Many-body wave scattering by small bodies and applications

A. G. Ramm
Mathematics Department, Kansas State University, Manhattan, Kansas 66506, USA
and TU Darmstadt, Germany

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A rigorous reduction of the many-body wave scattering problem to solving a linear algebraic system is given bypassing solving the usual system of integral equation. The limiting case of infinitely many small particles embedded into a medium is considered and the limiting equation for the field in the medium is derived. The impedance boundary conditions are imposed on the boundaries of small bodies. The case of Neumann boundary conditions acoustically hard particles is also considered. Applications to creating materials with a desired refraction coefficient are given. It is proved that by embedding a suitable number of small particles per unit volume of the original material with suitable boundary impedances, one can create a new material with any desired refraction coefficient. The governing equation is a scalar Helmholtz equation, which one obtains by Fourier transforming the wave equation. © 2007 American Institute of Physics.

I. INTRODUCTION

This paper can be considered as a continuation of Ref. 14, but it is essentially self-contained. It uses some of the ideas and results from Refs. 19, 11, 10, 21, and 20. Applications of our theory to creating materials with desired refraction coefficient are discussed in Refs. 7, 9, 8, and 15. Wave scattering by small bodies is a classical branch of science: it was originated by Rayleigh in 1871. In Ref. 3 one finds a discussion of wave scattering by a small particle. In Ref. 2 there is a review of the low frequency scattering theory, and formulas for scattering by small balls and ellipsoids are given. In Ref. 19 the theory is developed for small bodies of arbitrary shapes: explicit analytical formulas are given for calculating capacitances of the conductors of arbitrary shapes, electric and magnetic polarizability tensors for homogeneous bodies of arbitrary shapes, and $S$-matrix for acoustic and electromagnetic wave scattering by small bodies of arbitrary shapes with any desired accuracy. In Ref. 14 the many-body scattering problem was reduced to solving linear algebraic systems bypassing the usual study of a system of integral equations. In this paper we apply the approach proposed in Ref. 14 and study the limiting behavior of the scattering solution when the number of small bodies tends to infinity in such a way that the characteristic size $a$ of the small particles is related to their number $M$ so that $M=O(1/a)$ in Theorem 2 and $M=O(1/a^3)$ in Theorem 3. Sufficient conditions for convergence of the scattering solution in this limiting process are given. We prove that these conditions are, in some sense, also necessary for convergence. The limit of the scattering solution is a function that satisfies some differential or integral-differential equations. These equations describe the behavior of the wave field in the new medium, obtained in the limit.

There is a large literature on the calculation of the effective dielectric permittivity and magnetic permeability of the composite materials (Maxwell-Garnett and Bruggeman recipes and their numerous versions, see Refs. 22 and 5). In the literature mostly a uniform random distribution of the inclusions is assumed and the resulting homogenized medium is described by effective con-
stant dielectric permittivity and magnetic permeability, which can be tensors. In this work the propagation and scattering of scalar waves are discussed, and the “homogenized” medium is described not by a constant refraction coefficient but by a refraction coefficient which is a function of spatial variables.

In Ref. 4 boundary value problems were studied for positive operators for the Dirichlet boundary conditions in domains which are obtained from some domain by extracting many small bodies from it. We study in this paper the scattering problem in similar domains, but our operator is not positive and we use impedance boundary conditions. The methods in Ref. 4 do not seem to be applicable to our problem by the above reasons. Our assumptions lead to new physical phenomena. For example, the new material, created by embedding many small particles according to the recipe, given in Theorem 2, allows one to get a refraction coefficient with any desired real and imaginary parts, so that a desired absorption of energy in this material can be obtained.

Let us formulate the problem. Consider first a bounded domain \( D \subset \mathbb{R}^3 \) filled with a material with a known refraction coefficient \( n_0^2(x) \). The governing equation is

\[
L_0 u_0 := (\nabla^2 + k^2 n_0^2(x)) u_0 = 0 \quad \text{in } \mathbb{R}^3. \tag{1.1}
\]

We assume that \( n_0^2(x) = 1 \) in \( D' \subset \mathbb{R}^3 \setminus D \), \( k = \text{const.} > 0 \), and \( n_0^2 = \max_{x \in D} |n_0^2(x)| < \infty \). The operator \( L_0 \) can be written as a Schrödinger operator:

\[
L_0 = \nabla^2 + k^2 - q_0(x), \quad q_0(x) := k^2 \left[ 1 - n_0^2(x) \right]. \tag{1.2}
\]

and \( q_0 = 0 \) in \( D' \). One has

\[
n_0^2(x) = 1 - k^{-2} q_0(x), \tag{1.3}
\]

so there is a one-to-one correspondence between \( n_0^2(x) \) and \( q_0(x) \). If \( n_0^2(x) \) is known, then one knows the scattering solution:

\[
u_0(x) = e^{i k x} + A_0(\beta, \alpha) \frac{e^{i k r}}{r} + o \left( \frac{1}{r} \right), \quad r = |x| \to \infty, \quad \beta := \frac{x}{r}.
\]

The coefficient \( A_0(\beta, \alpha) \) is called the scattering amplitude, the unit vector \( \alpha \in \mathbb{S}^2 \) is given, \( \alpha \) is the direction of the incident plane wave \( e^{i k x} \), \( \mathbb{S}^2 \) is the unit sphere in \( \mathbb{R}^3 \), \( \beta \in \mathbb{S}^2 \) is the direction of the scattered wave, and \( k > 0 \) is a wave number, which we assume fixed throughout the paper. By this reason we do not show the \( k \) dependence of \( A \) and \( u_0 \). Let \( G(x, y) \) be the resolvent kernel of \( L_0 \) satisfying the radiation condition (or the limiting absorption principle)

\[
L_0 G(x, y) = -\delta(x - y) \quad \text{in } \mathbb{R}^3. \tag{1.4}
\]

This function \( G(x, y) \) is known because \( q_0(x) \) is known.

Consider now the scattering problem for many small bodies \( D_m \) embedded in \( D \), \( 1 \leq m \leq M \):

\[
L_0 u_m = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m, \tag{1.5}
\]

\[
u_m = u_0 + A_M(\beta, \alpha) \frac{e^{i k r}}{r} + o \left( \frac{1}{r} \right), \quad r = |x| \to \infty, \quad \begin{array}{c} x \cr r \end{array} = \beta. \tag{1.6}
\]
where \( u_0 \) is the solution of the scattering problem (1.3). Here \( N \) is the normal to \( S_m \) pointing out of \( D_m \), \( \xi_m \) is a complex number, the boundary impedance, \( \text{Im} \xi_m \leq 0 \), and \( S_m \) is uniformly \( C^{1,\lambda} \) with respect to \( m, 1 \leq m \leq M \). By \( C^{1,\lambda} \) surface we mean the surface with local equation \( x_3 = f(x_1, x_2) \), if \( f \in C^{1,\lambda}, \lambda > 0 \). We assume throughout this paper that

\[ n \xi f a \ll 1, \quad d \gg a, \]  

(1.8)  

where

\[ a = \frac{1}{2} \max_{m \neq j} \text{diam} D_m, \quad d = \min_{m \neq j} \text{dist}(D_m, D_j). \]  

(1.9)

By \( V_m := |D_m| \) the volume of \( D_m \) is denoted, and by \( |S_m| \) the surface area of \( S_m \) is denoted.

One can prove (see Sec. III) that problem (1.5)–(1.7) has at most one solution if \( \text{Im} \xi_m \leq 0 \), and \( 1 \leq m \leq M \), and \( \text{Im} q_0(x) \leq 0 \).

We look for the solution to problem (1.5)–(1.7) of the form

\[ u_M(x) = u_0(x) + \sum_{m=1}^{M} \int_{S_m} G(x,s) \sigma_m(s) ds, \]  

(1.10)

where \( \sigma_m \) should be found from the boundary conditions in Eq. (1.7). For any \( \sigma_m \), function (1.10) solves Eq. (1.5) and satisfies condition (1.6):

\[ A_M(\beta, \alpha) = \frac{1}{4\pi} \sum_{m=1}^{M} \int_{S_m} u_0(s, - \beta) \sigma_m(s) ds. \]  

(1.11)

Formula (1.11) follows from (1.6) and (1.10), and Ramm’s lemma [Ref. 12, formulas (5.1.31) and (5.1.36)]:

\[ G(x,y) = \frac{e^{i|y|}}{4\pi|y|} u_0(y, \alpha) + o\left( \frac{1}{|x|} \right), \quad \xi \to \infty, \quad \xi \in |x| = - \alpha, \]  

(1.12)

where \( u_0(x, \alpha) \) is the scattering solution. A similar formula was proved earlier in Ref. 18 (p. 46) for the resolvent kernel of the Laplacian in the exterior of a bounded obstacle (and even earlier, in Ref. 13, for some unbounded obstacles). The scattering amplitude for problem (1.5)–(1.7) is

\[ A(\beta, \alpha) = A_0(\beta, \alpha) + A_M(\beta, \alpha), \]  

(1.13)

where \( A_0 \) is defined in (1.3) and \( A_m \) is defined in (1.6). If \( ka \) is sufficiently small, then \( k^2 \) is not a Dirichlet eigenvalue of the operator \( \nabla^2 - q_0(x) \) in \( D_m, 1 \leq m \leq M \). If

\[ \text{Im} \xi_m \leq 0, \quad 1 \leq m \leq M; \quad \text{Im} q_0(x) \leq 0, \]  

(1.14)

then the unique solution to problem (1.5)–(1.7) can be found in form Eq. (1.10),

**Theorem 1:** Assume (1.8) and (1.14). Then problem (1.5)–(1.7) has a solution of the form in Eq. (1.10) and this is the unique solution of problem (1.5)–(1.7).

The proof of Theorem 1 is given in Sec. III.

Let

\[ g(x,y) := \frac{e^{i|x-y|}}{4\pi|x-y|}, \quad g_0(x,y) := \frac{1}{4\pi|x-y|}. \]  

(1.15)

Note that
\[ G(x, y) = g(x, y) - \int_D g(x, z)q_0(z)G(z, y)dz. \]  

(1.16)

113 We need two lemmas.

114 Lemma 1: If

115 \[ |t - x| \leq a, \quad |x - y| \geq d \gg a, \]  

(1.17)

116 then

117 \[ |g(t, y) - g(x, y)| \leq c \left( \frac{a}{d^2} + \frac{ka}{d} \right), \]  

(1.18)

118 where \( c > 0 \) stands for various positive constants independent of \( a \) and \( d \).

