Many-body wave scattering by small bodies
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Scattering problem by several bodies, small in comparison with the wavelength, is reduced to linear algebraic systems of equations, in contrast to the usual reduction to some integral equations. © 2007 American Institute of Physics.

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I. INTRODUCTION

Acoustic or electromagnetic (EM) wave scattering by one or several bodies is usually studied by reducing the problem to solving some integral equations. In this paper we show that if the bodies are small in comparison with the wavelength, then the scattering problem can be reduced to solving linear algebraic systems with matrices whose elements have physical meaning. These elements are electrical capacitances or elements of electric and magnetic polarizability tensors. The author has derived analytical explicit formulas allowing one to calculate these quantities for bodies of arbitrary shapes with arbitrary desired accuracy (see Ref. 3).

We derive these linear algebraic systems and give formulas for the elements of the matrices of these systems. There is a large literature on wave scattering by small bodies, see Ref. 3 and references therein. The theory was originated by Rayleigh, who understood that the main term in the scattered field is the dipole radiation if the body is small. Rayleigh did not give formulas for calculating the induced dipole moments for small bodies of arbitrary shapes. The dipole moments are uniquely defined by the polarizability tensors. Therefore, the formulas, derived by the author (see Ref. 3), allow one to calculate the dipole radiation for acoustic and EM wave scattering by small bodies of arbitrary shapes.

A well-known book deals with wave scattering by small spheres, for the most part. An old well-known paper develops a theory of scalar wave scattering by isotropic scatterers. A recent paper deals with EM wave scattering by small particles.

The basic novel point in our paper is a rigorous reduction of the many-body acoustic wave scattering problem to a linear algebraic system, whose matrix elements have physical meaning and their values are calculated analytically with any desired accuracy in the author’s earlier work. Compared with the work in Ref. 1, in our work the matrix elements are explicitly calculated. Compared with the work in Ref. 5, there is a big physical difference, especially in the case of scattering in a medium consisting of many small particles. The difference lies in the following: in acoustic wave scattering by small acoustically soft particles, the theory is developed in our current paper under the assumptions $a \ll \lambda, a \ll d$, where $\lambda$ is the wavelength, $a$ is the characteristic size of a particle, and $d$ is the smallest distance between two distinct particles. These assumptions allow one to have many small particles on the wavelength. So the medium is not a “rarefied gas” of particles.

In Ref. 5 the many-body scattering theory is developed for EM waves under much stronger assumptions: $a \ll \lambda \ll d$. The necessity of these stronger assumptions for EM wave scattering is explained and justified in Ref. 5 and is also motivated briefly in Sec. III.
II. ACOUSTIC WAVE SCATTERING BY SMALL BODIES

Let us start with acoustic wave scattering. Consider the problem

\[(\Delta + k^2)u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m, \]

\[u|_{S_m} = 0, \quad 1 \leq m \leq M, \quad S_m := \partial D_m, \]

\[u = u_0 + v, \]

where \(\Delta\) is the Laplacian, \(k > 0\) is the wave number, \(k = 2\pi/\lambda\), \(\lambda\) is the wavelength, \(D_m\) is a small body, \(S_m\) its boundary which is assumed uniformly with respect to \(m\) Lipschitz, and \(u_0\) is an incident field which solves Eq. (1). Often, \(u_0 = e^{ik\alpha \cdot x}\), where \(\alpha \in S^2\) is a given vector and \(S^2\) is the unit sphere.

Let us look for the solution of the form

\[u = u_0 + \sum_{m=1}^{M} \int_{S_m} g(x,s)\sigma_m(s) \, ds, \quad g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \]

where \(\sigma_m, 1 \leq m \leq M,\) are to be chosen so that the boundary conditions (2) hold. The function (5) satisfies Eqs. (1), (3), and (4) for any \(\sigma_m \in L^2(S_m)\). The scattering amplitude is

\[A(\alpha', \alpha) = \lim_{|x| \to \infty, \alpha' = \alpha} |x|e^{-ik|x|} = \sum_{m=1}^{M} \frac{1}{4\pi} \int_{S_m} e^{-ik\alpha' \cdot s} \sigma_m \, ds, \quad \alpha' := \frac{x}{|x|}. \]

