A simple proof of the uniqueness theorem in impedance tomography

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Abstract. Let $\nabla \cdot (\sigma(x)\nabla u) = 0$ in $D \subset \mathbb{R}^3$, where $D$ is a bounded domain with a smooth boundary. Suppose that $\sigma \geq m > 0$, $\sigma \in H^1(D)$, where $H^1$ is the Sobolev space. Let the set $\{u, \sigma u_N\}$ be given on $\Gamma$ for all $u \in H^{3/2}(\Gamma)$, where $u_N$ is the normal derivative of $u$ on $\Gamma$.

**Theorem.** The above data determine $\sigma(x)$ in $D$ uniquely.

I. INTRODUCTION

Recently much attention was given to the problem of impedance tomography. The problem consists in finding $\sigma(x)$, the conductivity of a body $D$ with a smooth boundary $\Gamma$, from the boundary measurements of the potential $u$ and the normal component of the corresponding current $j_N := \sigma u_N$, $u_N := \frac{\partial u}{\partial N}$ on $\Gamma$, $N$ is the unit outer normal to $\Gamma$. The governing equation is

$$f_\sigma u := \nabla \cdot (\sigma(x)\nabla u) = 0 \quad \text{in} \quad D \subset \mathbb{R}^3$$

and the data is the set

$$\{u, \sigma u_N\} \quad \text{on} \quad \Gamma \quad \text{for all} \quad u \in H^{3/2}(\Gamma).$$

The medical application of this problem can be described as follows: one applies a potential to a human body and measures the resulting $j_N = \sigma u_N$. Given the set of measurements $\{u, \sigma u_N\}$ for $u \in H^{3/2}(D)$ one wants to recover $\sigma(x)$.

The uniqueness of the solution to this problem has been proved for $\sigma$ piecewise analytic in [1], for $\sigma \in C^{\infty}$ in [2], and for $\sigma \in H^2$ in [5]. Methods given in [2] allow one to prove the uniqueness theorem for $\sigma \in W^{2,\infty}$, but not in $W^{2,2} = H^2$. Here $W^{2,2}$ is the Sobolev space, $H^2 = W^{2,2}$. The aim of this paper is to give a simple proof of the uniqueness theorem for $\sigma \in H^3$ using the ideas from [3]-[5] but in contrast with [3]-[5] (and with [2]) without a passage to the equation $(\Delta - q(x)) \omega = 0 \quad (*)$ where $\omega = \sigma\omega^{-1/2}(x)$, $q(x) := \sigma^{-1/2} \Delta \sigma^{1/2}$. The advantages in not passing to (*) are not only of methodological nature, and not only in a simplification of the argument but also in the following two points. First, the new approach may lead to further relaxation of the smoothness assumption on $\sigma$ (although at the moment the author did not have the final results of this nature). Secondly, the argument in this paper provides automatically the conclusion, that the data $\{u, \sigma u_N\} \forall u \in H^{3/2}(\Gamma)$ determine $\sigma$ and $\sigma u_N$ uniquely. In the previous works [1], [2], [7], [9] this uniqueness result was treated as a separate problem. In the approach given in the present paper the fact that $\sigma$ and $\sigma u_N$ are uniquely determined on $\Gamma$ by the boundary data follows from the Theorem. Indeed, if $\sigma \in H^3(D)$ is uniquely determined in $D$ (as stated in the Theorem) then $\sigma$ and $\sigma u_N$ on $\Gamma$ are uniquely determined by the embedding theorem as traces on $\Gamma$ of $\sigma$ and $\nabla \sigma \cdot N$. It would be interesting to find out if the uniqueness theorem holds for $\sigma \in L^{\infty}(\Omega)$. The case of piecewise smooth $\sigma(x)$ under additional restrictive assumptions that the discontinuity surface $\Gamma_0$ of $\sigma$ is a smooth connected manifold which divide $D$ into two parts $D_1$ with the boundary $\Gamma_1$ and $D_2$ with the boundary $\Gamma_1 \cup \Gamma$, and that $\sigma$ is known in $D_2$, is treated in [8]. In [10] a numerical method is given for 3D inverse scattering problem with data at fixed energy.

