1. (12 points) (a) $G$ is a set closed under a binary operation $\ast$. What properties make $<G, \ast>$ a group?

(b) Prove that $S = \{(a_1, a_2) : a_1 \in \mathbb{Q}^*, a_2 \in \mathbb{Q}\}$ is a group with binary operation

$$(a_1, a_2) \ast (b_1, b_2) = (a_1b_1, a_1b_2 + a_2b_1).$$

2. (12 points) Suppose that $G = \left\{ x, \frac{1}{x}, 1 - x, \frac{1}{1-x}, \ldots \right\}$ is the group of order six generated by the functions $f(x) = 1/x$ and $g(x) = 1 - x$ under composition, e.g $f \ast g = f(g(x)) = \frac{1}{1-x}$. Find the other elements of $G$ and complete the group table. Is $G$ isomorphic to $\mathbb{Z}_6$ or the group of permutations $S_3$? Justify your choice.

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3. (10 points) (a) What properties must a subset $S$ of a ring $R$ satisfy in order to be a subring?

(b) Prove that $T = \left\{ \begin{pmatrix} a & b \\ -3b & a \end{pmatrix} : a, b \in \mathbb{Q} \right\}$ is a subring of the ring $M_2(\mathbb{Q})$.

(c) Is $T$ a field? Why?

4. (12 points) (a) Suppose that $G$ is a finite group and $H \leq G$, then Lagrange’s theorem states that:

(b) Suppose that $G$ is a non-cyclic group of order 27. Show that $a^9 = e$ for all $a$ in $G$.

(c) How many non-isomorphic abelian groups of order 72 are there?
5. (10 points) Let $R$ be a commutative ring.
(a) Define what it means for a non-zero element $a$ of $R$ to be a zero divisor.
(b) Prove that if $ab$ is a zero divisor then $a$ or $b$ must be a zero divisor.
(c) Prove that if $R$ has no zero divisors then it has a cancellation law.

6. (28 points) Suppose that $R$ is a commutative ring with unity.
(a) What properties must a subset $I$ of $R$ satisfy in order to be an ideal?
(b) Define what it means for an ideal $I$ of $R$ to be prime
(c) Define what it means for an ideal $I$ of $R$ to be maximal.
(d) Show that $I_1 = 3\mathbb{Z} \times 7\mathbb{Z}$ is an ideal but not a prime ideal of $\mathbb{Z} \times \mathbb{Z}$.
(e) Is $I_2 = \{(a, b) : a, b \in \mathbb{Z}, 3|ab\}$ an ideal of $\mathbb{Z} \times \mathbb{Z}$? If yes give a proof, if not show why not.
(f) Prove that the ideal $I_3 = \{f \in \mathbb{Z}[x] : f(0) = 0\}$ of $\mathbb{Z}[x]$ is prime but not maximal.
7. (36 points) Circle True (T) or False (F).

T F (a) \( \mathbb{Z}[x] \) is an integral domain.
T F (b) If \( F \) is a field then \( F \times F \) is a field.
T F (c) \( \mathbb{Z}_{19} \) is an integral domain.
T F (d) In \( \mathbb{Q}[x] \) a non-zero prime ideal is always maximal.
T F (e) \( \mathbb{Z}_{10} \times \mathbb{Z}_3 \) and \( \mathbb{Z}_6 \times \mathbb{Z}_5 \) are isomorphic abelian groups.
T F (f) A subgroup of a cyclic group is cyclic.
T F (g) A non-cyclic group has at least one proper non-trivial cyclic subgroup.
T F (h) There are 8 units in \( \mathbb{Z}_{15} \).
T F (i) The non-zero elements of \( \mathbb{Z}_{31} \) form a cyclic group under multiplication.
T F (j) If \( |G| = 11 \) the \( G \simeq \mathbb{Z}_{11} \).
T F (k) The odd permutations in \( S_n \) form a normal subgroup of \( S_n \).
T F (l) \( (\mathbb{Z} \times \mathbb{Z}) / \langle (3,1) \rangle \simeq \mathbb{Z}_3 \times \mathbb{Z} \).
T F (m) The factor ring \( \mathbb{Z}_2[x] / \langle x \rangle \) has order 4.
T F (n) The operation \( a \ast b = 3/ab \) is not associative on \( \mathbb{R}^* \).
T F (o) A field does not have composite characteristic.
T F (p) \( \phi_g : G \rightarrow G \) given by \( \phi_g(x) = gx \) is a permutation of \( G \) for any fixed \( g \) in \( G \).
T F (q) In the quaternions \( ijk = -1 \).
T F (r) \( \mathbb{R}[x] / \langle x^2 + 4 \rangle \) is an integral domain.

8. (14 points) For the permutation \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 5 & 8 & 7 & 9 & 10 & 2 & 6 & 1 \end{pmatrix} \):

(a) Write \( \sigma \) as a product of disjoint cycles.

(c) Write \( \sigma \) as a product of transpositions.

(d) Is \( \sigma \) even, odd, neither or both?

(e) What is the order of \( \sigma \)?
9. (12 points) (a) If $G = \langle a \rangle$ is a group of order 50 then what is the order of $\langle a^{15} \rangle$?

(b) What is the order of $(25, 6)$ in $\mathbb{Z}_{30} \times \mathbb{Z}_{20}$?

(c) What is the order of the subgroup of $\mathbb{Z}_{45}$ generated by the set $\{12, 30\}$?

10. (7 points) Suppose that $\phi : R \to R'$ is a ring homomorphism. Prove that if $S'$ is a subring of $R'$ then its inverse image $\phi^{-1}[S']$ is a subring of $R$.

11. (12 points) Suppose that $F$ is a field.
   (a) Suppose that $f, g$ are non-zero polynomials in $F[x]$. What does the division algorithm in $F[x]$ say?

   (b) Find the quotient an remainder when $2x^3 + 3x + 1$ is divided by $3x + 2$ in $\mathbb{Z}_7[x]$.

   (c) Suppose that $I$ is a non-zero ideal in $F[x]$ and $g$ a non-zero polynomial in $I$ of minimal degree. Prove that $I = \langle g \rangle$. 
12. (17 points) (a) Define what it means for \( \phi : R \to R' \) to be a ring homomorphism.

(b) Verify that \( \phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_6 \) given by \( \phi(x, y) = 4x + 3y \pmod{6} \), is a ring homomorphism.

(c) What is the kernel, \( \ker(\phi) \), of the map in (b)?

(d) What is the image, \( \phi[\mathbb{Z} \times \mathbb{Z}] \), of the map in (b)?

(e) What does the fundamental homomorphism theorem say in the case of the map in (b)?

13. (18 points)

\[
R_1 = \mathbb{Z}_3[x], \quad R_2 = 2\mathbb{Z} \times \mathbb{R}, \quad R_3 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}.
\]

(i) Which of the above rings has a unity? Give each unity.

(ii) Describe the units for the ring(s) you picked in (i)

(iii) Which rings have zero divisors? Give an example in each case.