Theoretical and Practical Aspects of Singularity and Eigenmode Expansion Methods

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Abstract—In recent years a vast amount of literature has been written on singularity and eigenmode expansion methods (SEM and EEM). The main practical purpose is to produce a theory which allows us to identify a flying (or stationary) target from the observed transient field scattered by the target. (The other purposes include representation and computation of the transient field and construction of equivalent circuits according to the object's electromagnetic response.) This problem is discussed. Described are 1) the basic starting points of the engineering approach to the problem and the extent to which they are consistent from the mathematical point of view, 2) the material which has been rigorously established in the field, 3) the important points in practice, and 4) the unsolved mathematical problems in the field. This work may save efforts by other scientists and engineers, since the results demonstrate the kind of research that can be used in practice and the kind which is of mostly theoretical interest.

I. INTRODUCTION

A vast amount of literature has been written on singularity and eigenmode expansion methods (SEM and EEM) during the last decade. Engineers and physicists stimulated interest in the subject (see [1]-[3], [17], [18], and references given in these reviews). Mathematical analysis of the problems was initiated in [4], [5] and was considerably furthered by Agranovich (see [1]). Nevertheless, many questions in the theory are open and of considerable interest to engineers and mathematicians. The purpose of this paper can be summarized as follows. We are going to explain in a simple way the principal features of the singularity and eigenmode expansion methods and to formulate explicitly 1) what has been used by engineers without proof, 2) what is important for practice, 3) what has been rigorously established, and 4) what are the unsolved mathematical problems in this field.

The main results obtained in scalar wave scattering theory were generalized to electromagnetic wave scattering without much difficulty. For this reason—and also for simplicity—we restrict ourselves to the presentation of the theory for scalar wave scattering.

II. WHAT ARE SINGULARITY AND EIGENMODE EXPANSION METHODS

A. What Is EEM?

Consider the problem

$$\nabla^2 u + k^2 u = 0 \quad \text{in } \Omega = \mathbb{R}^3 \setminus D, \quad k^2 > 0$$

$$u|_{\partial \Omega} = 0$$

(1)

$$u = u_0 + v \equiv \exp \{ik(n, x)\} + v$$

(2)

(3)
and \( \psi \) satisfies the radiation condition
\[
\frac{\partial \psi}{\partial |x|} - ik\psi = o(|x|^{-1}) \quad \text{as } |x| \to \infty.
\] (4)

Here \( \Gamma \) is the smooth surface of a finite obstacle, \( \mathbb{R}^3 \backslash D \) is the exterior (with respect to \( D \)) domain, \( n \) is the unit vector in the direction of propagation of the wave. If we look for a solution of (1)-(4) of the form
\[
\psi = e^{ikx} u(y) dy,
\]
then
\[
u = u_0 + \int_{\Gamma} \frac{\exp (ikxy)}{4\pi xy} f(y) dy,
\]
\[
r_{xy} = |x - y|, \quad y \in \Gamma
\] (5)

forms a Riesz basis of \( L^2(\Gamma) \equiv H \). This means that any \( g \in H \) can be expanded in the series
\[
g = \sum_{j=1}^{\infty} g_j \phi_j
\] (8)

and
\[
c_1 \|g\|^2 \leq \sum_{j=1}^{\infty} |g_j|^2 \leq c_2 \|g\|^2, \quad c_1 > 0,
\] (9)

where \( \|g\| \) is the norm in \( L^2(\Gamma) \). The inequality (9) substitutes the Parseval equality for orthonormal bases. A complete system in \( H \) does not necessarily form a basis of \( H \) (example: \( H = L^2[0,1] \), the system \( \phi_j(x) = x^j, j = 0, 1, 2, \ldots \), not every \( g \in L^2[0,1] \) can be expanded in the series \( g(x) = \sum_{j=0}^{\infty} g_j x^j \).

