MATH 512  Intro to Modern Algebra – Exam II
Wednesday, October 23, 2013

Check that that you have all four pages - note that the pages are double-sided

1. (10 points) For the permutation $\sigma$ in $S_7$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 6 & 3 & 4 & 1 \end{pmatrix}.$$  

(a) The order of $\sigma$ is $10$.

$$\sigma = (1 \ 2 \ 3 \ 7 \ 4 \ 6) \quad \text{lcm}(5, 2) = 10$$

(b) Write $\sigma$ as a product of transpositions. Is $\sigma$ even, odd, or neither (circle one)?

$$\sigma = (1 \ 7 \ 4)(1 \ 5 \ 3)(4 \ 6) \quad \text{odd}$$

(c) The subgroup $<\sigma>$ has $504$ left cosets in $S_7$.

$$|S_7 : <\sigma>| = \frac{7!} {1 \cdot 10} = \frac{5040} {10} = 504$$

2. (11 points) The cyclic group $G = \langle a \rangle$ has order 20.

(a) In addition to the trivial subgroups $G$ and $\{e\}$ this group has $14$ proper subgroups.

Write out the elements in each of these subgroups.

Divisors of 20 are 1, 2, 4, 5, 10, 20

$\langle a^2 \rangle = \{e, a^2, a^4, a^6, a^8, a^{10}, a^{12}, a^{14}, a^{16}, a^{18}\}$

$\langle a^4 \rangle = \{e, a^4, a^8, a^{12}, a^{16}\}$

$\langle a^5 \rangle = \{e, a^5, a^{10}, a^{15}\}$

$\langle a^{10} \rangle = \{e, a^{10}\}$

(b) The order of $\langle a^{12} \rangle$ is $5$.

$$\text{gcd}(12, 20) = 4 \quad \langle a^{12} \rangle = \langle a^4 \rangle$$
3. (20 points) Circle True (T) or False (F).

T F (a) \( \mathbb{Q} \cong \mathbb{R} \).

F (b) The square \( \sigma^2 \) of a permutation \( \sigma \in S_n \) is always even.

T F (c) \( \{1, -1, i, -i\} \cong \mathbb{Z}_4 \).

T F (d) \( \mathbb{Z}_p \times \mathbb{Z}_p \cong \mathbb{Z}_{p^2} \).

F (e) If \( |G| = 11 \) then \( G \) is cyclic.

T F (f) If \( K < H < G \) with \( (G : H) = 10 \) and \( (H : K) = 5 \) then \( H \cong \mathbb{Z}_{20} \).

T F (g) \( 3\mathbb{Z} \cong 5\mathbb{Z} \).

F (h) The subgroups of a non-cyclic group are all non-cyclic.

T F (i) The Alternating group \( A_n \) always has two cosets in the symmetric group \( S_n \).

T F (j) The group of symmetries of the square, \( D_4 \), is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \).

4. (8 points) Suppose that \( H \) and \( K \) are isomorphic groups with \( H \) abelian. Prove that \( K \) is abelian.

Suppose that \( \phi : H \rightarrow K \) is an isomorphism.

Suppose that \( a, b \in K \). Since \( \phi \) is onto \( a = \phi(h_1) \) some \( h_1, h_2 \in H \)

\[ a = \phi(h_1) \]
\[ b = \phi(h_2) \]

\[ \phi \] has homomorphism property. \( H \) abelian.

\[ \phi \] is a homomorphism.

5. (8 points) \( H = \{e, (13)\} \) is a subgroup of \( S_3 = \{e, (12), (13), (23), (123), (132)\} \).

(a) List the distinct left cosets of \( H \).

\[ eH = \{e, e(13)\} \]
\[ (12)H = \{(12)e, (12)(13)\} \]
\[ (23)H = \{(23)e, (23)(13)\} \]

\[ (12)H = \{(12)e, (12)(13)\} = \{(12), (123)\} \]
\[ (132)H = \{(132)e, (132)(13)\} = \{(13), (123)\} \]

(b) Are the right cosets of \( H \) the same? No.

\[ H(12) = \{(e)(12), (13)(12)\} = \{(12), (123)\} \neq (12)H \]
\[ H(23) = \{(e)(23), (13)(23)\} = \{(23), (132)\} \neq (23)H \]
5. (18 points) (a) The index of $< 6 >$ in $\mathbb{Z}$ is $6$. 