In section II we formulate the new approach to the uniqueness theorem; in section III technical details are given.
II. Outline of the approach

Let $N_D(\ell_\sigma) := \{ u : \ell_\sigma u = 0 \text{ in } D, u \in H^2(D) \}$. Multiply (1) by a $v \in N_D(\ell_\sigma)$ and integrate over $D$ and then by parts to get

$$ \int_D \sigma(x) \nabla u \cdot \nabla v \, dx = \int_\Gamma \sigma u_N v_N \, ds. \tag{3} $$

The right side of (3) is known since $u$ and $\sigma u_N$ on $\Gamma$ are known for any $u \in N_D(\ell_\sigma)$. Therefore the set of integrals of $\sigma(x)$ times the functions from the set $\{ \nabla u \cdot \nabla v \} \forall u, v \in N_D(\ell_\sigma)$ is known. Let $0 < m \leq \sigma \in H^3, j = 1, 2$.

**LEMMA 1.** The set $\{ \nabla u \cdot \nabla v \} \forall u \in N_D(\ell_\sigma_1), \forall v \in N_D(\ell_\sigma_2)$ is complete in $L^2(D)$.

From Lemma 1 and (3) it follows that $\sigma(x)$ is uniquely determined, Lemma 1 is proved in section III.

Let us prove

**LEMMA 2.** If $\sigma_1$ and $\sigma_2$ generate the same set $\{ u_j, \sigma_j u_N \}, j = 1, 2$, then $\sigma_1 = \sigma_2$ in $D$.

**PROOF:** Subtract from $\ell_\sigma_1 u_1 = 0$ equation $\ell_\sigma_2 u_2 = 0$ to get

$$ \ell_\sigma_1 u = \ell_\sigma u_2, \quad \sigma := \sigma_2 - \sigma_1, \quad u := u_1 - u_2 \tag{4} $$

By the assumption $u_1 = u_2$ on $\Gamma$ implies $\sigma_1 u_1 N = \sigma_2 u_2 N$ on $\Gamma$. Choose $u_1 = u_2$ on $\Gamma$. Multiply (4) by $v_1 \in N_D(\ell_\sigma_1)$, integrate by parts, use the above remark, and get

$$ \int_\Gamma (v_1 \sigma_1 u_N - \sigma_1 v_1 u_N) \, ds + \int_D \ell_\sigma v_1 u \, dx = - \int_D \sigma v_2 \cdot \nabla v_1 \, dx + \int_\Gamma \sigma u_{2N} v_1 \, ds $$

or

$$ \int_D \sigma \nabla u_2 \cdot \nabla v_1 \, dx = \int_\Gamma \sigma (v_2 u_{2N} v_1 - \sigma_1 v_1 u_N v_1 - v_1 v_1 u_{1N} + \sigma_1 v_1 u_{2N} + \sigma_1 v_1 u_{1N} - \sigma_1 v_1 u_2) \, ds = 0 \tag{5} $$

where the underlined terms cancel each other.

Lemma 1 and equation (5) imply $\sigma = 0$. Lemma 2 is proved.

This concludes the outline of the approach to the uniqueness theorem.

**THEOREM.** If $0 < m \leq \sigma \in H^3(D)$ then the data $\{ u, \sigma u_N \}$ on $\Gamma \forall u \in H^{3/2}(\Gamma)$ determine $\sigma$ uniquely.

**PROOF:** If $u \in H^2(D)$ then $u \in H^{3/2}(\Gamma)$. The conclusion of the theorem follows from Lemma 2.

**REMARK:** The argument above and the conclusion of the Theorem remain valid for any $\sigma$ for which Lemma 1 is proved.