119 Lemma 2: If (1.17) holds, then

120 \[ |G(t, y) - G(x, y)| \leq c \left( \frac{a}{d^2} + \frac{ka}{d} \right). \]  

(1.19)

121 These lemmas are proved in Sec. III.

122 We denote by \( \tilde{D} \) an arbitrary subdomain of \( D \) independent of \( a \) by

123 \[ N_a' (\tilde{D}) := N(\tilde{D}) := \sum_{D_m \subset \tilde{D}} 1, \]  

124 the number of particles (small bodies) in \( \tilde{D} \), and assume that the small particle \( D_m \) shrinks to a

125 point \( x \in D \) as \( a \to 0 \). Since in the limiting process dependence on \( m \) disappears, we denote the

126 limiting point \( x \) without giving it any subindex. The functions \( h(x) \), \( \text{Im} h \leq 0 \), and \( N(x) \geq 0 \) in

127 Theorem 2 are arbitrary continuous functions which we can choose as we wish. They do not depend on \( a \). They determine the refraction coefficient of the new material, created by embedding \( a \) small particles, as \( a \to 0 \). We assume for simplicity that

130 \[ |S_m| = c_1 a^2, \quad J_m = c_2 a^3, \quad V_m = c_3 a^3, \quad M = O(a^{-1}), \quad d = O(a^{1/3}), \]  

131 where \( c_j, j = 1, 2, 3 \) are positive constants, independent of \( a \), \( |S_m| \) is the surface area of the bound-

132 ary of the \( m \)th body, \( V_m \) is its volume, and \( J_m = \int_{S_m} \int_{S_m} ds/dt/|s-t| \), where \( ds \) and \( dr \) are the surface

133 area elements, and \( s, t \) are points on \( S_m \). These assumptions are not repeated in the formulation of

134 Theorem 2.

136 Let us formulate our results under simplifying but physically reasonable assumptions (see Ref. 136 17).

137 \[ \text{Theorem 2: Assume that} \]

138 \[ \lim_{a \to 0} \frac{\xi_m J_m}{4 \pi |S_m|} = h(x), \]  

(1.20)

139 and for any subdomain \( \tilde{D} \subset D \) the following relation holds:

140 \[ \lim_{a \to 0} aN_a'(\tilde{D}) = \int_{\tilde{D}} N(x)dx. \]  

(1.21)

141 Under these assumptions, there exists the limit

142 \[ \lim_{M \to \infty} u_M(x) = u(x) := u(x, \alpha). \]  

(1.22)

143 This \( u(x) \) solves the equations
\[ u(x) = u_0(x) - \int_D G(x,y)p(y)u(y)dy \quad (1.23) \]

and

\[ Lu := [\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in} \ \mathbb{R}^3, \quad (1.24) \]

where the potential \( q \) is of the form

\[ q(x) = q_0(x) + p(x), \quad p(x) = \frac{4\pi c^2 N(x) h(x)}{c^2[1 + h(x)]}, \quad (1.25) \]

and \( u \) satisfies the radiation condition

\[ u = e^{ikx} + A(\beta, \alpha) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad (1.26) \]

where

\[ A(\beta, \alpha) = A_0(\beta, \alpha) - \frac{1}{4\pi} \int_D u_0(y, -\beta)p(y)u(y, \alpha)dy, \quad (1.27) \]

and \( u_0(y, -\beta) \) is the scattering solution defined in (1.3).

**Theorem 3:** Assume that \( \zeta_m = 0, 1 \leq m \leq M \), and the following limits exist:

\[ \lim_{a \to 0} \sum_{D_m \subseteq D} V_m \beta_{pj} = \int_D \beta_{pj}(y) \varphi(y)dy, \quad (1.28) \]

\[ \lim_{a \to 0} \sum_{D_m \subseteq D} V_m = \int_D \varphi(y)dy, \quad (1.29) \]

where \( \bar{D} \subseteq D \) are arbitrary, independent of \( a \), the functions \( \varphi(y) \geq 0 \) and \( \beta_{pj}(y) \) are continuous in \( D \), and \( \beta_{pj}^{(m)} \) is the magnetic polarizability tensor of the body \( D_m \), defined in (2.38) and (2.39), see below.

Then the function \( u_M(x) \), defined in (1.10), tends to the limit

\[ \lim_{M \to \infty} u_M(x) = U(x) = U(x, \alpha), \quad (1.30) \]

and \( U(x) \) solves the equation

\[ U(x) = u_0(x) + \int_D G(x,y)\Delta U(y)\varphi(y)dy - \sum_{p,j=1}^3 \int_D \frac{\partial G(x,y)}{\partial y_p} \frac{\partial U(y)}{\partial y_j} \beta_{pj}(y)\varphi(y)dy. \quad (1.31) \]

If all the small particles are balls of radius \( a > 0 \), then

\[ V_m = \frac{4\pi a^3}{3}, \quad |S_m| = 4\pi a^2, \quad J_m = 16\pi^2 a^3, \quad \int_{S_m} \frac{dr}{4\pi |s-r|} = a, \quad s \in S_m. \quad (1.32) \]

In this case...
Note that under the assumptions of Theorem 2, one has $M = O(a^{-1})$, while under the assumptions of Theorem 3, one has $M = O(a^{-3})$ [see formula (2.50) below]. Therefore, one needs many more particles to deal with the Neumann boundary condition, that is, with acoustically hard particles, than with the impedance boundary condition with large boundary impedance $\zeta = O(a^{-1})$. We discuss at the end of Sec. IV in more detail the question concerning the compatibility of assumption (1.8), namely, $d \gg a$, and the existence of limits (1.28) and (1.29). It will be shown that the assumption $d \gg a$ is compatible with the existence of limit (1.29) only if $\nu(y)$ is sufficiently small, and in this case the existence of limit (1.28) is also compatible with the assumption $d \gg a$.

In Sec. II, Theorems 2 and 3 are proved. In Sec. III, Theorem 1 and Lemmas 1 and 2 are proved. In Sec. IV some examples are given, the significance of the compatibility of the assumptions $d \gg a$ and (1.21), (1.28), and (1.29) is discussed, and a possible application of our results to creating materials with a desired refraction coefficient is described. In Sec. V some estimates of the effective field are given. These estimates imply the convergence of this field as $a \rightarrow 0$. The results of Sec. V are used in the proof of Theorem 2.

II. PROOF OF THEOREM 2

Let us look for the solution to problem (1.5)–(1.7) of the form

$$u_M = u_0(x) + \sum_{m=1}^{M} \int_{S_m} G(x,s)\sigma_m(s)ds,$$

where $G(x,y)$ is the resolvent kernel of $L_0$, see (1.4), and $\sigma_m$ are arbitrary functions at the moment.

For any $\sigma_m$, function (2.1) solves Eq. (1.5) and satisfies the radiation condition (1.6). Since problem (1.5)–(1.7) has at most one solution, function (2.1) is the unique solution to (1.5)–(1.7) provided that $\sigma_m$ are chosen so that the boundary conditions in Eq. (1.7) are satisfied. Since $\text{diam } D_m$, $1 \leq m \leq M$, are small, let us write (2.1) as

$$u_M = u_0(x) + \sum_{m=1}^{M} G(x,x_m)Q_m + \sum_{m=1}^{M} \int_{S_m} [G(x,s) - G(x,x_m)]\sigma_m(s)ds,$$

where $x_m \in D_m$ is a point inside $D_m$ and

$$Q_m := \int_{S_m} \sigma_m(s)ds.$$

The choice of $x_m \in D_m$ is arbitrary because $\text{diam } D_m \leq 2a$ is small. We will prove that $Q_m \neq 0$, give an analytic formula for $Q_m$ [formula (2.20) below], and approximate the field $u_M$ in (2.2) by the expression

$$u_M = u_0(x) + \sum_{m=1}^{M} G(x,x_m)Q_m.$$

The error of this approximate formula is of order $\max(a/\ell, \ell a)$, see estimate (2.7) below. Therefore this error tends to zero as $a \rightarrow 0$ since $d = O(a^{1/3})$. Let us estimate the term
\[ E_m := \int_{S_m} [G(x,s) - G(x,x_m)]\sigma_m(s)ds. \] (2.4)

203 By inequality (1.19) one gets

\[ |E_m| \leq c\left(\frac{a}{d^2} + \frac{ka}{d}\right)|Q_m|, \quad |x - x_m| \geq d \gg a. \] (2.5)

205 We will prove below that \( Q_m = O(a) \), see formula (2.20), and since \( |G(x,x_m)| \leq cd^{-1} \) if \( |x - x_m| \geq d > 0 \), one has

\[ |G(x,x_m)Q_m| = O\left(\frac{a}{d}\right). \] (2.6)

207 Let us prove that under our assumptions the term \( E_m \) is much smaller than \( O(a/d) \). Using again inequality (1.19), one gets

\[ |E_m| \leq c(ad^{-2} + kad^{-1})O(a). \] (2.7)

211 Therefore,

\[ |E_m| \leq \sum_{m=1}^{M} |G(x,x_m)Q_m| \ll \sum_{m=1}^{M} |G(x,x_m)Q_m| \] (2.9)

215 if \( |x - x_m| \geq d \gg a \), because \( M = O(1/a) \).

216 To find \( Q_m \), we use the boundary condition (1.7). Let us write \( u(x) := u_M(x) \) in a neighborhood of \( S_j \) as

\[ u_M(x) := u_e(x) + \int_{S_j} G(x,s)\sigma_j(s)ds, \quad |x - x_j| \leq 2a, \] (2.8)

where \( u_e \) is the effective field acting on the \( j \)th small particle from the outside:

\[ u_e(x) := u_M(x) - \int_{S_j} G(x,s)\sigma_j(s)ds = u_0(x) + \sum_{m \neq j} G(x,x_m)Q_m + O\left(\frac{a}{d}\right), \quad |x - x_j| \leq 2a, \] (2.9)

221 and \( u_e(x) := u_M(x) \), \( \min_m |x - x_m| \geq d \).

222 We neglect the error term \( O(a/d) \) in what follows. From (2.8) and (1.7) one gets

\[ 0 = u_{eN}(s) - \zeta u_e(s) + \frac{A_j\sigma_j - \sigma_j}{2} - \zeta T_j\sigma_j, \quad s \in S_j, \] (2.10)

224 where \( u_{eN}(s) \) is the normal derivative of \( u_e \) at the point \( s \in S_j \). One can rewrite this equation as

\[ \sigma_j = A_j\sigma_j - 2\zeta T_j\sigma_j - 2\zeta u_e(s) + 2u_{eN}(s). \] (2.11)

226 Here the operators \( A_j \) and \( T_j \) are defined as follows:

\[ T_j\sigma_j := \int_{S_j} G(s,t)\sigma_j(t)dt = \int_{S_j} \frac{\sigma_j(t)dt}{4\pi|s-t|}, \] (2.11)
\[ A_j \sigma_j := 2 \int_{S_j} \frac{\partial G(s,t)}{\partial N_s} \sigma_j(t) dt = 2 \int_{S_j} \frac{1}{\partial N_s 4 \pi |s-t|} \sigma_j(t) dt := A \sigma_j, \tag{2.12} \]

and we have used the following approximations:

\[ G(x,y) = g_0(x,y)[1 + O(|x-y|)], \quad |x-y| \to 0; \quad g_0(x,y) := \frac{1}{4 \pi |x-y|}, \tag{2.13} \]

\[ \frac{\partial G(x,y)}{\partial y_p} = \frac{\partial g_0}{\partial y_p}[1 + O(|x-y|)|\ln|x-y||], \quad |x-y| \to 0. \tag{2.14} \]

Note that [see Ref. 19, p. 96, formula (7.21)]

\[ \int_{S_j} A \sigma_j ds = - \int_{S_j} \sigma_j ds. \tag{2.15} \]

Indeed,

\[ \int_{S_j} ds \int_{S_j} \frac{\partial}{\partial N_s} \frac{1}{2 \pi |s-t|} \sigma_j(t) dt = \int_{S_j} dt \int_{S_j} ds \frac{\partial}{\partial N_s} \frac{1}{2 \pi |s-t|} = - \int_{S_j} \sigma_j(t) dt. \tag{2.15} \]

The integral

\[ \int_{S_j} \frac{1}{\partial N_s 2 \pi |s-t|} ds = - 1, \quad t \in S_j, \tag{2.16} \]

is well known in potential theory for surfaces \( S_j \in C^{1,\lambda} \).

Integrating (2.10) over \( S_j \) and using formula (2.15) and the divergence theorem, one gets

\[ Q_j = - \xi \mu_v(x_j)|S_j| - \xi \int_{S_j} ds \frac{\sigma_j(t) dt}{4 \pi |s-t|} + \int_{D_j} \Delta u_s dy. \tag{2.17} \]

The function \( u_s(y) \) is smooth, so

\[ \int_{D_j} \Delta u_s(y) dy = V_j \Delta u_s(x_j)[1 + o(1)], \quad a \to 0, \tag{2.18} \]

where \( V_j := |D_j| \) is the volume of \( D_j \) and we have used the smallness of the diameter of \( D_j \), that is, \( a \to 0, \)

the smallness of \( a \).

Let us write

\[ \int_{S_j} ds \int_{S_j} \frac{\sigma_j(t) dt}{4 \pi |s-t|} = \int_{S_j} dt \sigma_j(t) \int_{S_j} ds \frac{ds}{4 \pi |s-t|} = Q_j \int_{S_j} dt \int_{S_j} ds \frac{ds}{4 \pi |s-t|} = \frac{Q_j}{4 \pi |S_j|}, \tag{2.19} \]

Here we approximated the continuous on \( S_j \) function \( \int_{S_j} ds/|s-t| \) by its mean value

\[ 1/|S_j| \int_{S_j} ds/|s-t|. \tag{2.20} \]

If \( S_j \) is a sphere of radius \( a \), then
\[
\int_{|s-t|=a} \frac{ds}{|s-t|} = 4\pi a, \quad |t| = a, \quad (2.19)
\]

so in this case Eq. (2.18) is exact.