Let

\[a := \max_{1 \leq m \leq M} \text{diam} \, D_m \]

and

\[d := \min_{m \neq j} \text{dist}(D_m, D_j). \]

We assume

\[ka \ll 1, \quad a \ll d. \]

Then

\[e^{-ik\alpha' \cdot (s-x_m)} \approx 1 \quad \text{if} \quad x_m \in D, \]

so

\[A(\alpha', \alpha) = \sum_{m=1}^{M} \frac{1}{4\pi} \int_{S_m} \sigma_m \, ds := \sum_{m=1}^{M} \frac{Q_m}{4\pi} e^{-ik\alpha' \cdot x_m}, \quad Q_m := \int_{S_m} \sigma_m \, ds, \]

where \(x_m \in D_m\) and \(\alpha'\) is defined in Eq. (6). Since \(D_m\) is small, it does not matter which point \(x_m\) one takes in \(D_m\). The \(Q_m\) plays the role of the total charge on the surface \(S_m\).

If \(\min_m |x-x_m| \gg a\) and \(x_m \in D_m\), then
Let us derive a formula for $Q_m$. Using the boundary condition (2), one gets

$$0 = u_0(s_m) + \sum_{j \neq m} g(s_m, x_j) Q_j + \int_{S_m} g(s_m, s) \sigma_m(s) ds,$$

where $s_m \in S_m$.

Since $ka \ll 1$, one has

$$g(s_m, s) = g_0(s_m, s)[1 + O(ka)],$$

where

$$g_0(x, t) := \frac{1}{4\pi|x - t|}.$$ 

Therefore Eq. (10) is the equation for the electrostatic charge distribution $\sigma_m$ on the surface $S_m$ of a perfect conductor $D_m$, charged to the potential $U_m := -u_0(s_m) - \sum_{j \neq m} g(s_m, x_j) Q_j$.

This formula is valid with the error $O(ka + a/d)$ because of the assumption $d \gg a$ and the boundary condition $u=0$ on $S_m$. The total charge on $S_m$ is

$$Q_m = C_m U_m,$$

where $C_m$ is the electrical capacitance of the conductor with the shape $D_m$. The total charge is defined as

$$Q_m := \int_{S_m} \sigma_m ds.$$

Therefore, one gets

$$Q_m = C_m \left( -u_0(s_m) - \sum_{j \neq m} g(s_m, x_j) Q_j \right), \quad 1 \leq m \leq M,$$

where $C_m$ is the electrical capacitance of the perfect conductor with the boundary $S_m$.

Linear algebraic system (11) allows one to find $Q_j$, $1 \leq j \leq M$. If

$$\max_{1 \leq m \leq M} \sum_{j \neq m} \frac{C_m}{4\pi|s_m - x_j|} < 1,$$

then the matrix of the system (11) has diagonally dominant elements and, consequently, can be solved by iterations.

The approximate solution to the many-body scattering problem (1)–(4) is given by formula (9), where $Q_m$ are determined from the linear algebraic system (11).

Let us give a formula from Ref. 3 for the capacitance of a perfect conductor $D$ with the boundary $S$. Denote the area of $S$ by $|S|$. We assume that the conductor is placed in the medium with the dielectric permittivity $\varepsilon_0 = 1$. In this case the approximate formula for the capacitance is (see Ref. 3, p. 26):
\[
C^{(n)} = 4\pi|S|^2 \left\{ \frac{1}{2\pi} \int_S \int_{S(t)} ds dr \int_S \cdots \int_S \psi(t_1, t_1) \cdots \psi(t_{n-1}, t_n) dt_1 \cdots dt_n \right\}^{-1},
\]

and the error estimate of formula (13) is

\[
|C^{(n)} - C| = O(q^n), \quad 0 < q < 1,
\]

where \( q \) depends on the geometry of \( S \), and \( n = 1, 2, 3, \ldots \), is the approximation order.

