1. (12 points) (a) $G$ is a set closed under a binary operation $\ast$. What properties make $<G, \ast>$ a group?

(b) The set $S = \{a \in \mathbb{Q}, a \neq 1\}$ is closed under the commutative binary operation

$$a \ast b = (a - 1)(b - 1) + 1.$$ 

Prove that $<S, \ast>$ is an abelian group.

2. (4 points) The quotient and remainder when $3x^3 + x^2 + 2x + 3$ is divided by $2x^2 + 3$ in $\mathbb{Z}_5[x]$ are $q(x) =$ ____________ and $r(x) =$ ____________.
3. (6 points) (a) What properties must a non-empty subset $S$ of a ring $R$ satisfy in order to be a subring?

(b) Let $a$ be a fixed element of the ring $R$. Prove that $A_a = \{r \in R : ar = 0\}$ is a subring of $R$.

4. (11 points) (a) If $< a >$ has order 100 then $\langle a^{24} \rangle$ has order __________.

(b) The order of the group $\mathbb{Z}_{40} \times \mathbb{Z}_{60}$ is __________.

   The order of the element $(12, 28)$ in $\mathbb{Z}_{40} \times \mathbb{Z}_{60}$ is __________.

(c) The order of the subgroup of $\mathbb{Z}_{55}$ generated by the set $\{30, 50\}$ is __________.

5. (14 points) List the subgroups of the cyclic group $\mathbb{Z}_{45}$ and give their orders. Draw a subgroup diagram.
6. (5 points) Let $R$ be a (not necessarily commutative) ring with unity 1.
(a) Define what it means for an element $a$ of $R$ to be a unit.
(b) Prove that if $a$ and $b$ are units then $ab$ is a unit.

7. (12 points) (a) Define what it means for $\phi : G \to G'$ to be a group isomorphism.
(b) Find all the group isomorphisms $\phi : 2\mathbb{Z} \to 4\mathbb{Z}$.
(c) Define what it means for $\phi : R \to R'$ to be a ring homomorphism.
(d) Find all the ring homomorphisms $\phi : \mathbb{Z} \to \mathbb{Z}_6$. 
8. (36 points) Circle True (T) or False (F).
T  F  (a) \( \mathbb{Q}[x] \) is a field.
T  F  (b) If \( R \) is an integral domain then \( R[x] \) is an integral domain.
T  F  (c) If \( \mathbb{Z}_n \) has no zero divisors then it is a field.
T  F  (d) The binary operation \( a * b = 2a + 2b \) on \( \mathbb{R} \) is associative.
T  F  (e) \( \mathbb{Z}_2 \times \mathbb{Z}_{10} \) and \( \mathbb{Z}_4 \times \mathbb{Z}_5 \) are isomorphic abelian groups.
T  F  (f) Every group \( G \) with \( |G| \leq 5 \) is abelian.
T  F  (g) The symmetries of the regular octagon form a non-abelian group, \( D_8 \), of order 16.
T  F  (h) If a ring has prime characteristic then it is an integral domain.
T  F  (i) The non-zero elements of \( \mathbb{Z}_{15} \) form a group of order 14 under multiplication mod 15.
T  F  (j) A group of prime order is cyclic.
T  F  (k) There are \( \phi(21) = 12 \) units in \( \mathbb{Z}_{21} \).
T  F  (l) If \( [G : H] = 36 \) and \( |H| = 3 \) then \( |G| = 12 \).
T  F  (m) In the quaternions \( (1 - i + j)^{-1} = \frac{1}{2} + \frac{1}{3}i - \frac{1}{2}j \).
T  F  (n) In the quaternions \( i(1 + j)k = -1 + j \).
T  F  (o) \( 1 + 2x \) is a unit in \( \mathbb{Z}_4[x] \).
T  F  (p) The ring \( \mathbb{Z}_3[x]/\langle x^2 \rangle \) has nine elements, namely \( a + bx + \langle x^2 \rangle, a, b \in \mathbb{Z}_3 \).
T  F  (q) The ring \( \mathbb{Q}[x]/\langle x^2 - 2 \rangle \) is an integral domain.
T  F  (r) The ring \( \mathbb{R}[x]/\langle x^2 - 2 \rangle \) is a field.

9. (11 points) For the permutation \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 7 & 2 & 4 & 9 & 1 & 5 & 6 \end{pmatrix} \):

(a) Write \( \sigma \) as a product of disjoint cycles.

(c) Write \( \sigma \) as a product of transpositions.

(d) Is \( \sigma \) even, odd, neither or both?

(e) What is the order of \( \sigma \)?
10. (18 points) Suppose that $R$ is a commutative ring with unity.
(a) What properties must a non-empty subset $I$ of $R$ satisfy in order to be an ideal?

(b) Define what it means for an ideal $I$ of $R$ to be prime.

(c) Define what it means for an ideal $I$ of $R$ to be maximal.

(d) (i) Prove that $I_1 = 2\mathbb{Z} \times 3\mathbb{Z}$ is an ideal of $\mathbb{Z} \times \mathbb{Z}$.

(ii) Show that $I_1$ is not a prime ideal of $\mathbb{Z} \times \mathbb{Z}$.

(e) (i) Prove that $I_2 = \{f(x) \in \mathbb{Z}[x] : f(3) = 0\}$ is an ideal of $\mathbb{Z}[x]$.

(ii) Show that $I_2$ is prime.

(iii) Show that $I_2$ is not maximal.

(iv) Is $I_2$ a principal ideal in $\mathbb{Z}[x]$? Yes/No. If yes, give a generator: $I_2 = \langle \ldots \rangle$. 
11. (6 points) Find all non-isomorphic abelian groups of order 200 = 2^3 \cdot 5^2.

12. (24 points) The following are commutative rings with unities.

\[ R_1 = \mathbb{Z}_4 \times \mathbb{Z}, \quad R_2 = \mathbb{Z}_{12}, \quad R_3 = \mathbb{Z}_5[x], \quad R_4 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Q} \right\} \]

(i) Give each unity: \( 1_{R_1} = \), \( 1_{R_2} = \), \( 1_{R_3} = \), \( 1_{R_4} = \).

(ii) Give the characteristics: char(\( R_1 \)) = \( \), char(\( R_2 \)) = \( \), char(\( R_3 \)) = \( \), char(\( R_4 \)) = \( \).

(iii) Describe the units in each ring:

\( R_1 : \)

\( R_2 : \)

\( R_3 : \)

\( R_4 : \)

(iv) Which of the rings have zero divisors? Give an example in each case.

(v) Which of the rings if any are integral domains?

Which of the rings if any are fields?