From (2.16)–(2.18) one finds a formula for \( Q_j \):

\[
Q_j = -\frac{\xi_j\|S_j\|}{1 + \xi_j J_j/(4\pi|S_j|)} u_j(x_j). \quad (2.20)
\]

We neglected the term \( V_j \Delta u_j(x_j) = O(a^3) \) which is much smaller than \( |\xi_j||S_j| = O(a) \) as \( a \rightarrow 0 \), because \( |S_j| = O(a^2) \) and \( |\xi_j| = O(1/a) \). The quantity \( J_j = O(a^2) \). Therefore \( \xi_j J_j/(4\pi|S_j|) = O(1) \). We choose

\[
\xi_j = \frac{H(x_j)}{a}, \quad (2.21)
\]

where \( H(x) \) is a continuous function in \( D \), which we can choose as we wish subject to the condition \( \text{Im} H \leq 0 \) because \( \text{Im} \xi_j \leq 0 \).

If the small particles are all of the same shape and size, then

\[
|S_j| = c_j a^2, \quad J_j = c_j a^2, \quad (2.22)
\]

where \( c_1, c_2 > 0 \) are some constants independent of \( j \), \( 1 \leq j \leq M \).

Then

\[
\frac{\xi_j J_j}{4\pi|S_j|} = \frac{H(x_j)c_2}{4\pi c_1} = h(x_j), \quad (2.23)
\]

and

\[
\xi_j|S_j| = H(x_j)c_1 a. \quad (2.24)
\]

Formulas (2.2), (2.20), (2.22), and (2.23) imply

\[
u_M(x) = u_0(x) - \sum_{m=1}^{M} G(x,x_m) \frac{4\pi c_1 c_2}{1 + h(x_m)} h(x_m)a u_M(x_m), \quad (2.25)
\]

where \( |x-x_m| \geq d \gg a \), and we replaced \( u_j(x_m) \) by \( u_M(x_m) \) because their difference [see (2.9)] is of order \( O(a/d) \leq 1 \).

Indeed

\[
|u_M(x) - u_j(x)| \leq \int_{S_j} |G(x,s)||\sigma_j(s)|ds \leq \frac{c}{d} |Q_j| \leq \frac{c a}{d}, \quad |x-x_j| \geq d \gg a, \quad (2.26)
\]

where \( c, c' > 0 \) are some constants independent of \( a \).

If assumption (1.21) holds, then

\[
\lim_{a \rightarrow 0} \sum_{m=1}^{M} G(x,x_m) \frac{4\pi c_1 c_2}{1 + h(x_m)} h(x_m)a u_M(x_m)a \leq \int_D G(x,y) \frac{4\pi c_1 c_2}{1 + h(y)} h(y)u_M(y)N(y)dy. \quad (2.27)
\]

To pass to the limit in (2.26), one uses Lemmas 5 and 6 of Sec. V and the following lemma.

Lemma 3: Assume that \( x_m \in D_m, \text{diam} D_m \ll 2a, f \) is a continuous function in \( D \) with a possible exception of a point \( y_0 \) in a neighborhood of which it is absolutely integrable, for example, it admits an estimate \( |f(y)| \leq C|y-y_0|^b, \quad b < 3, \) and assume that
\[
\lim_{a \to 0} a \sum_{m=1}^{M} f(y_m) a = \int_D f(y) N(y) dy.
\] (2.28)

for any subdomain \( \tilde{D} \subset D \), where \( N(x) \) is a continuous function. Then there exists the limit

\[
\lim_{a \to 0} \sum_{m=1}^{M} f(y_m) a = \int_D f(y) N(y) dy.
\] (2.27)

Remark 1: In our case \( f(y) = G(x, y)(4\pi c^2 h(y)/c_2^2 [1 + h(y)]) u_M(y) \) and (2.27) is assumption (1.21).

Proof of Lemma 3: Let \( D = \bigcup_{p=1}^{P} \Delta_p \), where \( \Delta_p \) and \( \Delta_q \) do not intersect each other, \( \Delta_p \) is the closure of the domain \( \Delta_p \), and \( \lim_{p \to 0} \max_y \operatorname{diam} \Delta_p = 0 \). Choose any point \( y^{(p)} \in \Delta_p \) and note that

\[
\sup_{y_m \in D_m, D_m \subset \Delta_p} |f(y^{(p)}) - f(y_m)| < \varepsilon_p \to 0 \quad \text{as diam} \Delta_p \to 0.
\] (2.29)

Therefore

\[
\lim_{a \to 0} \sum_{m=1}^{M} f(y_m) a = \lim_{a \to 0} \sum_{p=1}^{P} a \sum_{m=1}^{M} f(y_m) = \sum_{p=1}^{P} \left[ f(y^{(p)}) + O(\varepsilon_p) \right] \lim_{a \to 0} \sum_{m=1}^{M} f(y_m) a
\] (2.28)

\[
= \sum_{p=1}^{P} \left[ f(y^{(p)}) + O(\varepsilon_p) \right] \left[ N(y^{(p)}) + O(\varepsilon_p) \right] \Delta_p.
\] (2.30)

Then

\[
\lim_{P \to \infty} \sum_{p=1}^{P} \left[ f(y^{(p)}) + O(\varepsilon_p) \right] \left[ N(y^{(p)}) + O(\varepsilon_p) \right] \Delta_p = \int_D f(y) N(y) dy.
\] (2.31)

In the above argument we assumed that \( f \) is continuous in \( D \). If \( f \) has an integrable singularity at a point \( x_0 \), then we choose a ball \( B(x_0, \delta) \) centered at \( x_0 \) of radius \( \delta \) such that \( \sup_{0 < \rho < \delta, y \in D \setminus B(x_0, \delta)} |f(y)| dy < \varepsilon \), where \( \varepsilon > 0 \) is an arbitrary small fixed number. Then

\[
\sup_{0 < \rho < \delta, y \in D \setminus B(x_0, \delta)} |f(y)| ||N(y)|| dy < c \varepsilon,
\] (2.32)

where \( c = \max_{y \in D \setminus B(x_0, \delta)} |N(y)| > 0 \) is a constant independent of \( \varepsilon \). Now we apply the above argument to the region \( D \setminus B(x_0, \delta) \), where \( f \) is continuous and get

\[
\lim_{a \to 0} \sum_{m=1}^{M} f(y_m) a = \int_{D \setminus B(x_0, \delta)} f(y) N(y) dy.
\] (2.33)

The left side of (2.28) in the case of \( f \) having integrable singularity at the point \( x_0 \) and continuous in \( D \setminus x_0 \) is understood as the limit of the expression on the left of (2.32) as \( \delta \to 0 \). This yields (2.28). Lemma 3 is proved. □

Passing to the limit \( M \to \infty \), or \( a \to 0 \), in Eq. (2.24) and using Lemma 3, one gets

\[
u(x) = u_0(x) - \int_D G(x, y) p(y) u(y) dy,
\] (2.34)
Applying the operator $L_0 = \nabla^2 + k^2 - q_0(x)$ to this equation and using (1.3) and (1.4), one obtains Eq. (1.24). Formulas (1.26) and (1.27) follow from the above equation and from formula (1.12). This concludes the proof of Theorem 2. 

Remark 2: It is possible (and not difficult) to generalize Theorem 2 to the case of particles with different shapes. Since this does not lead to an essentially new result, we do not go into detail. In Refs. 6, 11, and 19 one can find analytical formulas for the $S$-matrix for wave scattering by small bodies of arbitrary shapes.

Proof of Theorem 3: Now we assume $\xi_m = 0, 1 \leq m \leq M$, which means that all the small particles are acoustically hard. In this case Eq. (2.10) takes the form

$$
\sigma_j = \lambda_j \sigma_j + 2u_{e_n}(s), \quad s \in S_j, \quad 1 \leq j \leq M, \tag{2.33}
$$

where

$$
u_n(x) := u_0(x) + \sum_{m \neq j} \int_{S_m} G(x, s) \sigma_m(s) ds.
$$

We cannot use approximation (2.2) because the quantity $Q_m$ now is of the same order of magnitude as the integral $\int S_m \left[ G(x, s) - G(x, x_m) \right] \sigma_m(s) ds$, or even smaller than this integral. This is established below. While under the assumptions of Theorem 2 we had $Q_m = O(a)$, now, under the assumptions of Theorem 3, we have $Q_m = O(k^2 a^3)$, which is a much smaller quantity than $O(a)$ because $ka \ll 1$. To estimate the order of magnitude of $Q_m$, we integrate (2.33) over $S_j$ and use (2.15). The result is

$$
Q_j \approx \int_{S_j} u_{e_n} \, ds = \int_{D_j} \Delta u_e \, dx = \Delta u_e(\xi_j) V_j, \tag{2.35}
$$

where $V_j$ is the volume of $D_j$, and we have used the assumption $d \gg a$. This assumption allows one to claim that $u_e(x)$ is practically constant in the domain $D_j$ in the absence of the $j$th particle. Differentiation with respect to $x$ brings a factor $k$. Since we assume that $k > 0$ is fixed, this factor is not important for our argument, but to make the dimensionality of the term $V_j \Delta u_e$ clear, we may write $V_j \Delta u_e = O(k^2 a^3)$. This quantity has dimensionality of length since $ka$ is dimensionless. We now prove that the term $E_m := \int S_m \left[ G(x, s) - G(x, x_m) \right] \sigma_m(s) ds$, which was neglected under the assumptions of Theorem 2, because it was much smaller than $G(x, x_m) Q_m$, is now, under the assumption $\xi_m = 0, 1 \leq m \leq M$, of the same order of magnitude as $|G(x, x_m) Q_m|$, namely, $O(k^2 a^3 d^{-1})$, or even larger. We have

$$
\int_{S_m} \left[ G(x, s) - G(x, x_m) \right] \sigma_m(s) ds = \int_{S_m} \nabla_s G(x, a) \big|_{y=x_m} (s - x_m) \sigma_m(s) ds, \quad |x - x_m| \gg d \gg a, \tag{2.36}
$$

where we have used the assumption $|x - x_m| \gg a$ and kept the main term in Taylor’s expansion of the function $G(x, s) - G(x, x_m)$.

Recall that

$$
\int_{S_m} (s - x_m)^p \sigma_m(s) ds = - V_m \beta^{(m)}_{ij} \frac{\partial u_e(y)}{\partial y_j} \big|_{y=x_m}, \quad \tag{2.37}
$$

where one sums up index $j = 1, 2, 3, \beta^{(m)}_{ij}$ is the magnetic polarizability tensor defined in Ref. 19, [p. 55, formulas (5.13)–(5.15) and p. 62, formula (5.62)], and $(s - x_m)_p$ is the $p$th component of the vector $s - x_m$. 

$$
Namely, if

$$\sigma = A \sigma - 2N_j, \quad (2.38)$$

then

$$\int_s \sigma(s) ds = V \beta_{pj}, \quad (2.39)$$

where $V$ is the volume of the body with boundary $S$, $N_j$ is the $j$th component of the exterior unit normal $N$ to $S$, the role of the point $x_m$ from Eq. (2.37) is played by the origin, which is located inside $S$, and the role of $S_m$ is played by $S$. Equation (2.33) with $j=m$ can be written as

$$\sigma_m = A_m \sigma_m - 2N(m) \left( - \frac{\partial u(x)}{\partial y_j} \bigg|_{y=x_m} \right), \quad (2.40)$$

where one sums up over $j$ (but not over $m$). Compare (2.40) and (2.38) and get (2.37).