III. Proof of Lemma 1

It is proved in [3]-[5] that (1) has a solution of the form

$$ u = \sigma^{-1/2}(x) \exp(i \theta \cdot x)(1 + r(x, \theta)), \quad \theta \cdot \theta = 0, \quad \theta \in \mathbb{C}^3 \tag{6} $$

where

$$ \| r \|_{L^\infty(D)} \to 0 \quad \text{as} \quad |\theta| \to \infty, \quad \theta \cdot \theta = 0, \quad \theta \in \mathbb{C}^3. \tag{7} $$
Consider $\nabla u \cdot \nabla v, u \in N_D(\ell_1), v \in N_D(\ell_2)$. One has

$$
\nabla u \cdot \nabla v = \exp\{i(\theta_1 + \theta_2) \cdot x\}(\sigma_1\sigma_2)^{-1/2} \left[ i\theta_1(1 + r_1) - \frac{\nabla \sigma_1}{2\sigma_1}(1 + r_1) + \nabla r_1 \right]
\times \left[ i\theta_2(1 + r_2) - \frac{\nabla \sigma_2}{2\sigma_2}(1 + r_2) + \nabla r_2 \right].
$$

(8)

Let us choose $\theta_1 + \theta_2 = p$ where $p \in \mathbb{R}^3$ is an arbitrary real vector, $\theta_1 \cdot \theta_1 = \theta_2 \cdot \theta_2 = 0$, $\theta_1, \theta_2 \in \mathbb{C}^3$, $|\theta_1| \to \infty, |\theta_2| \to \infty$. The product of the brackets gives

$$
-\theta_1 \cdot \theta_2 - \frac{i\theta_1 \cdot \nabla \sigma_2}{2\sigma_2} + i\theta_1 \cdot \nabla r_2 - \frac{i\theta_2 \cdot \nabla \sigma_1}{2\sigma_1} + i\theta_2 \cdot \nabla r_1 + \cdots
$$

(9)

where $\cdots$ denote the terms of lower order as $|\theta_1| \to \infty, |\theta_2| \to \infty$. Since $\theta_2 = p - \theta_1, \theta_2 \cdot \theta_1 = p \cdot \theta_1$, one can write (9) as

$$
-\theta_1 \cdot \theta_2 - \frac{i\theta_1 \cdot \nabla \sigma_2}{2\sigma_2} + i\theta_1 \cdot \nabla r_2 - \frac{i\theta_2 \cdot \nabla \sigma_1}{2\sigma_1} + i\theta_2 \cdot \nabla r_1 - i\theta_1 \cdot \nabla r_1 + \cdots
$$

(10)

Assume that

$$
0 = \int_D f \nabla u \cdot \nabla v dx \quad \forall u \in N_D(\ell_1), \quad \forall v \in N_D(\ell_2).
$$

(11)

From (8)-(11) one concludes, that

$$
0 = \int_D dx f(x)(\sigma_1\sigma_2)^{-1/2} \exp(ip \cdot x) \left[ \theta_1 \cdot \left(-p - \frac{i\nabla \sigma_2}{2\sigma_2} + i\nabla r_2 + \frac{i\nabla \sigma_1}{2\sigma_1} - i\nabla r_1\right) + ip \left(\nabla r_1 - \frac{\nabla \sigma_1}{2\sigma_1}\right) + \cdots \right]
$$

(12)

Let $f(\sigma_1\sigma_2)^{-1/2} := h(x), |\theta_1| \to \infty, \theta_1 \cdot \theta_1 = 0, \theta_1 \in \mathbb{C}^3$. Until now the assumption $\sigma \in H^3$ has not been used explicitly. At this moment we use this assumption in the following way.

