give a method for finding such a $q$. The idea of the solution of this ISP can be outlined as follows: start with the known formula

$$-4\pi A(\beta) = \int_D e^{-ik\beta \cdot x} u(x)q(x)dx,$$  \hspace{1cm} (12.2)

where $u(x) := u(x, \alpha, k) := u_q$ is the scattering solution:

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in} \ \mathbb{R}^3,$$  \hspace{1cm} (12.3)
u satisfies (11.8), $\alpha \in S^2$ and $k > 0$ are fixed. Denote

$$h(x) := h_q := u(x)q(x),$$

so

$$A(\beta) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h_q(x) dx. \quad (12.5)$$

**Step 1.** Given $f(\beta)$ and $\varepsilon > 0$, find an $h \in L^2(D)$ such that

$$\|f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx\| < \varepsilon. \quad (12.6)$$

There are infinitely many such $h$. Existence of such $h$ follows from Lemma 1.

**Lemma 1.** The set $\{\int_D e^{-ik\beta \cdot x} h(x) dx \}_{\forall h \in L^2(D)}$ is total (complete) in $L^2(S^2)$.

In $73$ some analytical formulas are given for calculating an $h$ satisfying (12.6).

**Step 2.** Given $h \in L^2(D)$, find a $q \in L^2(D)$ such that

$$\|h - q(x)u_q(x)\|_{L^2(D)} < \varepsilon. \quad (12.7)$$

This is possible because of Lemma 2.

**Lemma 2.** The set $\{q(x)u_q(x)\}_{\forall q \in L^2(D)}$ is total in $L^2(D)$.

Let an arbitrary $h \in L^2(D)$ be given. Consider the function

$$q(x) := \frac{h(x)}{u_0(x)} - \frac{1}{\int_D g(x, y) h(y) dy} \frac{h(x)}{\psi(x)}, \quad g := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \quad (12.8)$$

The scattering solution $u = u_q$ solves the equation

$$u(x) = u_0(x) - \int_D g(x, y) q(y) u(y) dy, \quad u_0 := e^{ik\alpha \cdot x}. \quad (12.9)$$

If the function $q(x)$, defined in (12.8), belongs to $L^2(D)$, then define

$$u_q(x) := u_0(x) - \int_D g(x, y) h(y) dy. \quad (12.10)$$

The function (12.10) satisfies equation (12.9) because formula (12.8) implies $q(x)u_q(x) = h(x)$. Thus, the function (12.10) is a scattering solution. The corresponding scattering amplitude

$$A_q(\beta) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot y} h(y) dy. \quad (12.11)$$

Thus, inequality (12.6) is satisfied, so formula (12.8) gives a solution to ISP.

Suppose now that the function $q(x)$, defined by (12.8), does not belong to $L^2(D)$. This can happen because the denominator $\psi$ in (12.8) may vanish. Note that small in $L^2(D)$-norm perturbations of $h$ preserve inequality (12.6), possibly with $2\varepsilon$ in place of $\varepsilon$. Since the set of polynomials is total in $L^2(D)$, one may assume without
loss of generality that \( h(x) \) is a polynomial. In this case the function \( \psi(x) \) is infinitely smooth in \( D \) (even analytic). Consider the set of its zeros \( N := \{ x : \psi(x) = 0, x \in D \} \). Let \( \psi := \psi_1 + i\psi_2 \), where \( \psi_1 = \text{Re} \psi, \psi_2 = \text{Im} \psi \). Small perturbations of \( h \) allow one to have \( \psi \) such that vectors \( \nabla \psi_1, \nabla \psi_2 \) are linearly independent on \( N \), which we assume. In this case \( N \) is a smooth curve \( C \), defined as the intersection of two smooth surfaces:

\[
C := \{ x : \psi_1(x) = 0, \psi_2(x) = 0, x \in D \}. \tag{12.12}
\]

Let us prove that for an arbitrary small \( \delta > 0 \) there is a \( h_\delta \), \( \|h_\delta - h\|_{L^2(D)} < \delta \), such that \( q_\delta \in L^\infty(D) \), where

\[
q_\delta(x) := \begin{cases} 
\frac{h_\delta(x)}{\|\psi_1\|_D} g(x,y)h(y)dy & \text{in } D_\delta, \\
0 & \text{in } N_\delta.
\end{cases} \tag{12.13}
\]

Here \( D_\delta := D \setminus N_\delta, N_\delta := \{ x : |\psi(x)| \geq \delta > 0 \} \). The function (12.13) solves our inverse scattering problem ISP because the inequality \( \|h_\delta - h\|_{L^2(D)} < \delta \) for sufficiently small \( \delta \) implies the inequality

\[
\|f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h_\delta(x)dx\| < 2\varepsilon. \tag{12.14}
\]