Formulas for the tensor $\beta_{pj} = \alpha_{pj}(\gamma)|_{y=1}$ for bodies of arbitrary shapes were derived in Ref. 19, p. 55, formula (5.15). In the cited formula, $\alpha_{pj}(\gamma)$ is the polarizability tensor of a dielectric body with a constant dielectric permittivity $\epsilon_j$ and $S$ is the surface of this body. The parameter $\gamma = (\epsilon_j - \epsilon_i)/(\epsilon_i + \epsilon_j)$, where $\epsilon_i$ is the dielectric permittivity of the surrounding medium. The case $\gamma = -1$ occurs when $\epsilon_i = 0$. This is the case, for example, in the problem of calculation of the magnetic dipole moment of a superconductor placed in a homogeneous magnetic field: in the superconductor the magnetic induction vector $B = 0$, which means that the magnetic permeability $\mu_i$ of such body is zero, $\mu_i = 0$, see Ref. 3. That is why the tensor $\beta_{pj}$ is called the magnetic polarizability tensor in Ref. 19.

From (2.36) and (2.37) it follows that

$$\int_{S_m} [G(x,s) - G(x,x_m)] \sigma_m(s) ds = - \frac{\partial G(x,y)}{\partial y_p} \bigg|_{y=x_m} \frac{\partial u(x)}{\partial y_j} \bigg|_{y=x_m} V m \beta_{pj}^{(m)}, \quad (2.41)$$

where one sums up over the repeated indices $p,j$, but not over $m$. The quantity on the right of (2.41) is of the order $O\left(k^2 a^{-1} d^3 \right)$ if $kd \gg 1$, that is, of the same order as $|G(x,x_m)Q_m|$, provided that $|x-x_m| \gg d \gg a$, and it is of the order $O(k a d^2)$ if $kd \ll 1$. Indeed, $\beta_{pj}^{(m)} = O(1)$, $V_m = O(a^3)$, and

$$|\nabla_y G(x,y)| \leq c \max \left( \frac{k}{d}, \frac{1}{d^3} \right) \text{ for } |x-y| \gg d \gg a,$$

where $c > 0$ is a constant independent of $d$.

We have

$$G(x,y) = g(x,y) - \int_D g(x,z) q_0(z) G(z,y) dz := g - TG,$$

where $T$ is compact as an operator in $L^p(D), p \geq 1$, under our assumptions, namely, $D \subset R^3$ is a bounded domain, and $q_0(x)$ is a bounded piecewise-continuous function. From this equation we get

$$\nabla_y G(x,y) = \nabla_y g(x,y) - T \nabla_y G.$$

Clearly,
\[\nabla_y g(x,y) = g(x,y) \left( ik \frac{1}{|x-y|} \right) \frac{y - x}{|x-y|},\]

so

\[|\nabla_y g(x,y)| \leq 2 \max \left( \frac{k}{4 \pi d}, \frac{1}{4 \pi d^2} \right) = \frac{1}{2 \pi} \max \left( \frac{k}{d^2} \right), \quad |x-y| \gg d > 0.\]

Thus

\[|\nabla_y G(x,y)| \leq |\nabla_y g(x,y)| \left[ 1 + c \int_D \frac{1}{|x-z|} |\nabla_y G(z,y)| dz \frac{|x-y|^2}{|z| |x-y| - 1} \right] = |\nabla_y g|(1 + c),\]

where

\[I := \int_D |\nabla_y G(z,x)| \frac{dz}{|x-z| \sqrt{1 + k^2 |x-y|^2}} \leq c \int_D \frac{dz}{|x-z| \sqrt{1 + k^2 d^2}}.\]

One has

\[\int_D \frac{dz}{|x-z| \sqrt{1 + k^2 d^2}} \leq c \|x-y\|,\]

and

\[\sup_{x,y \in D} \|x-y\| \leq c,\]

where \(c = c(D)\) is a constant. Therefore

\[I \leq \frac{c}{\sqrt{1 + k^2 d^2}} \leq c,\]

and

\[|\nabla_y G(x,y)| \leq c \max \left( \frac{k}{d^2} \right) \frac{1}{\sqrt{1 + k^2 d^2}} \leq c \max \left( \frac{k}{d^2} \right),\]

as claimed.

If \(\frac{k}{d} \geq \frac{1}{d^2}\), i.e., \(kd \gg 1\), then \(|\nabla_y G(x,y)| \leq c \frac{k}{d}, \quad |x-y| \gg d > 0.\)

If \(\frac{k}{d} < \frac{1}{d^2}\), i.e., \(kd \ll 1\), then \(|\nabla_y G(x,y)| \leq \frac{c}{d^2}, \quad |x-y| \gg d > 0.\)

Therefore, the right side of (2.41) is \(O(k^2 a^3 / d)\) if \(kd \gg 1\), in which case it is of the same order as the term \(G(x,x_m)Q_m\) if \(kd < 1\), then the right side of (2.41) is \(O(ka^3 / d^2)\), in which case it may become larger than the term \(G(x,x_m)Q_m\) because the ratio \((ka^3 / d^2) / (k^2 a^3 / d) = 1/kd \gg 1\) provided that \(kd < 1\).

Writing field (2.1) in the form

\[u_M(x) = u_0(x) + \sum_{m=1}^M G(x,x_m)Q_m + \sum_{m=1}^M \int_{S_m} [G(x,s) - G(x,x_m)] \sigma_m(s) ds\]

and using formulas (2.35) and (2.41), one gets

\[u_M(x) = u_0(x) + \sum_{m=1}^M G(x,x_m)Q_m + \sum_{m=1}^M \int_{S_m} [G(x,s) - G(x,x_m)] \sigma_m(s) ds\]
Applying the operator \( L_0 \), sufficient that

\[
 u_M(x) = u_0(x) + \sum_{m=1}^{M} G(x,x_m) \Delta u_v(x_m) V_m = \sum_{m=1}^{M} \partial G(x,x_m) \frac{\partial u_v(x_m)}{\partial y_j} V_m \beta^{(m)}_{pj}(x_m), \tag{2.43}
\]

and over the repeated indices \( p, j \), one sums up.

Let \( a \to 0 \), \( M \to \infty \). We want to give sufficient conditions for passing to this limit in (2.43).

**Lemma 4**: Assume that for any subdomain \( \widehat{D} \subset D \) the following limits exist:

\[
 \lim_{a \to 0} \sum_{D_m \subset \widehat{D}} V_m \beta^{(m)}_{pj}(x_m) = \int_{\widehat{D}} \beta_{pj}(y) \nu(y) dy, \tag{2.44}
\]

\[
 \lim_{a \to 0} \sum_{D_m \subset \widehat{D}} V_m = \int_{\widehat{D}} \nu(y) dy. \tag{2.45}
\]

Then the limiting form of Eq. (2.43) is

\[
 \mathcal{U}(x) = u_0(x) + \int_{\widehat{D}} G(x,y) \Delta \mathcal{U}(y) \nu(y) dy - \int_{\widehat{D}} \frac{\partial G(x,y)}{\partial y_j} \beta_{pj}(y) \nu(y) dy, \tag{2.46}
\]

where one sums up over the repeated indices \( p, j \).

**Remark 3**: If one assumes that \( \nu(y) \) vanishes near the boundary \( S \) of \( D \) and integrates the last integral in (2.46) by parts, one gets

\[
 \mathcal{U}(x) = u_0(x) + \int_{\widehat{D}} G(x,y) \left\{ \Delta \mathcal{U}(y) \nu(y) + \sum_{p,j=1}^{3} \frac{\partial}{\partial y_p} \left( \frac{\partial \mathcal{U}(y)}{\partial y_j} \beta_{pj}(y) \nu(y) \right) \right\} dy. \tag{2.47}
\]

Applying the operator \( L_0 \) to both sides of (2.47) and using (1.4) one gets

\[
 L_0 \mathcal{U} + \nu(y) \Delta \mathcal{U}(x) + \sum_{p,j=1}^{3} \frac{\partial}{\partial y_p} \left( \frac{\partial \mathcal{U}(y)}{\partial y_j} \beta_{pj}(y) \nu(y) \right) = 0. \tag{2.47'}
\]

**Remark 4**: If all the small particles are identical, then \( V_m = c_1 a^3 \), where the positive constant \( c_1 \) does not depend on \( m \), and \( \beta^{(m)}_{pj} = \beta_{pj} \). Then

\[
 \lim_{a \to 0} \sum_{D_m \subset \widehat{D}} V_m = \lim_{a \to 0} \left[ c_1 a^3 \sum_{D_m \subset \widehat{D}} 1 \right] = \lim_{a \to 0} [c_1 a^3 N(\widehat{D})], \tag{2.48}
\]

where \( N(\widehat{D}) \) is the number of small particles in the domain \( \widehat{D} \). For limit (2.48) to exist, it is sufficient that

\[
 \mathcal{N}(\widehat{D}) = \int_{\widehat{D}} \nu(y) dy, \tag{2.49}
\]

where \( \nu(y) \geq 0 \) is a continuous function, and the limit in (2.48) is equal to \( \int_{\widehat{D}} \nu(y) dy \).

One can write (2.49), with \( |\widehat{D}| = dy \), \( y \in \widehat{D} \), as

\[
 N(y) dy = \frac{\nu(y)}{c_1 a^3} dy, \tag{2.50}
\]

where \( N(y) \) is defined by the above formula. In contrast to Theorem 2, where \( M = O(1/a) \), we now have \( M = O(1/a^7) \).

Similarly,
We have replaced

\[
\lim_{a \to 0} \sum_{D_m \subset D} V_m \beta^{(m)}_p(x_j) = \lim_{a \to 0} [c_i a^j \beta_p N(\overline{D})] = \beta_p \int_D v(y)dy.
\] (2.51)

We gave in this remark some practically realizable sufficient conditions for the existence of limits.

Let us verify that if limits (2.44) and (2.45) exist, then the limit of the right side of Eq. (2.43) exists, and, denoting this limit by

\[
\mathcal{U}(x) = \lim_{a \to 0} u_M(x),
\]

one obtains the limiting form of Eq. (2.43):

\[
\mathcal{U}(x) = u_0(x) + \int_D G(x,y)\Delta \mathcal{U}(y)v(y)dy - \int_D \sum_{p,j=1}^3 \frac{\partial G(x,y)}{\partial y_p} \frac{\partial \mathcal{U}(y)}{\partial y_j} \beta_p(y)v(y)dy,
\]

which is Eq. (2.46).

We took into account that

\[
\lim_{a \to 0} u_\epsilon(x) = \mathcal{U}(x).
\]

This is so because, as \(a \to 0\), the input of a single particle into the field \(\mathcal{U}(x)\) tends to zero.

Let us verify the existence of the limit of the right side of Eq. (2.43). We use, as in the proof of Lemma 3, a representation of \(D\) of the form \(D = \bigcup_{p=1}^P \Delta_p\), and assume that

\[
\lim_{P \to \infty} \max_{1 \leq p \leq P} \text{diam } \Delta_p = 0.
\] (2.53)

Then

\[
\lim_{a \to 0} \sum_{m=1}^M G(x,y_m)\Delta u_\epsilon(x_m)V_m = \lim_{a \to 0} \sum_{p=0}^P \sum_{D_m \subset \Delta_p} G(x,y_m)\Delta u_\epsilon(x_m)V_m
\]

\[
= \sum_{p=1}^P G(x,y^{(p)})\Delta u_\epsilon(y^{(p)})(1 + e_p)\lim_{a \to 0} \sum_{D_m \subset \Delta_p} V_m
\]

\[
= \sum_{p=1}^P G(x,y^{(p)})\Delta u_\epsilon(y^{(p)})(1 + e_p e')v(y^{(p)})(1 + e'_p)|\Delta_p|,
\] (2.54)

where

\[
\lim_{P \to \infty} \max_{p} |e_p| + |e'_p| = 0.
\] (2.55)

Let \(P \to \infty\) in (2.54) and use (2.55) to get

\[
\lim_{P \to \infty} \sum_{p=1}^P G(x,y^{(p)})\Delta u_\epsilon(y^{(p)}|\Delta_p|(1 + e_p + e'_p) = \int_D G(x,y)\Delta \mathcal{U}(y)v(y)dy.
\] (2.56)

We have replaced \(u_\epsilon(y^{(p)})\) in the limit \(P \to \infty\) by \(\mathcal{U}(y)\), because

\[
\mathcal{U}(y) - u_\epsilon(y) = \int_{S_m} G(y,t)\sigma(t)dt = o(1) \quad \text{as } a \to 0, \quad |y - x_m| \geq d.
\] (2.57)

From (2.54) and (2.56) one gets
The singular points \( x = y \in D \) of \( G(x, y) \) are treated as in the proof of Theorem 2. The function \( |G(x, y)| \leq c|x-y|^{-1} \) as \( |x-y| \to 0 \), so \( |G(x, y)| \in L^1(D) \) as a function of \( y \) for any \( x \in D \).