If the boundary condition

\[
u_N = \zeta u \quad \text{on} \quad S_m
\]

is imposed in place of the Dirichlet condition (2), and \( \zeta \) is the impedance, then \( C_m \) in Eq. (11) is replaced by

\[
C_{m,\zeta} = \frac{C_m}{1 + C_m(\zeta|S|)^{-1}},
\]

see Ref. 3, p. 97.

If

\[
u|_{S_m} = 0, \quad 1 \leq m \leq M,
\]

then the formula for the solution to problems (1), (17), (3), and (4), is

\[
u(x) = u_0(x) + \sum_{m=1}^M q(x-x_m) V_m \left[ \frac{\partial^2 u(x_m)}{\partial x_m^2} + \sum_{p,q=1}^M \beta_{pq,m} \frac{\partial u(x_m)}{\partial x_m} (x-x_m)_p \right],
\]

where \((x-x_m)_p\) is the \(p\)th coordinate of the vector \(x-x_m\), \(\partial/\partial x_m\) is the derivative with respect to the \(q\)th coordinate of \(x\) calculated at the point \(x_m\), and \(\beta_{pq,m}\) is the magnetic polarizability tensor of \(D_m\), defined by the formula (Ref. 1, p. 98)

\[
V_m \beta_{pq,m} = \int_{S_m} s_p \sigma(s) ds,
\]

where \(V_m\) is the volume of \(D_m\), the function \(\sigma\) solves the equation

\[
\sigma = A \sigma - 2 N_q,
\]

\(N\) is the exterior unit normal to \(S_m\), and

\[
A \sigma = \int_{S_m} \frac{\partial}{\partial N} 2\pi r_{st} \sigma(t) dt, \quad r_{st} = |s-t|,
\]

The formulas for the tensor \(\beta_{pq,m}\), analogous to the formulas (13) and (14) for the capacitance, are derived in Ref. 3 [p. 55, formula (5.15)]. The unknown quantities \(\Delta u(x_m)\) and \(\partial u(x_m)/\partial x_m\), \(1 \leq m \leq M, 1 \leq q \leq 3\), in Eq. (18) can be found from the following linear algebraic system, analogous to Eq. (11)
\[ \Delta u(x_m) = \Delta u_0(x_m) - k^2 \sum_{j+m,j=1}^M \hat{g}(x_{m+}, x_j) V_j \left[ \Delta u(x_j) + \sum_{p,q=1}^3 \beta_{pq,j} \frac{\partial u(x_j)}{\partial x_{j,q}} \frac{(x_m - x_j)_p}{|x_m - x_j|} \right], \quad (19) \]

\[ \frac{\partial u(x_m)}{\partial x_{m,q}} = \frac{\partial u_0(x_m)}{\partial x_{m,q}} + \sum_{j+m,j=1}^M \frac{\partial \hat{g}(x_{m+}, x_j)}{\partial x_{m,q}} V_j \left[ \Delta u(x_j) + \sum_{p,q=1}^3 \beta_{pq,j} \frac{\partial u(x_j)}{\partial x_{j,q}} \frac{(x_m - x_j)_p}{|x_m - x_j|} \right], \quad (20) \]

In Eq. (19) we have used the equation

\[ \Delta g(x,y) = -k^2 g(x,y), \]

which holds if \( x \neq y \).

From the linear algebraic systems (19) and (20), one finds the unknowns \( \Delta u(x_m) \) and \( \frac{\partial u(x_m)}{\partial x_{m,q}} \), \( 1 \leq m \leq M, \ 1 \leq q \leq 3 \).

If conditions (7) hold, then systems (19) and (20) have a unique solution which can be obtained by iterations.

This completes the description of our method for solving many-body scattering problem for small bodies and acoustic (scalar) waves.