Let us now prove the existence of \( h_\delta \), \( \|h_\delta - h\|_{L^2(D)} < \delta \), such that \( q_\delta \in L^\infty(D) \), where \( q_\delta \) is defined in (12.13). The \( h_\delta \) is given by the formula:

\[
h_\delta(x) = \begin{cases} 
0 & \text{in } N_\delta, \\
h & \text{in } D_\delta.
\end{cases} \tag{12.15}
\]

To prove this, it is sufficient to prove that

\[
\psi_\delta := u(x) - \int_D g(x,y)h_\delta(y)dy
\]

satisfies the inequality

\[
\min_{x \in D_\delta} |\psi_\delta(x)| \geq c(\delta) > 0, \tag{12.16}
\]

because \( h_\delta(x) \) is a bounded function in \( D \).

To prove (12.16) use the triangle inequality:

\[
|\psi_\delta(x)| \geq |u_0(x) - \int_D g(x,y)h(y)dy| - \int_D |g(x,y)||h(y) - h_\delta(y)|dy \\
\geq \delta - c \int_{N_\delta} \frac{dy}{|x - y|}, \quad x \in D_\delta,
\]

where \( c > 0 \) is a constant independent of \( \delta \),

\[
c = \frac{1}{4\pi} \max_{y \in N_\delta} |h(y)| \leq \frac{1}{4\pi} \max_{y \in D} |h(y)|.
\]

Let us prove that

\[
I_\delta := \int_{N_\delta} \frac{dy}{|x - y|} \leq c\delta^2 \ln \delta, \quad x \in D_\delta, \quad \delta \to 0, \tag{12.17}
\]
where \( c > 0 \) stands for various constants independent of \( \delta \). If (12.18) is proved, then (12.16) is established, and this proves that \( q_\delta(x) \) solves the ISP. To prove (12.18) choose the new coordinates \( s_1, s_2, s_3 \), such that \( s_1 = \psi_1(x), s_2 = \psi_2(x), s_3 = x_3 \), where the origin \( O \) is on \( N \), i.e., on the curve \( C \), \( x_1 \) and \( x_2 \) axes are in the plane orthogonal to the curve \( C \) at the origin. This plane contains vectors \( \nabla \psi_1 \) and \( \nabla \psi_2 \) calculated at the origin, and \( x_3 \) axis is directed along the vector product \( [\nabla \psi_1, \nabla \psi_2] \). The set \( N_\delta \) is a tubular neighborhood of \( C \), and we consider a part of this neighborhood near the origin. The set \( N_\delta \) is a union of similar parts and for each of them the argument is the same. Consider the Jacobian \( J \) of the transformation \((x_1, x_2, x_3) \rightarrow (s_1, s_2, s_3)\):

\[
J = \frac{\partial (\psi_1, \psi_2, \psi_3)}{\partial (x_1, x_2, x_3)} = \begin{vmatrix}
\psi_{1,1} & \psi_{1,2} & \psi_{1,3} \\
\psi_{2,1} & \psi_{2,2} & \psi_{2,3} \\
0 & 0 & 1
\end{vmatrix} \neq 0, \quad (12.19)
\]

where \( \psi_{i,j} := \frac{\partial \psi_i}{\partial x_j} \), and we have used the assumption that the vectors \( \nabla \psi_1 \) and \( \nabla \psi_2 \) are linearly independent, so that

\[
\begin{vmatrix}
\psi_{1,1} & \psi_{1,2} \\
\psi_{2,1} & \psi_{2,2}
\end{vmatrix} \neq 0
\]

in our coordinates. Thus

\[
|J| + |J^{-1}| \leq c \quad \text{on} \quad N, \quad (12.20)
\]

and, by continuity, in \( N_\delta \) for a small \( \delta \). The integral (12.18) can be written in the new coordinates as

\[
I_\delta \leq \int_{0 \leq s_3 \leq c_3} \int_0^3 dp \rho \int_0^{c_3} \frac{ds_3}{\sqrt{\rho^2 + s_3^2}}, \quad (12.21)
\]

where we have used the estimate (12.12) and (12.20) and the following estimate:

\[
c_1(\psi_1^2 + \psi_2^2 + y_3^2) \leq |y|^2 \leq c_2(\psi_1^2 + \psi_2^2 + y_3^2) \quad \text{in} \quad N_\delta. \quad (12.22)
\]

We have:

\[
\int_0^{c_3} \frac{ds_3}{\sqrt{\rho^2 + s_3^2}} = \ln(s_3 + \sqrt{\rho^2 + s_3^2}) \bigg|_0^{c_3} \leq c \ln \frac{1}{\rho} \leq c \ln \frac{1}{\delta}.
\]

Thus,

\[
I_\delta \leq c_3 \delta^2 |\ln \delta|,
\]

and (12.16) is verified.