Similar arguments, applied to the last sum in (2.43), lead to the formula

\[
\lim_{\varepsilon \to 0} \sum_{m=1}^{M} \frac{\partial G(x, x_m) \partial u_{x_m}}{\partial y_p} \partial y_j \beta_{x_m}^{(m)}(x_m) = \int_{D} \frac{\partial G(x, y)}{\partial y_p} \frac{\partial \beta}{\partial y_j}(y) \nu(y) dy,
\]

where one sums up over the repeated indices \( p, j \).

In Sec. IV we discuss the compatibility of the condition \( d \gg a \) and the existence of limit (1.29).

### III. AUXILIARY RESULTS

In this section we prove Theorem 1 and Lemmas 1 and 2.

**Proof of Lemma 1**: Let us start with the following observations:

\[
\|t - y\| - |x - y| \leq |t - y - (x - y)| = |t - x| \leq a,
\]

\[
\sup_{a \leq s \leq a} |e^{is} - 1| \leq a,
\]

\[
|e^{ik|t-y|} - e^{ik|x-y|}| = |e^{ik|t-y|-|x-y|)} - 1| \leq ka,
\]

where the last inequality follows from (3.2).

One has

\[
|G(t, y) - g(x, y)| = \frac{|x - y| e^{ik|t-y|} - |t - y| e^{ik|x-y|}}{|x - y|} \leq \frac{|x - y| - |t - y||}{|x - y|} + \frac{|t - y||e^{ik|x-y|} - e^{ik|t-y|}}{|x - y||t - y|}
\]

\[
\leq \frac{a}{|x - y||t - y|} + \frac{ka}{|x - y|} \leq \frac{a}{d^2(1 - ald)} + \frac{ka}{d} \leq O\left(\frac{a}{d^2}\right) + \frac{ka}{d}.
\]

**Lemma 1** is proved.

**Proof of Lemma 2**: Let us start with the equation

\[
G(x, y) = g(x, y) - \int_{D} g(x, z)q_{00}(z)G(z, y)dz,
\]

where \( q_{00} \) is defined in (1.2). From (3.5) one gets

\[
|G(t, y) - G(x, y)| \leq |g(t, y) - g(x, y)| + \left| \int_{D} [g(t, z) - g(x, z)]q_{00}(z)G(z, y)dz \right|
\]

\[
\leq O\left(\frac{a}{d^2}\right) + \frac{ka}{d} + c \int_{D} |g(t, z) - g(x, z)| \frac{dz}{|z - y|}.
\]

Here we have used Lemma 1 and the estimates

\[
\sup_{z \in D} |q_{00}(z)| \leq c_4, \quad |G(z, y)| \leq c_5|z - y|^{-1},
\]

where \( c_4, c_5 > 0 \) are some constants.
Let us estimate the integral
\[
I := \int_D \frac{|g(t,z) - g(x,z)|}{|z - y|} \, dz = \int_{|x-z| > d/4} \frac{|g(t,z) - g(x,z)|}{|z - y|} \, dz + \int_{|x-z| \leq d/4} \frac{|g(t,z) - g(x,z)|}{|z - y|} \, dz
\]
\[= I_1 + I_2. \tag{3.8} \]
By Lemma 1, which is applied to \(I_1\) with \(d\) replaced by \(d/4\), one gets
\[
I_1 \leq c \left( \frac{a}{d^2} + \frac{ka}{d} \right) \int_{|x-z| > d/4} \frac{dz}{|z-y|} \leq c \left( \frac{a}{d^2} + \frac{ka}{d} \right). \tag{3.9}
\]
Here and below we do not write \(z \in D\) under the integration sign to simplify the notations.

Let us estimate \(I_2\):
\[
I_2 \leq \frac{1}{4\pi} \int_{|x-z| \leq d/4} \frac{dz |e^{ik|t-z|} - e^{ik|x-z|}|}{|z-y||t-z||x-z|}. \tag{3.10}
\]
One has
\[
|e^{ik|t-z|} - e^{ik|x-z|}| \leq \|x-z| - |t-z|\| + |t-z| |e^{ik|x-z|} - e^{ik|x-z|}| \leq |x-t| + |t-z| |t-z| - |x-z| \leq |x-t| + k|t-z| |t-x|. \tag{3.11}
\]
Thus, with \(|x-y| \geq d \gg a\) and \(|t-x| \leq a\), one has
\[
I_2 \leq \frac{1}{4\pi} \int_{|x-z| \leq d/4} \frac{dz (|x-t| + k|t-z| |t-x|)}{|z-y||t-z||x-z|} \leq \frac{|t-x|}{4\pi} \left( \int_{|x-z| \leq d/4} \frac{dz}{|z-y||t-z||x-z|} \right) \leq c a \left( \frac{1}{d^2} + \frac{k}{d} \right). \tag{3.12}
\]
From (3.9) and (3.12), estimate (1.19) follows. Lemma 2 is proved.

**Proof of Theorem 1**: Let us first prove that if the conditions in Eq. (1.14) hold, then problem (1.5)–(1.7) has at most one solution. It is sufficient to prove that the homogeneous problem
\[
(\nabla^2 + k^2 - q_0)u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m, \tag{3.13}
\]
\[
\frac{\partial u}{\partial r} -iku = o \left( \frac{1}{r} \right), \quad u = O \left( \frac{1}{r} \right), \quad r = |x| \to \infty, \tag{3.14}
\]
\[
u_N = \zeta_m u \quad \text{on} \quad S_m, \quad 1 \leq m \leq M \tag{3.15}
\]
has only the trivial solution if the conditions in Eq. (1.14) hold.

Taking the complex conjugate of (3.13)–(3.15) one gets
\[
(\nabla^2 + k^2 - \tilde{q}_0(x))\tilde{u} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m, \tag{3.16}
\]
\[
\frac{\partial \tilde{u}}{\partial r} + ik\tilde{u} = o \left( \frac{1}{r} \right), \quad \tilde{u} = O \left( \frac{1}{r} \right), \quad r = |x| \to \infty, \tag{3.17}
\]
\( \vec{u}_N = \xi_m \vec{u} \quad \text{on} \quad S_m, \quad 1 \leq m \leq M. \)  

Multiply (3.13) by \( \vec{u} \), (3.16) by \( u \), subtract from the first equation the second one, and integrate over the region \( (\mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m) \cap B_R = D_R \), where \( B_R \) is the ball centered at the origin of radius \( R \). Using Green’s formula, one gets

\[
0 = \int_{D_R} [\vec{u} \nabla^2 u - u \nabla^2 \vec{u} - (q_0 - \vec{q}_0)[|u|^2]] dx = -2i \int_{D_R} \text{Im} \ q_0(x)|u|^2 dx + \int_{|x|=R} \left( \frac{\partial \vec{u}}{\partial r} - u \frac{\partial \vec{u}}{\partial r} \right) ds \\
- \sum_{m=1}^M \int_{S_m} \left( \vec{u} \frac{\partial u}{\partial n} - u \frac{\partial \vec{u}}{\partial n} \right) ds. 
\]

Using (3.17) and (3.18), one rewrites (3.19) as follows:

\[
0 = -2i \int_{D_R} \text{Im} \ q_0(x)|u|^2 dx + 2ik \int_{|x|=R} |u|^2 dx + o(1) - 2i \sum_{m=1}^M \int_{S_m} \xi_m |u|^2 ds. 
\]

Letting \( R \to \infty \) and taking into account that \( q_0(x) = 0 \) in \( D' = \mathbb{R}^3 \setminus D \), one gets

\[
0 \leq \int_{D' \setminus \bigcup_{m=1}^M D_m} \text{Im} \ q_0(x)|u|^2 dx + \sum_{m=1}^M \int_{S_m} \text{Im} \ \xi_m |u|^2 ds - k \limsup_{R \to \infty} \int_{|x|=R} |u|^2 dx. 
\]

Since all the terms on the right side of this relation are nonpositive by the assumptions in Eq. (1.14), it follows that

\[
\limsup_{R \to \infty} \int_{|x|=R} |u|^2 dx = 0. 
\]

This implies that \( u = 0 \) (see, Ref. 12, p. 231).

Thus, uniqueness of the solution to problem (1.5)–(1.7) is proved.

Let us prove the existence of the solution to (1.5)–(1.7) of form (1.10). The existence of the solution of form (1.10) will be established if one proves the existence of \( \sigma_m, 1 \leq m \leq M \), such that the boundary condition (1.7) is satisfied:

\[
u_e N - \xi_m \mu_e + \frac{A \sigma_j - \sigma_j}{2} - \xi_j \mu_e \sigma_j = 0, \quad 1 \leq j \leq M. \]

Here \( \nu_e \), which depends on \( j \), is defined by the formula

\[
u_e := u - \int_{S_j} G(x, s) \sigma_j(s) ds = u_0 + \sum_{m \neq j} \int_{S_m} G(x, s) \sigma_m(s) ds. \]

Under our assumptions, \( S_m \in C^{1,\lambda} \) uniformly with respect to \( m \). Therefore Eq. (3.22) is of Fredholm type in the space \( L^2(S_m) \). The corresponding homogeneous equation, i.e., the equation with \( u_0 = 0 \), cannot have a nontrivial solution because such a solution would generate by formula (1.10) with \( u_0 = 0 \) a function \( u_M(x) = \sum_{m=1}^M \int_{S_m} G(x, s) \sigma_m(s) ds \), which would solve the homogeneous problem (1.5)–(1.7). We have already proved that such a function has to be zero in \( \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m \).

Thus, \( u_M|_{S_m} = 0, 1 \leq m \leq M \), and \( u_M \) solves the problem

\[
L_0 \mu_M = 0 \quad \text{in} \quad D_m, \quad u_M|_{S_m} = 0, \quad 1 \leq m \leq M. 
\]

If \( \text{diam } D_m \leq 2a \) is sufficiently small, then problem (3.24) has only the trivial solution for every \( m, 1 \leq m \leq M \). Therefore \( u_M = 0 \) in \( D_m \) and in \( \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m \). Therefore, by the formula for the jumps of the normal derivatives of the single layer potential,
we conclude that $\sigma_m = 0$, $1 \leq m \leq M$. This implies the existence of the solution to problem (1.5)–(1.7) of form (1.10).

Theorem 1 is proved. 

Let us return to the assumptions of Theorem 2, namely,

$$\xi_m = O\left(\frac{1}{a}\right), \quad aN(\Delta_b(y)) = N(y)|\Delta_b(y)|((1 + o(1)).$$

where $\Delta_b(y)$ is a cube, centered at the point $y$ with the side $b > 0$, and $o(1)$ is related to the limiting process $b \to 0$.

Under these assumptions let us establish an estimate for the function $u_M = u_M - u_0$, which is uniform with respect to $M \to \infty$, or $a \to 0$. From this estimate it follows that $u_M$ converges, as $a \to 0$, in $L^2(\mathbb{R}^3, (1 + |x|)^{1-\gamma})$, where $\gamma > 0$ is an arbitrary fixed constant. The function $u_M$ satisfies the radiation condition at infinity. The function $u_0 \in H^2_{\text{loc}}(\mathbb{R}^3)$ solves the equation $L_0 u_0 = 0$ in $\mathbb{R}^3$.

Let $D_e := \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m$ and $S' := \bigcup_{m=1}^M S_m$. Let

$$\|v\| := \left(\int_{D_e} |v(x)|^2(1 + |x|)^{1-\gamma} dx\right)^{1/2}, \quad \|v\| = \sum_{m=1}^M \left(\int_{S_m} |v_m|^2 + |v|^2\right) ds^{1/2}. $$

The estimate we wish to prove is

$$\|v_M\| \leq c\|u_0\|.$$  

Here and below $c > 0$ stand for various constants independent of $a$. An inequality similar to (3.25) was used in Ref. 16, where a theorem, I have called “modified Rayleigh conjecture,” is proved.