If the assumption made is true, then (6) can be solved by the Picard formula
\[
f(x) = -\sum_{j=1}^{\infty} \frac{u_0}{\lambda_j^{1/2}} \phi_j(x)
\] (10)

where the coefficients are uniquely defined by the equality
\[
u_0 = \sum_{j=1}^{\infty} u_0 / \phi_j.
\] (11)

This method of solution of the scattering problem (1)-(4) is called the EEM. It was used without mathematical analysis by engineers [11, 3]. The questions which immediately arise can be formulated as follows: 1) Is it true that the nonself-adjoint operator \( A(k) \) has eigenvectors (e.g., the Volterra operator has no eigenvectors)? 2) Is it true that the set of eigenvectors of \( A(k) \) forms a Riesz basis of \( H \)? 3) Suppose that the set of eigenvectors (\( \Xi \) eigensystem) of \( A(k) \) does not form a Riesz basis of \( H \); is it true that the root system of \( A(k) \) forms a Riesz basis of \( H' \)?

Let us explain the root system. Let \( A \) be a linear operator on \( H, A \phi = \lambda \phi, \phi \neq 0 \). Then \( \phi \) is an eigenfunction of \( A \) corresponding to the eigenvalue \( \lambda \). Consider equation \( A\phi_1 - \lambda_1 \phi_1 = \phi \). If this equation is solvable, \( \phi_1 \) is called a root vector of \( A \) corresponding to eigenvalue \( \lambda \) and eigenvector \( \phi \). If \( \phi_1 \) exists, consider the equation \( A\phi_2 - \lambda_2 \phi_2 = \phi_1, k > 1 \). It is known [7] that only a finite number \( r \) of root vectors \( \phi_1, \ldots, \phi_r \) associated with \( \phi \) exists. The chain \( (\phi, \phi_1, \ldots, \phi_r) \) is called a Jordan chain with the length \( r + 1 \). The union of all root vectors of a linear operator \( A \) corresponding to all eigenvalues of \( A \) is called the root system of \( A \). It is well-known from linear algebra that the eigensystem of a nonself-adjoint operator may not form a basis. For example, if the operator \( A \) is an operator in \( \mathbb{R}^2 \) with the matrix
\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
then \( A \) has only one eigenvector, so that the eigensystem of \( A \) does not form a basis of \( \mathbb{R}^2 \). It is also known that the root system of any linear operator (matrix) in \( \mathbb{R}^n \) forms a basis of \( \mathbb{R}^n \). Of course, in \( \mathbb{R}^n \) any basis is a Riesz basis. In a Hilbert space (infinite dimensional space) this is not true. For practice, it is important to have affirmative answers for questions 1) and 3). Indeed, if the eigensystem of \( A(k) \) does not form a Riesz basis but its root system forms a Riesz basis of \( H \), then it is still possible to solve (6) using the root system of \( A \). This explains the significance of the root systems in EEM and SEM.

B. What Is SEM?

In order to explain what SEM is, consider the problem
\[
u_{tt} - \nabla^2 \nu = 0 \quad \text{in } \Omega, \quad \nu|_{r} = 0, \quad \nu|_{r=0} = 0, \quad \nu_t|_{r=0} = f(x),
\] (12)

where \( f \in C_0^\infty \) (a smooth function which vanishes for large \( |x| \)). If \( G(x, y, -p^2) \) is the Green function of the problem
\[
(-\nabla^2 + p^2) G = \delta(x - y) \quad \text{in } \Omega, \quad G|_{r} = 0, \Re p > 0
\] (13)

then the solution of (12) can be written as
\[
u = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \exp (pt) \bar{u}(x, p) dp,
\] (14)

where
\[
\bar{u}(x, p) = \int_{\Omega} G(x, y, -p^2)f(y) dy.
\] (15)

Suppose that \( \bar{u}(x, p) \) is meromorphic in \( p \) on the whole complex plane \( p \) (this is actually true [81]-[121], and suppose
the following estimate is valid
\[
|\tilde{u}(x, p)| \leq \frac{c}{1 + |p|^a}, \quad a > 1/2,
\]
\[
c = \text{const} > 0, \quad \text{Re } p > -A,
\] (16)
\[
|\text{Im } p| > N_A,
\]
where \(A > 0\) is arbitrary.