Let \( \phi \in H_0^2(D) \), where \( H_0^2(D) \) is the closure of \( C_0^\infty(D) \) functions in the norm of the Sobolev space \( H^2(D) \). If one replaces \( h \) by \( h + (\nabla^2 + k^2) \phi \) in (12.11), then \( A_\phi(\beta) \) remains unchanged because

\[
\int_D e^{-ik\beta \cdot x}(\nabla^2 + k^2)\phi(x)dx = 0
\]

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\[
\int_D e^{-ik\beta \cdot x}(\nabla^2 + k^2)\phi(x)dx = 0
\]
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for \( \phi \in H_0^2(D) \).

One has

\[
\int_D g(x, y)[h(y) + (\nabla^2 + k^2)\phi]dy = \int_D g(x, y)h(y)dy - \phi(x). \tag{12.23}
\]

Thus, the potential

\[
q_\phi(x) := \frac{h(x) + (\nabla^2 + k^2)\phi(x)}{u_0(x) - \int_D g(x, y)h(y)dy + \phi(x)} \tag{12.24}
\]

generates the same scattering amplitude as the potential (12.8) for any \( \phi \in H_0^2(D) \).

Choosing a suitable \( \phi \), one can get a potential with a desired property.

Let us prove that \( \phi \in H_0^2(D) \) can be always chosen so that the potential \( q_\phi(x) \) has the property \( \text{Im} \ q_\phi \leq 0 \), which physically corresponds to an absorption of the energy.

Denote \( L := \nabla^2 + k^2 \), \( \phi_1 = \text{Re} \phi \), \( \phi_2 := \text{Im} \phi \),

\[
\psi := u_0 - \int_D g(x, y)h(y)dy = \psi_1 + i\psi_2,
\]

\[
\psi_1 := \text{Re} \psi, \quad \psi_2 := \text{Im} \psi. \]

Then

\[
q_\phi = \frac{[h_1 + L\phi_1 + i(h_2 + L\phi_2)][\psi_1 + \phi_1 - i(\psi_2 + \phi_2)]}{|\psi + \phi|^2}. \tag{12.25}
\]

Thus, \( \text{Im} \ q_\phi \leq 0 \) if and only if

\[
-(h_1 + L\phi_1)(\psi_2 + \phi_2) + (h_2 + L\phi_2)(\psi_1 + \phi_1) \leq 0. \tag{12.26}
\]

Choose \( \phi_1 \) and \( \phi_2 \) so that

\[
L\phi_1 + h_1 = \psi_2 + \phi_2, \quad L\phi_2 + h_2 = -(\psi_1 + \phi_1) \tag{12.27}
\]

in \( D \). Eliminate \( \phi_2 \) and get

\[
L^2\phi_1 + \phi_1 + Lh_1 - L\psi_2 + h_2 + \psi_1 = 0 \quad \text{in} \ D. \tag{12.28}
\]

The operator \( L^2 + I \) is elliptic, positive definite, of order four, with boundary conditions

\[
\phi_1 = \phi_1N = 0 \quad \text{on} \ S, \tag{12.29}
\]

because \( \phi_1 \in H_0^2(D) \). Therefore problems (12.28) - (12.29) have a unique solution \( \phi_1 \in H_0^2(D) \). If \( \phi_1 \) solves (12.28) - (12.29), then

\[
\phi_2 := L\phi_1 + h_1 - \psi_2 \tag{12.30}
\]

solves the second equation (12.27). Function (12.30) solves the equation

\[
L^2\phi_2 + \phi_2 + Lh_2 + L\psi_1 - h_1 + \psi_2 = 0 \quad \text{in} \ D. \tag{12.31}
\]

One may ask if \( \phi \) can be chosen so that \( \text{Im} \ q_\phi = 0 \). A sufficient condition for this is the following one:

\[
(h_1 + L\phi_1)(\psi_2 + \phi_2) = (h_2 + L\phi_2)(\psi_1 + \phi_1). \tag{12.32}
\]
There are $\phi_1$ and $\phi_2$ in $H^1_0(D)$ such that (12.32) is satisfied. For example, $\phi_1$ and $\phi_2$ can be found from the equations

\begin{align*}
L\phi_1 + h_1 &= \psi_1 + \phi_1, & \phi_1 &\in H^1_0(D), \\
L\phi_2 + h_2 &= \psi_2 + \phi_2, & \phi_2 &\in H^1_0(D),
\end{align*}

(12.33) (12.34)

provided that $k^2 - 1$ is not a Dirichlet eigenvalue of the Laplacian in $D$. This can be assumed without loss of generality because if $k^2 - 1$ is such an eigenvalue, then it will not be such an eigenvalue in $D_\delta$ for a small $\delta > 0$ (see 25). However, this argument leaves open the existence of $\phi_1, \phi_2 \in H^2_0(D)$ for which (12.32) holds.