$0+6\mathbb{Z}, \ 1+6\mathbb{Z}, \ 2+6\mathbb{Z}, \ 3+6\mathbb{Z}, \ 4+6\mathbb{Z}, \ 5+6\mathbb{Z}$ 

(b) A single generator for the subgroup of $\mathbb{Z}$ generated by the set $\{45, 30, 20\}$ is $5$.

$\text{gcd}(45, 30, 20) = \text{gcd}(5, 5, 5) = 5$

(c) The order of the element $(4, 5, 6)$ in the group $\mathbb{Z}_{24} \times \mathbb{Z}_{20} \times \mathbb{Z}_9$ is $12$.

$k(4, 5, 6) = (4k, 5k, 6k) = (0, 0, 0) \Rightarrow 24|4k \Rightarrow 61k \Rightarrow 12|k$

(d) A generator for the cyclic group $\mathbb{Z}_7 \times \mathbb{Z}_5$ is $(1, 1)$.

$k(1, 1) = (0, 0) \Rightarrow 7k = 5k \Rightarrow 35|k$

$\text{gcd}(9, 11) \text{ and } 7 \times a, 5 \times b \Rightarrow \text{gcd} = 1$.

(e) Show that $\mathbb{Z}_6 \times \mathbb{Z}_{10}$ is not cyclic.

$lcm(6, 10) = 30 \Rightarrow (9, 11) \notin \mathbb{Z}_6 \times \mathbb{Z}_{10}$

so $\mathbb{Z}_6 \times \mathbb{Z}_{10}$ does not contain an element of order $60$.

6. (9 points) (a) $\phi(x) = \frac{e^x + 1}{e^x - 1}$ is an isomorphism $\phi : \mathbb{R} \to \mathbb{R}^{\text{pos}}$.

Your map should be an obvious bijection but check the additional property making it an isomorphism.

$\phi(x+y) = e^{x+y} = \frac{e^x + 1}{e^x - 1} \cdot \frac{e^y + 1}{e^y - 1}$

(b) $H = \left\{ \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) : a \in \mathbb{Z} \right\}$ is a group under multiplication. Prove that $H$ is isomorphic to $2\mathbb{Z}$.

Your map should be an obvious bijection but check the additional property making it an isomorphism.

$\phi : H \to 2\mathbb{Z} \quad \phi \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right) = 2a$

$\phi \left( \left( \begin{array}{c} a \\ b \end{array} \right) \right) = \phi \left( \left( \begin{array}{c} 1+a+b \\ 1+b \end{array} \right) \right) = \phi \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right) + \phi \left( \left( \begin{array}{c} b \\ 1 \end{array} \right) \right)$
7. (7 points) (a) If \( H \) is a subgroup of a finite group \( G \) then Lagrange's Theorem states that:

\[
|H| \text{ divides } |G|
\]

(b) Suppose that the group \( |G| = 27 \) contains an \( a \) with \( a^9 \neq e \). Show that \( G \) must be cyclic.

Consider \( H = \langle a \rangle \leq G \)

Lagrange says:

\[
|H| \mid 27 = 3^3 \implies |H| = 1, 3, 9, \text{ or } 27
\]

\[
|H| = 1 \implies a = e \implies a^9 = e \\
|H| = 3 \implies a = e \implies a^9 = e \\
|H| = 9 \implies a^9 = e
\]

and \( H = \langle a^9 \rangle = G \)

8. (9 points) (a) What three properties make a relation \( \sim \) on a set \( S \) an equivalence relation?

\( \forall a, b, c \in S \)

i) \( a \sim a \)
ii) \( a \sim b \implies b \sim a \)
iii) \( a \sim b, b \sim c \implies a \sim c \)

(b) Suppose that \( G \) is a group. For \( a, b \in G \) define \( a \sim b \) if \( a = xbx^{-1} \) for some \( x \) in \( G \).
Show that \( \sim \) is an equivalence relation on \( G \).

i) Reflexive: \( a = eae^{-1} \implies a \sim a \).

ii) Symmetric: Suppose \( a \sim b \implies a = xbx^{-1} \text{ some } x \in G \)

\[
\implies x^ax = b
\]

\[
\implies b = (x^a)a(x^a)^{-1}, \quad x^a \in G
\]

\( \implies b \sim a \).

iii) Transitive: Suppose that \( a \sim b, b \sim c \)

\[
\Rightarrow a = xbx^{-1} \text{ some } x, y \in G
\]

\[
b = ycy^{-1}
\]

\[
\Rightarrow a = x(ycy^{-1})x^{-1}
\]

\[
= (xy)c(xy)^{-1}, \quad xy \in G
\]

\( \Rightarrow a \sim c \).