This estimate follows from the Lax-Phillips result [12] and the arguments given in [9], [10], [13]. Under the assumptions made, the contour of integration in (14) can be moved to the left, and one gets
\[
u = \sum_{j=1}^{N} \sum_{m=1}^{b_j} c_j^{(m)} \frac{t^{m-1}}{(m-1)!} \exp(p_j t) + o(\exp(-|\text{Re } p_N| t))
\] (17)
where \(p_j\) are the poles of \(G(x, y, -p^2)\) and \(b_j\) is the multiplicity of the pole \(p_j\). The order of the remainder in (17) can be explained if one takes in (14) \(c < -|\text{Re } p_N|\). The sum in (17) is the input made by residues at the poles \(p_j, j \leq N\), the remainder is the estimate of integral (14) with \(c < -|\text{Re } p_N|\).

Note that
\[
\text{Res}_{p=p_j} \frac{\exp(p t)}{(p-p_j)^m} = \frac{t^{m-1}}{(m-1)!} \exp(p_j t).
\]

Other arguments which do not use analytic properties of \(\tilde{u}(x, p)\) are given in [22]. Expansion (17) is a form of the SEM expansion. Actually, such expansions were known for a long time for various concrete problems of mathematical physics (especially in cases when the solution can be represented explicitly in the form of series). The main difficulty is to prove (16), which allows us to move the contour of integration in (14). If only the meromorphic nature of \(\tilde{u}(x, p)\) as a function of \(p\) is established, then no SEM expansions of the type (17) can be proved in general, because there is a possibility that poles \(-\epsilon_n + i\omega_n\) with very small \(\epsilon_n > 0\) and very large \(\omega_n\) can exist. The general Mittag-Leffler representation cannot be applied for the derivation of SEM expansion (17), because this representation uses special system of expanding contours, while the derivation of (17) requires the possibility of moving our particular contour \((c - i\infty, c + i\infty)\) to the left. From a practical point of view, the SEM expansion is used at present according to the following scheme.

Suppose that only a few terms in (17) are essential, e.g., \(1-3\). This will be true if \(|\text{Re } p_j| \geq |\text{Re } p_3|\) for \(j > 3\). Then in experiments the transient field \(u(x, t)\) is measured, and each \(p_j, j = 1, 2, 3\) is determined. It is assumed that the location of these complex poles of the Green function \(G(x, y, -p^2)\) can give information enough to identify the obstacle (the scatterer \(D\)). This assumption has not been backed theoretically. Nevertheless, if there is a finite set of scatters (say, flying targets), it is possible to believe that a one-to-one correspondence can be established empirically between the scatters and the corresponding complex poles.

An interesting inverse problem can be formulated in connection with this question. Given a set of complex numbers \(\{p_j\}, \text{Re } p_j < 0\), is it possible to find a scatterer such that the Green function corresponding to this scatterer has complex poles \(\{p_j\}\)? Does the set \(\{p_j\}\) uniquely determine the scatterer? What restrictions must be imposed on the set \(\{p_j\}\), \(\text{Re } p_j < 0\) in order that this set will be the set of complex poles of the Green function of a scatterer?

If the scatterer is a star-like body (i.e., there exists a point 0 inside the body such that every point of the surface of the body can be seen from 0) and the boundary condition is the Dirichlet condition then the set \(\{p_j\}\) must satisfy the condition \(|\text{Re } p_j| > a \ln |\text{Im } p_j| + b, a > 0 [12]\). It seems that no other information on the problem is available.

From a practical point of view this problem may not be as important as it seems. First, only a few complex poles are available. It seems hopeless to make any general conclusions about the scatterer from this information without severe restrictions on the set of scatters. (For example, if it is known \(a \text{ priori}\) that the scatterer is a ball, it is possible to determine its radius from the above information because the complex poles of the Green function for the domain exterior to a ball depend on the radius of the ball and can be calculated analytically so that the radius can be determined.) That is why this author thinks that, from a practical point of view, in order to use the SEM for identification of scatters it is more useful to work out tables of responses of the typical scatterers than to try to develop a theory of the posed (which is very interesting from a theoretical point of view) inverse problem.