Let us outline the proof of inequality (3.25).

Step 1. If $M = O(1/a)$, then the right side of (3.25) is bounded as $a \to 0$.

Indeed, the number of small particles is $M = O(1/a)$ and $u_0$ is $C^2_{\text{loc}}(\mathbb{R}^3)$, so that $u_0$ and $u_{0N}$ are bounded in $L^2(S_m)$ uniformly with respect to $m$, $1 \leq m \leq M$. Thus,

$$\|u_0\| \leq \frac{c}{a} \max_{1 \leq m \leq M} \left(\int_{S_m} (|u_{0m}|^2 + |u_0|^2) ds^{1/2}\right) \leq \frac{c}{a} |S_m|^{1/2} \leq c,$$

where $c > 0$ stand for various constants independent of $a$.

Step 2. If inequality (3.25) is false, then there is a sequence $u_0^{(n)}$, $\|u_0^{(n)}\| = 1$, such that $\|v_M^{(n)}\| \to n$.

Define $w^{(n)} := u^{(n)}/\|u^{(n)}\|$. Then

$$\|w^{(n)}\| = 1.$$  

From the weak compactness of bounded sets in $L^2$, it follows that one may select a subsequence, denoted again $w^{(n)}$, such that $w^{(n)}$ converges weakly in $L^2_{\text{loc}}(D_e)$ to a function $w$. The function $w^{(n)}$ solves the problem

$$L_0 w^{(n)} = 0 \quad \text{in} \quad D_e,$$

$$w^{(n)} - \xi_n w^{(n)} = \left(\xi_n u_0^{(n)} - u_0^{(n)}\right)/\|u^{(n)}\| \quad \text{on} \quad S_m, \quad 1 \leq m \leq M,$$

and $w^{(n)}$ satisfies the radiation condition.

It follows from (3.27) that $\|\nabla^2 w^{(n)}\| < c$, so $\|w^{(n)}\|_{H^2_{\text{loc}}(D_e)} < c$, where $H^2_{\text{loc}}(D_e)$ is the Sobolev space (see, for example, Ref. 1, where the theory of these spaces is presented). Thus, one may assume, using the compactness of the embedding from $H^2_{\text{loc}}$ into $L^2_{\text{loc}}$, that $w^{(n)}$ converges to $w$.
strongly in $L^2_{\text{loc}}(D_x)$. This and Eq. (3.27) imply that $w^{(n)}$ converges to $w$ strongly in $H^2_{\text{loc}}(D_x)$, so that $w$ solves Eq. (3.27) and satisfies the radiation condition and the homogeneous boundary condition (3.27), that is, $w_N - \xi_m w = 0$ on $S_m$. Therefore, by the already proved uniqueness theorem (see the proof of Theorem 1), we conclude that $w = 0$. The terms $u^{(n)}_0/\|v^{(n)}\|$ and $u^{(n)}_0/\|v^{(0)}\|$ tend to zero as $n \to \infty$, because $\|v^{(0)}\| > n$. Therefore, the limiting function $w$ satisfies the homogeneous boundary condition $w_N = \xi_m w$ on $S_m$, $1 \leq m \leq M$.

Let us prove that $|w^{(n)}(x)| < c/|x|$, $|x| > R$, where $R > 0$ is sufficiently large and $c > 0$ does not depend on $n$.

For $w^{(0)}$ one has a representation by the Green formula in the region $|x| > R$, where $R > 0$ is large enough, so that the ball $B_R := \{x: |x| < R\}$ contains $D$. Namely,

$$w^{(n)}(x) = \int_{|x|=R} (w^{(n)}_r g(x,s) - g_r(x,s)w^{(n)})(s)ds, \quad |x| > R, \quad (3.28)$$

where the derivatives with respect to $r$ are the derivatives along the normal to the sphere $S_R := \{s: |s| = R\}$, and $g$ is defined in (1.15). It follows from (3.28) that $|w^{(n)}(x)| < c/|x|$ for $|x| > R$, where $c > 0$ is a constant independent of $n$, because local convergence in $H^2$ implies that the $L^2(S_R)$-norms of $w^{(n)}$ and of $w^{(n)}_r$ are bounded uniformly with respect to $n$.

Therefore

$$\lim_{n \to \infty} \|w^{(n)} - w\| = 0, \quad (3.29)$$

because on compact sets $\lim_{n \to \infty} \|w^{(n)} - w\|_{H^2_{\text{loc}}(D_x)} = 0$, and near infinity the inequality $|w^{(n)}(x)| < c/|x|$ implies that

$$\int_{|x| > R} |w^{(n)}(x)|^2(1 + |x|)^{-1-\gamma}dx = O(R^{-\gamma}) \to 0, \quad R \to \infty,$$

so that (3.29) holds. Because of the uniqueness of the limit, not only a subsequence of $w^{(n)}$ but the sequence itself converges to $w$ as $n \to \infty$.

This leads to a contradiction, because $w = 0$ and (3.26) together with (3.29) imply $\|w\| = 1$.

This contradiction proves inequality (3.25).

From inequality (3.25) and step 1, one concludes that $u_M$ contains a weakly convergent in $L^2_{\text{loc}}(D_x)$ subsequence. By the arguments, similar to that given above, this subsequence converges in $L^2(R^3, (1 + |x|)^{-1-\gamma})$. Its limit solves Eq. (1.24).

The relation $M = O(1/a)$ plays an important role in our proof of Theorem 2 and in step 1 in the above argument.

**IV. APPLICATION TO CREATING SMART MATERIALS**

Let us ask the following question: Can one make a material with a desired refraction coefficient $n^2(x)$ in a bounded domain $D \subseteq R^3$, filled by a material with a known refraction coefficient $n_0^2(x)$, for example, $n_0^2(x) = n_0^2 = \text{const}$ in $D$, by embedding into $D$ a number of small particles, each of which is defined by its shape and its boundary impedance?

We give an affirmative answer to this question. Moreover, we give explicit formulas for the number of small particles of characteristic size $a$ to be embedded in the domain $D$ around a point $x \in \Delta$, where $\Delta \subseteq D$ is a small cube, centered at $x$, with a side $b \gg a$, and for the boundary impedances $\xi_m$ of these particles (see Theorem 2). Specifically, given the original refraction coefficient $n_0^2(x)$ in $D$ and the desired refraction coefficient $n^2(x)$, we calculate $q_0(x) = k^2 - k^2 n_0^2(x)$ and $q(x) = k^2 - k^2 n^2(x)$, and then calculate $p(x) = q(x) - q_0(x) = k^2 [n^2(x) - n_0^2(x)]$. Then we find (nonuniquely) three functions: $N(x) \geq 0$, a real-valued function $h_1(x)$, and a nonpositive function $h_2(x)$. Define $h(x) = h_1(x) + ih_2(x)$. How does one find these functions is explained below. If $N(x)$ and $h(x) = h_1(x) + ih_2(x)$ are found, then the boundary impedance $\xi(x)$ is defined by formula (4.2) and the number of particles in any small cube $\Delta \subseteq D$ is found by formula (1.21):
\[ \mathcal{N}(\Delta) = \frac{1}{a} \int_{\Delta} N(x) \, dx, \]

where \( a \) is the size of one particle.

Consider first the particles satisfying the assumptions of Theorem 2. More specifically, suppose that all the particles are balls of the same radius \( a \). In this case

\[ |S_m| = 4\pi a^2, \quad V = \frac{4\pi}{3} a^3, \quad J_m = \int_{|s|<a} \int_{|s-t|} \frac{ds \, dt}{|s-t|} = 16\pi^2 a^3, \]

so

\[ c_1 = 4\pi, \quad c_2 = 16\pi^2, \quad c_3 = \frac{4\pi}{3}, \quad \frac{4\pi c_1^2}{c_2} = 4\pi, \]

and formula (1.25) yields

\[ p(x) = \frac{4\pi N(x)h(x)}{1 + h(x)}, \quad (4.1) \]

where \( h(x) \) is defined by the choice of the boundary impedances by formula (1.20):

\[ \zeta(x) = \frac{h(x)}{a}, \quad (4.2) \]

and \( N(x) \) is defined by formula (1.21).

If the original refraction coefficient is \( n_0^2(x) \), then the corresponding potential is \( q_0(x) \)

\[ = k_0^2[1 - n_0^2(x)] \] by formula (1.2). If the desired refraction coefficient in \( D \) is \( n^2(x) \), then the corresponding potential is \( q(x) = k^2[1 - n^2(x)] \), so

\[ p(x) = q(x) - q_0(x) = k^2(n_0^2(x) - n^2(x)). \quad (4.3) \]

To create a material with the desired refraction coefficient \( n^2(x) \), it is sufficient to choose \( N(x) \) and \( h(x) \) so that (4.1) holds with \( p(x) \) defined in (4.3). If the new material with the refraction coefficient \( n^2(x) \) has some absorption, that is, \( \text{Im } n^2(x) \not\equiv 0 \), and \( \text{Im } n_0^2=0 \), then \( \text{Im } p(x) \not\equiv 0 \). Let us prove

\[ \text{that any function } p(x) \text{ in } D \text{ with } \text{Im } p \not\equiv 0 \text{, can be obtained (in many ways, nonuniquely) by } \text{formula (4.1) with some choices of a non-negative function } N(x) \text{ and a function } h(x) \text{ with } \text{Im } h \not\equiv 0. \quad (4.3) \]

Let \( p(x) = p_1(x) + ip_2(x) \), \( p_2(x) \not\equiv 0 \), and \( h(x) = h_1(x) + ih_2(x) \), \( h_2(x) \not\equiv 0 \). Assume that \( p(x) \) is given. Then (4.1) implies

\[ p_1 + ip_2 = 4\pi \frac{(h_1 + ih_2)(1 + h_1 - ih_2)}{(1+h_1)^2 + h_2^2} N(x). \quad (4.4) \]

Thus

\[ p_1 = 4\pi \frac{h_1 + h_2^2}{(1 + h_1)^2 + h_2^2} N(x), \quad p_2 = 4\pi \frac{h_2}{(1 + h_1)^2 + h_2^2} N(x). \quad (4.5) \]

There are many choices of the three functions: \( N(x) \equiv 0 \), \( h_2(x) \equiv 0 \), and a real-valued function \( h_1(x) \) such that the relations in Eq. (4.5) hold. For example, if \( p_1 > 0 \) and \( p_2 \not= 0 \), then one can choose
It is a simple matter to check that the relations in Eq. (4.5) hold with choice (4.6). Since one has three functions $h_1(x)$, $h_2(x) \equiv 0$, and $N(x) \equiv 0$ to satisfy the two equations in Eq. (4.5) with $p_2(x) \equiv 0$, there are many ways to do this. A particular choice of $h(x) = h_1(x) + ih_2(x)$ and $N(x)$ yields the surface impedance $\zeta(x)$ of the particles to be embedded around each point $x \in D$. 

Equation (4.2) provides the number of particles per unit volume around the point $x$. 

namely, by formula (1.21) this number is $N(x)/a$, so that the number of particles to be embedded is obtained in the volume $dx$ around point $x$ is equal to $N(x)/adx$. The smallest distance $d$ between the embedded particles should satisfy the inequality $d \geq a$. One may try to take practically $d \geq 10a$. 

Example 1: Suppose that the elementary subdomain $\Delta_p$, used in the proof of Lemma 3, is a cube with the side $b \gg d$, $x \in \Delta_p$. Let, for example, $b = 10^{-2}$ cm, $d = 10^{-3}$ cm, and $a = 10^{-5}$ cm. Then there are $(b/d)^3 = 10^3$ small particles in $\Delta_p$ around a point $x$, the center of $\Delta_p$. The function $N(x)$ in Eq. (4.5) is obtained from the formula $N(x)/a|b|^3 = 10^3$ [use (1.21) with $D = \Delta_p$], so $N(x)$ is equal to $10^{3-3} \times 10^3 \times 10^0 = 10^3$. The number of small particles embedded in the cube $\Delta_p$ around point $x$ is equal to $10^3$ in this example. The relative volume of these particles in $\Delta_p$ is $10^3 \times 4 \pi 10^{-15} \times 10^6 = 4.18 \times 10^{-6}$, so it is quite small, which is in full agreement with our theory.