It follows from [3] that if $\sigma \in H^3$ then, in addition to (7), the equation

$$
\| \nabla r \|_{L^\infty(D_1)} \to 0 \quad \text{as} \quad |\theta| \to \infty, \quad \theta \cdot \theta = 0, \quad \theta \in \mathbb{C}^3
$$

(13)

holds. If $|\theta_1| \to \infty$, then $|\theta_2| \to \infty$, so that for $|\theta_1| \to \infty$ it follows from (12) and (13) that

$$
\theta_1^0 \cdot p \int_D dx \exp(ip \cdot x) h(x) = \theta_1^0 \cdot Q(p) \quad \forall p \in \mathbb{R}^3
$$

(14)

where $\theta_1^0 := \theta_1|\theta_1|^{-1}, h(x) := (\sigma_1\sigma_2)^{-1/2} f(x),

$$
Q(p) := (i/2) \int_D dx \exp(ip \cdot x) h(x)(\sigma_1^{-1}\nabla \sigma_1 - \sigma_2^{-1}\nabla \sigma_2).
$$

(15)

Take $\theta_1^0 = a + ib, a, b \in \mathbb{R}^3, a \cdot b = 0, |a| = |b| = 2^{-1/2}$. If $(a + ib) \cdot A = 0$ for a certain vector $A \in \mathbb{C}^3$ which does not depend on $a, b$, and for all $a, b \in \mathbb{R}^3$ such that $a \cdot b = 0, |a| = |b| = 2^{-1/2}$, then $A = 0$. Indeed, $A = \alpha + i\beta, \alpha, \beta \in \mathbb{R}^3$. Write

$$
0 = (a + ib)(\alpha + i\beta) = a \cdot \alpha - b \cdot \beta + i(b \cdot \alpha + a \cdot \beta).
$$

(16)

Choose coordinates so that $\alpha$ is directed along $e_3$, the unit vector along $x_3$ axis, choose $a = e_3/\sqrt{2}$, and write (16) as

$$
0 = (\alpha_3 + i\beta_3)/\sqrt{2} - b \cdot \beta, \quad \beta := \beta_1 e_1 + \beta_2 e_2.
$$

(17)
Here we took into account that $b \cdot e_3 = 0$ since $b \cdot a = 0$. Since $a, b, \alpha, \beta \in H^2$ one concludes from (17), by taking real and imaginary part, that

$$\beta_3 = 0, \quad \alpha_3 / \sqrt{2} = b \cdot \hat{\beta}. \quad (18)$$

Since $b = (e_1 \cos \phi + e_2 \sin \phi) / \sqrt{2}$ where $\phi \in [0, 2\pi)$ is arbitrary, one concludes from (18) that $\alpha_3 = 0$ and $\hat{\beta} = 0$. Thus (14) implies that

$$\int h(p) = Q(p) \quad \forall p \in R^3. \quad (19)$$

Note that $Q(p) = \frac{1}{2} \tilde{\sigma} m$, where $m(z) := \sigma^{-1} \nabla \sigma_1 - \sigma^{-1}_2 \nabla \sigma_2$ is a vector-function, $m \in H^2$ if $\sigma_j \in H^3$. Therefore (19) implies that $h = 0$ and Lemma 1 is proved.

Let us explain the last conclusion:

**LEMMA 3.** If (19) holds then $h = 0$.

**PROOF:** If (19) holds then

$$\nabla h = \frac{1}{2} \tilde{h} m, \quad \frac{\partial h}{\partial x_j} = \frac{1}{2} \tilde{h} m_j, \quad 1 \leq j \leq 3. \quad (20)$$

Since $h$ is compactly supported, one can write, e.g.,

$$h(z_1, z_2, z_3) = \frac{1}{2} \int_{a_1}^{z_1} h(t, z_2, z_3)m_1(t, z_2, z_3)dt. \quad (21)$$

If $m_1 \in L^1$ then (21), being a homogeneous Volterra equation, implies $h = 0$. In our case $m_1 \in H^2$, so $m_1 \in L^1$. Lemma 3 is proved.

References