C. Questions Concerning EEM and SEM

The following questions arise naturally in connection with the EEM and SEM methods.

1) Does the root system of the integral operators in diffraction theory (e.g., operator \(A(k)\) in (6)) form a Riesz basis of \(H\)?

2) When does the root system coincide with the eigen-system?

3) Do the complex poles of the Green function depend continuously on the obstacle? In more detail, suppose that \(x_j = x_j(t_1, t_2), 0 < t_1, t_2 < 1\) are parametric equations of \(\Gamma\) and \(y_j = y_j(t_1, t_2) + \epsilon_j(t_1, t_2), 0 < t_1, t_2 < 1, 0 < \epsilon < 1\) are parametric equations of the surface of a perturbed scatterer. Let us assume that \(x_j(t), y_j(t) \in C^\infty(\Delta\setminus0), t = t_1, t_2, \Delta = \{t_1 < t_2 < 1\}\). Let us fix an arbitrary number \(R > 0\) and let \(p_j, 1 < j < n(R)\) be the complex poles of the unperturbed Green function which lie in the circle \(|p_j| < R\). Let \(p(\epsilon)\) be the complex poles of the perturbed Green function. Our question can now be formulated as follows. Is it true that \(p(\epsilon) \to p_j\), \(\epsilon \to 0\), uniformly in \(1 < j < n(R)\) provided that the numeration of \(p(\epsilon)\) is properly done?

4) How can one calculate the complex poles?

5) What are sufficient conditions for the validity of SEM expansion (17)?

6) Is it possible to calculate complex poles via calculation of zeros of some functions?

III. WHAT HAS BEEN RIGOROUSLY ESTABLISHED

In this section we give answers to questions 1)-6) of Section II-C. No proofs will be given, but the results obtained will be formulated and references will be given. Proofs are omitted for three reasons: 1) they are difficult for engineers, 2) they
In order to formulate the answer to question 1) we must explain what a Riesz basis with brackets is. Let \( \{ f_j \} \) be an orthonormal basis of \( H \), \( \{ h_j \} \) be a complete and minimal system in \( H \). (A complete system \( \{ h_j \} \) is called minimal if the system \( \{ h_j \} \setminus h_k \) is not complete for any \( k, k = 1, 2, 3, \ldots \). In other words, if we remove any element \( h_k \) of our system, we obtain an incomplete system.) Let \( m_1 < m_2 < \cdots \) be an arbitrary increasing sequence of integers; \( F_j \) is the linear space with the basis \( \{ f_{m_j - 1}, f_{m_j - 1} + 1, \cdots, f_{m_j - 1} \} ; H_j \) is the linear space with the basis \( \{ h_{m_j - 1}, \cdots, h_{m_j - 1} \} \). Suppose that there exists a linear bounded operator \( B \) with bounded \( B^{-1} \), defined on all \( H \), such that \( BH_j = F_j, j = 1, 2, \ldots \). Then the system \( \{ h_j \} \) is called a Riesz basis of \( H \) with brackets. This definition is equivalent (see [14]) to the following. Let \( P_j \) be projectors in \( H \) onto \( H_j \). Suppose that, for any \( f \in H \),

\[
c_1 \| f \|^2 \leq \sum_{j=1}^{\infty} \| P_j f \|^2 \leq c_2 \| f \|^2, \quad c_1 > 0,
\]

then the system \( \{ h_j \} \) is called a Riesz basis of \( H \) with brackets.

It is proved that the root system of operator \( A(k) \) (see (6)) forms a Riesz basis with brackets [6]. The same is true for the operator arising in the exterior Neumann boundary value problem [1], [6]. The same is also true for the electrodynamics scattering problem [1]. Recently, the author proved that the root system of \( A(k) \) forms a Riesz basis without brackets of \( H = L^2(\Gamma) \) under some assumption [21].

2) If the integral operator of the diffraction problem is normal, then its eigen system coincides with its root system. This is the case in the problem (1)-(4) for a spherical surface \( \Gamma \) and for a linear antenna. First this was observed in [5]. An operator is called normal if \( A^*A = AA^* \), where \( A^* \) is the adjoint operator. The condition \( AA^* = A^*A \) is the condition on \( \Gamma \) provided that the kernel of the integral operator \( A \) is given (see [5] for details and [19]).