Assumption (1.8), specifically, $d \geq a$, is compatible with requirement (1.21). Indeed, if one denotes by $N(D)$ the left side of (1.21), then $N(D) = O(1/a)$ for any $D \subset D$.

Let us assume that $D$ is a unit cube, and denote by $N(D)$ the left side of (1.21). The assumption $d \geq a$ implies that the number $N(D)$ of particles in $D$ is $O(1/d^3)$. These relations are compatible if and only if $O(1/a) = O(1/d^3)$, i.e., $d = O(a^{1/3})$. Therefore, it is possible to have $a \rightarrow 0$, $d \rightarrow 0$ and Eq. (1.21) satisfied.

Let us discuss the new material properties, specifically, anisotropy, when acoustically hard particles are embedded in the domain $D$, and the assumptions of Theorem 3 are valid. The physical situation is now quite different from the one in Theorem 2. From the physical point of view one can anticipate the drastic difference because the wave scattering by one small acoustically soft particle of the characteristic size $a$ is isotropic and the scattering amplitude is of order $a$, while the wave scattering by a small acoustically hard particle is anisotropic and the corresponding scattering amplitude is of order $k^2a^3$, (see Ref. 19, Chap. 7). We assume that $ka \ll 1$, say, $ka < 0.1$, so that the quantity $k^2a^3 = (ka)^2a$ is 100 times less than $a$.

Example 2: Let us assume again that the small particles are all balls of the same radius $a$. Then

$$V_m = \frac{4}{3} \pi a^3, \quad \nu(y)|\Delta_p| = \frac{4}{3} \pi a^3 N(\Delta_p).$$

where $N(\Delta_p)$ is the number of small particles in a small cube $\Delta_p$ centered at the point $y$. If $b$ is the size of the edge of the cube $\Delta_p$, then $\nu(y) = 4.18(a^3/b^3)N(\Delta_p)$, where $4.18$ is an approximate value of $4\pi/3$. The magnetic polarizability tensor $\beta_{pj}$ of a ball of radius $a$ is $\beta_{pj} = -\frac{3}{2} \delta_{pj}$, while the electric polarizability tensor of a perfectly conducting ball is $3 \delta_{ij}$, where

$$\delta_{pj} = \begin{cases} 1, & p = j \\ 0, & p \neq j \end{cases}.$$
where \( \nu(y) \) is the function defined in (1.29). Thus, in contrast to Example 1, where \( N(\Delta_p) = O(1/a) \), we now have \( N(\Delta_p) = O(1/a^3) \). The relative volume of the small particles in Example 2 is not negligible and does not go to zero as \( a \to 0 \), in contrast to Example 1.

Let us discuss the compatibility of the condition \( d \gg a \) and the existence of limits (1.28) and (1.29). If the condition \( d \gg a \) is compatible with the existence of limit (1.29), then it is compatible with the existence of limit (1.28). If limit (1.29) exists, then \( a^2 N(\bar{D}) = O(1) \), so \( N(\bar{D}) = O(a^{-3}) \). On the other hand, \( N(\bar{D}) = O(d^{-3}) \). These relations, in general, are not compatible because \( d \gg a \). Let us argue more precisely. Let \( \bar{D} = \Delta_p \), where \( \Delta_p \) is a cube with the edge of size \( b \). Let us assume that the small particles in \( \Delta_p \) are identical and their characteristic size is \( a \). If (1.29) holds, where \( \nu(y) \) is continuous and if \( b \) is small, then the right side of (1.29) equals \( \nu(y)b^3 \), \( y \in \Delta_p = \bar{D} \). The left side of (1.29) equals \( c_3 a^3 N(\Delta_p) \), where \( V = c_3 a^3 \) is the volume of one particle; \( c_3 = 4\pi/3 \) if the particle is a ball of radius \( a \). Thus \( N(\Delta_p) = (1/c_3) \nu(y)b^3/a^3 \). On the other hand, \( N(\Delta_p) = b^3/d^3 \), provided that one assumes that the centers of the small particles are at the uniform grid, so that there are \( b/d \) centers on the segment of length \( b \). If \( (1/c_3) \nu(y)b^3/a^3 = b^3/d^3 \), then \( a/d = (\nu(y)/c_3)^{1/3} \). Therefore the condition \( d \gg a \) is satisfied only if \( (\nu(y)/c_3)^{1/3} \ll 1 \), say, \( (\nu(y)/c_3)^{1/3} \ll 0.1 \). The number \( c_3 \) depends on the shape of the particle. If the particles are balls of radius \( a \), then \( c_3 = 4.18 \). Therefore \( \nu(y) \ll 4 \times 10^{-3} \).

The conclusion is as follows: The condition \( d \gg a \) is compatible with the existence of limit (1.29) only if the function \( \nu(y) \) in (1.29) is sufficiently small. In general, Eq. (1.31) cannot be reduced to a local differential equation for \( U(x) \). However, if \( \nu(y) \) is small, one may use perturbation theory to study Eq. (1.31). However, under an additional assumption, reasonable from the physical point of view, one can reduce integral-differential equation (1.31) to a differential equation. Namely, let us assume that \( \nu(y) \) is a continuously differentiable function in \( D \) which vanishes near the boundary \( S \).

Under this assumption one can integrate by parts the last integral in (1.31) and get

\[
U(x) = u_0(x) + \int_D G(x,y) \left[ \Delta U(y) \nu(y) + \sum_{p=1}^{3} \frac{\partial}{\partial y_p} \left( \frac{\partial U(y)}{\partial y_p} \beta_{pj}(y) \nu(y) \right) \right].
\]

Let us apply the operator \( L_0 = \nabla^2 + k^2 - q_0(x) \) to (4.8) and use (1.4) to get

\[
[\nabla^2 + k^2 - q_0(y)] U + \nu(y) \nabla^2 U(x) + \sum_{p=1}^{3} \frac{\partial}{\partial y_p} \left( \frac{\partial U(x)}{\partial y_p} \beta_{pj}(y) \nu(y) \right) = 0,
\]

where \( U(x) \) satisfies the radiation condition of type (1.6). This is an elliptic equation and the perturbation \( \mathcal{P} \) of the operator \( L_0 \) is

\[
\mathcal{P} U := \nu(x) \nabla^2 U(x) + \sum_{p=1}^{3} \frac{\partial}{\partial y_p} \left( \frac{\partial U(x)}{\partial y_p} \beta_{pj}(y) \nu(x) \right).
\]

This perturbation is the sum of the terms with positive small coefficient \( \nu(y) \) in front of the second derivatives of \( U(x) \) and a term with the first order derivatives of \( U(x) \):
waves is described by the integral-differential equation (4.8) or [under the additional assumption
on \( \nu(y) \), namely, \( \nu(y) \) vanishes near the boundary \( S \) of \( D \)] by the differential equation (4.9) with
variable coefficients in front of the senior (second order) derivatives and the terms with the first
order derivatives.

The role of the compatibility of the assumption \( d \gg a \) and of assumption (1.29) is quite
important. Although passing to the limit \( a \to 0 \), justified in the proof of Theorem 3, is based on
assumptions (1.28) and (1.29), without the assumption \( d \gg a \) one cannot expect, in general, that the
effective field \( u_e(x) \), acting on any single particle, is practically constant on the distances of the
order \( 2a \). This physical assumption is important for our theory.

From the mathematical point of view, if \( \nu(x) \) is not sufficiently small, then the existence of the
unique solution to Eq. (4.8) or of the solution to Eq. (4.9), satisfying the radiation condition, is not
guaranteed.

If, on the other hand, the quantity

\[
\sup_{x \in \mathbb{R}^3} (|\nu(x)| + |\nabla \nu(x)|) \ll 1,
\]

that is, this quantity is sufficiently small, then one can argue that the norm of the integral operator
in (4.8) in \( L^2(D) \) is small, so that Eq. (4.8) has a unique solution in \( L^2(D) \). This solution admits a
natural extension to the whole space \( \mathbb{R}^3 \) by the right side of (4.8) because \( \nu(y) \) vanishes outside \( D \).
Since \( G(x,y) \) satisfies the radiation condition, the solution to (4.8) also satisfies this condition.
Without the assumption that \(|\nu(x)| + |\nabla \nu(x)|\) is sufficiently small, one cannot use the above argu-
ment.

With this assumption one may solve Eq. (4.8) by iterations and find in this way an approxi-
mate solution to this equation. The first iteration yields the following approximate solution to Eq.
(4.8):

\[
\mathcal{U}(x) = u_0(x) + \int_D G(x,y) \left[ \Delta u_0(y) \nu(y) + \sum_{p,j=1}^3 \frac{\partial}{\partial y_p} \left( \frac{\partial u_0(y)}{\partial y_j} \beta_{pj}(y) \nu(y) \right) \right] \, dy. \tag{4.12}
\]

Formula (4.12) gives the correction to the solution \( u_0(x) \) of the unperturbed scattering problem,
i.e., the scattering problem in the absence of small bodies. Since one has

\[
\Delta u_0 = -k^2 n_0^2(x) u_0,
\]

(4.12) can be rewritten as

\[
\mathcal{U}(x) = u_0(x) - k^2 \int_D G(x,y) n_0^2(y) u_0(y) \nu(y) \, dy + \int_D G(x,y) \sum_{p,j=1}^3 \frac{\partial}{\partial y_p} \left( \frac{\partial u_0(y)}{\partial y_j} \beta_{pj}(y) \nu(y) \right) \, dy.
\]

In Ref. 4, Chap. 3, Sec. III, the Neumann problem for the Helmholtz equation with \( n_0^2(x) = 1 \) was
studied in the domain, similar to the one in Eq. (1.5), and it was proved under the assumptions
used in Ref. 4 that the main term of the asymptotics of the solution, as the relative volume of the
particles tends to zero, is the incident field, while the next term is proportional to this relative
volume.

V. CONVERGENCE AND COMPACTNESS ESTIMATES

Here we prove that the function \( u_e(x) \) converges in \( C(\mathbb{R}^3) \) to the limit \( u(x) \), satisfying the 762
integral equation
\[ u(x) = u_0(x) - \int_D G(x, y) p(y) u(y) dy, \]

where \( p(x) \) is defined in (1.25).

Let us outline the steps of our proof.

**Step 1.** Let \( \Delta_j \) be the cube with the side \( b \gg a^{1/6} \) and center at the point \( y^{(j)} \). The union of the cubes \( \Delta_j, 0 \leq j \leq J, \) is a partition of \( D \). Since \( a > 0 \) is small and \( d = O(a^{1/3}) \), one has \( b \gg d \).

**Lemma 5:** Assume \( d = O(a^{1/3}) \), \( \sigma = O \left( \frac{1}{a} \right) \), \( |S_m| = O(a^2) \), \( N(\Delta) = O \left( \frac{b^3}{a} \right) \).

**Then**

\[ \lim_{b \to 0} \lim_{a \to 0} \sum_{S_{m}} \int_{s_{m}} G(x, s) \sigma_{m}(s) ds = 0. \]

We denote by \( \varepsilon(a, b) \) the sum in (5.2).