III. ELECTROMAGNETIC WAVE SCATTERING BY SMALL BODIES

In the problem of EM wave scattering by many small bodies, we assume

\[ a \ll \lambda \ll d. \quad (21) \]

This assumption is more restrictive than Eq. (7). The reason is that in EM theory the fields are obtained by an application of first order differential operators, for instance, \( \nabla \times \), to potentials, such as the vector potential. Applying this operator and calculating the field in the far zone, one neglects the term \( |1/|x - x_m|| \) compared with the term \( k \). This means that the following inequality is assumed:

\[ \frac{1}{d} \ll \frac{1}{\lambda}, \]

or

\[ d \gg \lambda. \]

In the acoustic wave theory, the potential itself \( \int g(x,s) \sigma ds \) has physical meaning; it is the acoustic pressure, and this pressure is studied. Therefore, the condition \( d \gg \lambda \) does not appear.

Condition (7) allows one to have many small particles on the distance of order \( \lambda \), while condition (21), namely, the inequality \( d \gg \lambda \), does not allow this. Recall that \( d \) is the minimal distance between two neighboring particles. The formula for the scattering amplitude, analogous to Eq. (8), for EM wave scattering by small bodies is (see Ref. 4):

\[ A(\theta', \theta) = \frac{1}{4\pi} \sum_{m=1}^M S_m d_m e^{-ik\theta' \cdot x_m}. \quad (22) \]

Here

\[ \mathcal{U} = \begin{pmatrix} E \\ H \end{pmatrix} \]

is a six-component vector, \( S_m \) is a \( 6 \times 6 \) matrix, the scattering matrix, \( \varepsilon_0 \) and \( \mu_0 \) are dielectric and magnetic parameters of the medium, in which the body \( D_m \) is placed, and \( \theta, \theta' \) are the unit vectors in the direction of the incident and scattered waves, respectively. These vectors were denoted \( \alpha \) and \( \alpha' \) in Sec. II. We have changed the notations because in EM theory \( \alpha \) denotes the polarizability tensor.
The formula for $S$ is (cf. Ref. 2)

$$S_m(E, H) = \frac{k^2 V_m}{4\pi} \left( \begin{array}{c} \alpha E - \theta'(\theta', \alpha E) - \frac{\mu_0^{1/2}}{e_0^{1/2}} [\theta', \vec{\beta} H] \\ \left( \frac{e_0}{\mu_0} \right)^{1/2} [\theta', \alpha E] - \mu_0 [\vec{\beta} - \theta'(\theta', \vec{\beta} H)] \end{array} \right). \tag{23}$$

Here $V_m$ is the volume of $D_m$, $\alpha$ is the electric polarizability tensor of $D_m$, and $\vec{\beta}$ is the magnetic polarizability tensor of $D_m$. In Ref. 3 (pp. 54 and 55) the author derives analytical formulas for the calculation of the polarizability tensors $\alpha$ and $\beta$,

$$\vec{\beta} := \alpha(\gamma) + \beta, \quad \beta := \alpha_i(-1), \quad \gamma := \frac{e-\varepsilon_0}{e+\varepsilon_0} \tag{24}$$

Tensors $\beta$ and $\vec{\beta}$ are expressed through the polarizability tensor $\alpha = \alpha(\gamma)$. One has $\gamma = -1$ if $e = 0$. Here $[\cdot, \cdot]$ is the vector product; $\langle \cdot, \cdot \rangle$ is the scalar product.

The analytic formula from Ref. 3 [p. 54, formula (5.9)] for the tensor $\alpha = \alpha_i(\gamma)$, $1 \leq i, j \leq 3$, that we referred to above is analogous to formulas (13) and (14) for the electrical capacitance. The incident direction $\theta$ enters via the vectors $E$ and $H$, which depend on $\theta$. These vectors are calculated in formula (23) at the point $x_m$. The values of these vectors are determined from a linear algebraic system of equations. This system is derived similarly to the derivation of the systems (11), (19), and (20). We do not write down this system since it would take much space, but the ideas are the same as the ones used in the derivations of (11), (19), and (20).

IV. CONCLUSIONS

In this paper it is shown how to reduce rigorously the many-body scattering problem to linear algebraic system in the case when the bodies are small in comparison with the wavelength. The theory is constructed for acoustic and EM wave scatterings. The basic physical assumptions are Eq. (7) for acoustic scattering and Eq. (21) for EM scattering. In the recent paper, we have applied the theory, developed in Sec. II, to the problem of creating wave-focusing materials by embedding many small particles into a medium.