3) The answer to question 3) from Section II-C is affirmative (see [6] for details).

4) A general method (with the proof of its convergence) for calculating the complex poles of the Green's functions in diffraction and potential scattering theory was given in [4], [15], [6]. From a practical point of view there are two nontrivial points in performing this method: 1) calculation of \( b_{ij}(k) \) by formula (23) and 2) numerical solution of (24). For both steps there are methods available in the literature on numerical analysis.

5) Sufficient conditions for the validity of SEM expansion (17) were given in Section II.

6) The set of the complex poles of the Green function of the problem (1)-(4) coincides with the set of the complex zeros of the eigenvalues \( \rho_n(k) \) of the operator \( A(k) \):

\[
A(k)\phi_n = \rho_n(k)\phi_n, \quad n = 1, 2, \ldots.
\]

Indeed, let \( G = R(x, y)/(k - z)^\mu + \cdots \), i.e., \( z \) is a pole of the Green function \( G \), \( G |_{\Gamma} = 0 \),

\[
G = g - \int_{\Gamma} g(x, s, k)\mu(s, y, k) ds, \quad \mu = \frac{\exp (ikr_{xy})}{4\pi r_{xy}}.
\]

Multiplying (26) by \( (k - z)^\mu \) and taking \( k \to z \), we get

\[
\int_{\Gamma} g(x, s, z) \frac{\partial R(s, y)}{\partial N_s} ds = 0, \quad x \in \Gamma.
\]

The kernel \( R(t, y) \) is degenerate. Thus a function \( \phi \neq 0 \) exists
such that
\[ \int g(x, s; z) \phi(s) \, ds = 0 \]  \tag{28}

This means that \( k = z \) is a zero of some of the functions \( \rho_n(k) \). Conversely, if \( \phi \equiv 0 \) is a solution of (28), then
\[ u = \int \frac{g(x, s; z) \phi(s) \, ds}{(\Delta + \lambda^2)} \]  \tag{29}
is a solution of the problem
\[ (\Delta^2 + \lambda^2) u = 0 \quad \text{in} \quad \Omega, \quad u |_{\Gamma} = 0 \]  \tag{30}
with the outgoing asymptotics at infinity. Hence \( u \equiv 0 \) in \( \Omega \)
if \( z \) is not a pole of \( G \). Since \( z^2 \) is complex, \( u(x) \equiv 0 \) in \( D \) (as a solution of the homogeneous interior problem). By the boundary value jump relation \( \phi \equiv 0 \). This contradiction proves that \( z \) is a pole of \( G \). A variational method for calculation of eigenvalues of nonselfadjoint compact operators is given in [6]. In [23] a variational principle for resonances is given.

IV. OPEN PROBLEMS

1) The inverse problem formulated in Section II is of interest. It is very interesting to have partial answers: what information about the geometry of a scatterer can be obtained from the location of the complex poles.

2) There is a conjecture [3] that the complex poles of the Green function of the problem (1)-(4) for a convex smooth compact boundary are simple. It would be interesting to prove it or to give a counterexample. For impedance boundary condition there can be poles of order \( > 1 \).

3) It would be interesting to numerically test the method described in Section III for question 6 in some practical problems.

4) In [16] some properties of the purely real poles \( R e \rho_j < 0, \text{Im} \rho_j = 0 \) were established. It would be interesting to tell what information about the geometry of an obstacle can be obtained from the location of the purely real poles. In the physics literature the complex plane \( k = ip \) is usually used. On this plane the purely real complex poles are purely imaginary, \( R e \lambda_j = 0, \text{Im} \lambda_j < 0 \).

5) It would be interesting to find an asymptotic distribution of the complex poles \( \rho_j \) with the minimal real parts as \( j \to \infty \).

6) Is it possible to pass to the limit \( N \to \infty \) in (17) and obtain a convergent series?

REFERENCES