**Step 2.** Let us write

\[ u_M(x) = u_0(x) + \sum_{x_m \in \Delta_0} \sum_{x \in \Delta_0} G(x, x_m) Q_m + \varepsilon(a, b) + \eta(a), \]

where

\[ \lim_{a \to 0} \eta(a) = 0, \lim_{b \to 0} \varepsilon(a, b) = 0, \]

and by (2.20),

\[ Q_m = -f(x_m) a u_e(x_m), \]

where \( f(x) \) is a continuous function in \( D \), \( f(x_m) = \text{const} \ h(x_m) [1 + h(x_m)] \) in (2.24). We may replace \( u_M(x) \) by \( u_e(x) \) and neglect the terms \( \varepsilon(a, b) \) and \( \eta(a) \) in (5.3) because of (5.4). Then (5.3) takes the form

\[ u_e(x) = u_0(x) - \sum_{x_m \in \Delta_0} \sum_{x \in \Delta_0} G(x, x_m) f(x_m) u_e(x_m) a. \]

We prove that, as \( a \to 0 \), the set \( \{u_e\} \) is uniformly bounded and equicontinuous in the space \( C(\mathbb{R}^3) \) and uniformly small at infinity. This implies convergence, as \( a \to 0 \), in \( C(\mathbb{R}^3) \) of a subsequence, denoted \( \{u_e\} \) again. The limit \( u(x) \) of this subsequence satisfies Eq. (5.14) (see below), which has a unique solution in \( C(\mathbb{R}^3) \). Thus, the whole set \( \{u_e\} \) converges to \( u(x) \) as \( a \to 0 \).

Let us now give the details of these two steps.

**Proof of Lemma 5:** Write the sum in (5.2) as

\[ \Sigma := \sum_{x_m \in \Delta_0} + \sum_{|x_m - x| > a^{1/6}} + \sum_{|x_m - x| < a^{1/6}}, \]

where \( \{\Delta_j\} \) is a partition of \( D \) into a union of nonintersecting cubes with side \( b \) and center \( y^{(j)} \), and \( \Delta_0 \) is the cube, containing point \( x \). In the proof of Lemma 5 we only deal with this cube \( \Delta_0 \), and we assume the origin at the point \( x \). Note that \( a^{1/6} = O(d^{1/2}) \gg d \).

Let us estimate \( \Sigma_2 \), taking \( x = 0 \) to be the origin:
\[
\lim_{a \to 0} |\Sigma_2| \leq c \sum_{d \in \mathbb{N} \setminus \{a\}^{1/6}} \left| \sigma_m(s) \right| ds = \frac{ca^2}{a} \sum_{d \in \mathbb{N} \setminus \{a\}^{1/6}} \frac{1}{d} \leq \frac{ca^2}{a} \int_{a^{1/6}}^1 \frac{r^2 dr}{d} 
\]

(5.8)

where \( c > 0 \) stands for various constants independent of \( a \), and we have used the assumption \( d = O(a^{1/3}) \).

We will prove that \( |\Sigma_1| \leq c \sum_{d \in \mathbb{N} \setminus \{a\}^{1/6}} \frac{1}{d} \leq \frac{ca^2}{a} \int_{a^{1/6}}^1 \frac{r^2 dr}{d} \),

\[
|\Sigma_1| \leq \frac{ca^2}{a} \sum_{d \in \mathbb{N} \setminus \{a\}^{1/6}} \frac{1}{d} \leq \frac{ca^2}{a} \int_{a^{1/6}}^1 \frac{r^2 dr}{d} 
\]

(5.9)

Proof: Let us estimate \( \Sigma_1 \):

\[
|\Sigma_1| \leq \frac{ca^2}{a} \sum_{d \in \mathbb{N} \setminus \{a\}^{1/6}} \frac{1}{d} \leq \frac{ca^2}{a} \int_{a^{1/6}}^1 \frac{r^2 dr}{d} 
\]

(5.11)

From (5.8) and (5.9) Lemma 5 follows.

Lemma 6: If \( \sigma_m = O(1/a) \) and \( d = O(a^{1/3}) \), then

(a) \( \sup_{0 < r < a, x \in D} |\sigma_m(x) - \sigma_m(x + \Delta x)| \to 0 \),

(b) \( \sup_{0 < r < a, x \in D} |\sigma_m(x + \Delta x)| \to 0 \),

(5.10)

Proof: Let us prove (a):

\[
\left| \sum_{m=1}^M \int_{S_m} G(x,y) \sigma_m dy \right| \leq \frac{ca^2}{a} \sum_{d \in \mathbb{N} \setminus \{a\}^{1/6}} \frac{1}{d} \leq \frac{ca^2}{a} \int_{a^{1/6}}^1 \frac{r^2 dr}{d} 
\]

(5.12)

Thus, (b) is proved.

Corollary 1: If \( \sigma_m = O(1/a) \) and \( d = O(a^{1/3}) \), then the set \( \{u(x)\}_{0 < r < a} \) contains a convergent in \( C(D) \) subsequence:

\[
\lim_{n \to 0} \|u_{e,n}(x) - u(x)\|_{C(D)} = 0. 
\]

(5.13)

We will prove that \( u(x) = \lim_{n \to 0} u_{e,x}(x) \) satisfies the equation

\[
u(x) = u_0(x) - \int_D G(x,y)p(y)u(y)dy, \quad p(x) = \frac{4\pi h(x)}{1 + h(x)} N(x). 
\]

(5.14)

Equation (5.14) has a unique solution in \( C(D) \). Therefore every subsequence \( u_{e,n} \) converges to the same limit \( u(x) \). Thus, (5.13) holds for the set \( u_e \) and not only for its subsequence.

To prove that Eq. (5.13) has a unique solution, we note that if \( u_0 = 0 \), then the solution to (5.14) solves the Schrödinger equation.

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and satisfies the radiation condition. Since $k^2 > 0$ and $\Im q \leq 0$, Theorem 1, proved in Sec. III,
implies that $u = 0$.

Let us derive Eq. (5.14) for the limit $u(x)$ of $u_t(x)$ as $a \to 0$. Taking the origin at $x$, one gets

$$
\sum_{x_m \in \Delta_0} G(x, x_m) u_m(x_m) a = \sum_{\Delta_j \Delta_0} G(x, y^{(j)}) u_f(y^{(j)}) (1 + \varepsilon_j) a \sum_{x_m \in \Delta_j} 1
$$

$$
= \sum_{\Delta_j \Delta_0} G(x, y^{(j)}) u_f(y^{(j)}) u_j(y^{(j)}) (1 + \varepsilon_j) a N(j)
$$

$$
= \sum_{\Delta_j \Delta_0} G(x, y^{(j)}) u_f(y^{(j)}) (1 + \varepsilon_j) a N(y^{(j)}) |\Delta_j|.
$$

Here $\lim_{a \to 0} \max_j \varepsilon_j = 0$ if $u_t(x)$ is continuous in $D$, because $f(y), N(y)$, and $G(x, y)$ are continuous functions in $D \setminus \Delta_0$. $G$ is integral in $D$, and we have used relation (1.21). For any fixed $a$ the function $u_j(x)$, defined in (5.6), is continuous in $D$. Thus, as $\max_j \text{diam}|\Delta_j| \to 0$, the right side of (5.16) tends to the limit

$$
\int_D G(x, y) f(y) N(y) u(x) dy,
$$

being the Riemann sum for this integral, and $u(x) = \lim_{a \to 0} u_j(x)$, where the limit can be understood as the limit in $C(D)$-norm.

Thus, we have proved that $\zeta_m = h(x_m)/a$, $\sigma_m = O(1/a)$, $d = O(a^{1/3})$, and $N(\Delta)$, 1/2 $\Delta N(\Delta) dx [1 + o(1)]$ as $a \to 0$ imply (5.10), the existence of the limit $\lim_{a \to 0} u_j(x) = u(x)$, and

Eq. (5.14).

On the other hand, if (5.10) holds, and

$$
\zeta_m = \frac{h(x_m)}{a}, \quad d = O(a^{1/3}), \quad N(\Delta) = \frac{1}{a} \int_\Delta N(x) dx [1 + o(1)] \quad \text{as } a \to 0,
$$

then one can prove that $\sigma_m = O(1/a)$.

Let us sketch this proof.

The function $\sigma_j = \sigma_j, j = 1, 2, \ldots, M$ solves Eq. (2.10). Let us study the asymptotics, as $a \to 0$, of the solution to Eq. (2.10) assuming that $u_j(x_N)$ and $u_t(x)$ are continuous functions on $S_j$, and $\zeta_j = h_j/a$. The choice of $\zeta_j$ is in our hands, and we will see that the assumptions about $u_j$ are justified. The equation is

$$
\sigma = A \sigma - 2 \zeta T \sigma + 2 u_{j_N}(s) - 2 \zeta u_j(s), \quad \zeta := \zeta_j, \quad A := A_j, \quad T := T_j.
$$

Let $\zeta^{-1} := \tau$. We have

$$
T \sigma = \frac{(A - I) \sigma}{2} + \tau u_{j_N} - u_{j_0}, \quad \tau \to 0.
$$

The operator $T: H^0 \to H^1$ is an isomorphism if $S_j$ is a sphere of a sufficiently small radius. Here $H^0 = H^0(S_j)$ are the usual Sobolev spaces.

We wish to prove that the main term of the asymptotics of the solution to (5.18) as $\tau \to 0$ is

$$
\sigma_0 = - T^{-1} u_j.
$$

Equation (5.19) implies $\sigma_0 = O(1/a)$. This can be seen from the following argument. The equation

$$
\sigma_{j_0} = - T^{-1} u.
$$
\[ \int_{S_j} \frac{\sigma_0(t)dt}{4\pi|s-t|} = T\sigma_0 = -u_e \]  
(5.20)

is the equation for the electrostatic charge density \( \sigma_0 \) on the surface \( S_j \) of the perfect conductor \( D_j \). Charged to the constant potential \( -u_e \). This potential can be considered constant on \( S_j \) because it is very small. The total charge \( Q_0 = \int_{S_j} \sigma_0 ds = C u_e \), where \( C = O(a) \) is the electric capacitance of the conductor with the surface \( S_j \). Since the surface area is \( O(a^2) \) and \( \int_{S_j} \sigma_0 ds = O(a) \), it follows that \( \sigma_0 = O(1/a) \). If \( S_j \) is a sphere of radius \( a \), then \( \sigma_0 = \text{const} = -u_e/a \). For an arbitrary smooth surface \( S_j \) such that \( \text{diam} S_j \leq 2a \), diffeomorphic to a sphere, the estimate \( \sigma_0 \) follows from the Hopf lemma (the strong maximum principle), which guarantees that the surface charge density does not vanish on \( S_j \).

Finally, let us check that \( \sigma_0 = -T^{-1}u_e \) is the main term of the asymptotics of the solution \( \sigma \) to (5.18) as \( \tau \to 0 \).

The operator \( T, \) \( T\sigma = \int_{S_j} \sigma(t)dt/4\pi|s-t| \) is self-adjoint and positive in \( H^0 = L^2(S_j) \). The quadratic form \( \langle T\sigma, \sigma \rangle = \langle (T\sigma, \sigma)_{L^2(S_j)} \rangle \) defines an inner product, and the corresponding norm is equivalent to the norm in \( H^{-1/2} = H^{-1/2}(S_j) \) (see its definition in Ref. 1). Let \( \sigma_0 = -T^{-1}u_e \). Then Eq. (5.18) implies

\[
(T(\sigma - \sigma_0), \sigma - \sigma_0) = -\frac{\tau}{2}\|\sigma - \sigma_0\|^2 - \frac{\tau}{2} \sigma_0, \sigma - \sigma_0 + \tau u_e, \sigma - \sigma_0 \]  
(5.21)

\[
+ \frac{\tau}{2} \langle (A - I)\sigma_0, \sigma - \sigma_0 \rangle. \]

If \( S_j \) is convex, then \( \langle (A - I)\sigma, \sigma \rangle \leq 0. Therefore (5.21) implies (5.22) where \( c > 0 \) is a constant independent of \( \tau, \sigma_0 = c(\sigma_0) \). As \( \tau \to 0 \), one gets from (5.22) the inequality

\[
\|\sigma - \sigma_0\|_{-1/2} \leq c \tau, \quad \tau \to 0. \]
(5.23)

Thus, our claim is verified. One can prove that the norm in (5.23) can be replaced by the norm of \( H^{1/2}(S_j) \) provided that \( u_e \) and \( u_e N \) are sufficiently smooth, but we do not go into detail.

The convexity of \( S_j \) is not necessary for our argument, it just simplifies it. Our conclusions hold without this convexity assumption because \( A : H^1 \to H^{3/2} \) for smooth \( S_j \).