13. Creating materials with desired refraction coefficient

Let $D \subset \mathbb{R}^3$ be a bounded domain filled with a material with known refraction coefficient $n_0^2(x)$, so that the wave scattering problem is described by the equations

\begin{align*}
L_0 u_0 := \left[\nabla^2 + k^2 n_0^2(x)\right] u_0(x, \alpha) &= 0 \quad \text{in } \mathbb{R}^3, \\
u_0(x, \alpha) &= e^{ikr}A_0(\beta, \alpha) \frac{e^{ik\beta}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \beta := \frac{x}{r}.
\end{align*}

(13.1) (13.2)

Here $k > 0$ is fixed, $n_0^2(x)$ is a bounded function, such that

$$
\max_{x \in \mathbb{R}^3} |n_0(x)| = n_0 < \infty, \quad \text{Im} \, n_0(x) \geq 0,
$$

so that absorption is possible, and

$$
n_0^2(x) = 1 \quad \text{in } D' := \mathbb{R}^3 \backslash D.
$$

(13.3)

The question is:

*Is it possible to create a material in $D$ with a desired refraction coefficient $n^2(x)$ by embedding small particles into $D$?*

If yes, what is the number $N(\Delta)$ of the particles of characteristic size $a$ that should be embedded in a small cube $\Delta \subset D$, centered at a point $x \in D$, and what should be the properties of these small particles?

A positive answer to the above question is given in 77 and this answer requires to solve a many-body wave scattering problem for small particles embedded in a medium. The theory was presented in 72–83.

One of the results can be stated as follows. Assume that the small bodies $D_m$, $1 \leq m \leq M$, are all balls of radius $a$ and that the following limit exists:

$$
\lim_{a \to 0} aN(\hat{D}) = \int_{\hat{D}} N(x) dx
$$

(13.4)

for any subdomain $\hat{D} \subset D$, where $N(x) \geq 0$ is a continuous function in $D$, $N(x) = 0$ in $D'$. 
Let us consider the scattering problem:

\[ L_0 u_a = 0 \text{ in } \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m, \]  

\[ u_{aN} = \zeta_m u_a \text{ on } S_m := \partial D_m, \]  

\[ u_a = u_0 + A_a(\beta, \alpha) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \frac{x}{r} = \beta, \]  

and assume that

\[ \zeta_m = \frac{h(x_m)}{a}, \]  

where \( x_m \) is the center of the ball \( D_m \), and \( h(x), \text{Im} h(x) \leq 0 \), is an a priori given arbitrary continuous in \( D \) function, \( n(x) = 0 \) in \( D' \), \( x_m \to x \) as \( a \to 0 \).

Finally assume that

\[ d = O\left(\frac{a^{1/3}}{2}\right), \quad a \to 0, \]  

where \( d \) is the smallest distance between two distinct particles (balls).

**Theorem 6.** Under the above assumptions there exists the following limit:

\[ \lim_{a \to 0} u_a(x) = u(x). \]

This limit solves the scattering problem

\[ \left[ \nabla^2 + k^2 n^2(x) \right] u = 0 \text{ in } \mathbb{R}^3, \]  

\[ u(x) = u_0(x) + A(\beta, \alpha) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad \frac{x}{r} = \beta, \]  

where

\[ n^2(x) = n_0^2(x) - k^{-2} p(x) \text{ in } D, \quad n^2(x) = 1 \text{ in } D', \]  

where

\[ p(x) := \frac{4\pi N(x) h(x)}{1 + h(x)}. \]  

**Corollary 1.** Given \( n_0^2 \) and an arbitrary continuous function \( n^2(x) \) in \( D \), \( \text{Im} n^2(x) \leq 0 \), one can find (non-uniquely) three functions \( N(x) \geq 0, h_2 := \text{Im} h(x) \leq 0, \) and \( h_1(x) = \text{Re} h(x) \), such that (13.12) holds with \( p(x) \) defined in (13.13).

The result of Theorem 6 was generalized to the case of small particles of arbitrary shapes in 77.
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