1. (18 points) (a) \( G \) is a set closed under a binary operation \(*\). What three properties make \(<G,*>\) a group?

**Associative:** \( \forall a,b,c \in G \quad (a * b) * c = a * (b * c) \)

**Identity:** \( \exists e \in G \quad x * e = x = e * x \quad \forall x \in G \)

**Inverses:** For any \( a \in G \), \( \exists a^{-1} \in G \) s.t. \( a * a^{-1} = e = a^{-1} * a \).

(b) The set \( \mathbb{R} \) is closed under the binary operation

\[ a * b = \sqrt[3]{a^3 + b^3} + 1. \]

Prove that \(<\mathbb{R},*>\) is a group.

**Associative:**

\[ (a * b) * c = \sqrt[3]{a^3 + b^3 + 1} * c = \sqrt[3]{a^3 + b^3 + 1} * c = \sqrt[3]{a^3 + (b^3 + c^3 + 1) + 1} = \sqrt[3]{a^3 + b^3 + c^3 + 2} \]

\[ a * (b * c) = a * \sqrt[3]{b^3 + c^3 + 1} = \sqrt[3]{a^3 + (b^3 + c^3 + 1) + 1} = \sqrt[3]{a^3 + b^3 + c^3 + 2} \]

**Identity:** \( a * e = a \leftrightarrow \sqrt[3]{a^3 + e^3 + 1} = a \)  
By symmetry also

\[ e^3 = 1 \]

\[ e = -1 \in \mathbb{R} \]

**Inverses:** \( a * a^{-1} = e \leftrightarrow \sqrt[3]{a^3 + (a^{-1})^3} + 1 = -1 \)  
By symmetry also

\[ a^3 + (a^{-1})^3 + 1 = -1 \]

\[ (a^{-1})^3 = -2 - a^3 \]

\[ a^{-1} = -\sqrt[3]{2 + a^3} \in \mathbb{R} \]
2. (18 points)

\[ H = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{Z} \right\} \]

(a) Show that the set \( H \) is closed under matrix multiplication.

\[ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix} \in H \Rightarrow \begin{bmatrix} 1 & b+b' \\ 0 & 1 \end{bmatrix} \in H \]

(b) \( H \) contains an identity under multiplication, namely \( e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

(c) Show the additional property needed to prove that \( < H, \cdot > \) is a subgroup of the group \( < GL(2, \mathbb{Q}), \cdot > \).

\[ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in H \Rightarrow \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix} \in H \]

(d) \( \phi \left( \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) = 2b \) gives a bijection \( \phi : H \rightarrow 2\mathbb{Z} \). (Plainly a bijection)

(e) Show that \( < H, \cdot > \) is isomorphic to \( < 2\mathbb{Z}, + > \).

\[ \phi \left( \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix} \right) = \phi \left( \begin{bmatrix} 1 & b+b' \\ 0 & 1 \end{bmatrix} \right) \]

\[ = 2(b+b') \]

\[ = 2b + 2b' \]

\[ = \phi \left( \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) + \phi \left( \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix} \right) \]

3. (10 points) Let \( < G, * > \) be a group. For a fixed element \( g \) in \( G \) define a \( \phi : G \rightarrow G \) by

\[ \phi(x) = g \ast x. \]

(i) Show that \( \phi \) is onto.

Suppose \( y \in G \). We need to show there exists \( x \in G \) such that \( \phi(x) = y \).

So \( \phi(g \ast y) = y \)

(ii) Show that \( \phi \) is one-to-one.

\[ \phi(x) = \phi(y), \quad x, y \in G \]

\[ \Rightarrow g \ast x = g \ast y \]

\[ \Rightarrow x = y \quad \text{Cancellation law or multiply both sides by } g^{-1} \]
4. (18 points) (a) Let \(< S, * >\) and \(< S', *' >\) be binary algebraic structures and \(\phi : S \to S'\) a bijection (i.e. one-to-one, onto mapping). What additional property makes \(\phi\) an isomorphism?

\[ \forall x, y \in S \quad \phi(x * y) = \phi(x)' \phi(y) \]

(b) Show that the bijection \(\phi(x) = e^x\) gives an isomorphism from \(< \mathbb{R}, + >\) to \(< \mathbb{R}^+, \cdot >\).

\[ \phi(x + y) = e^{x+y} = e^x \cdot e^y = \phi(x) \cdot \phi(y) \]

(c) Prove that if \(\phi : S \to S'\) is an isomorphism for \(< S, * >\) and \(< S', *' >\) and * is associative on S then *' is associative on S'.

Take \(a, b, c \in S\) since \(\phi\) is one-to-one \(a, b, c \in S\) and \(\phi(a) = a, \phi(b) = b, \phi(c) = c\)

\[ (a *' b) *' c = (\phi(a) *' \phi(b)) *' \phi(c) \]
\[ = \phi(a * b) *' \phi(c) \quad \phi \text{ is a homomorphism} \]
\[ = \phi((a * b) * c) \quad \phi \text{ is a homomorphism} \]
\[ = \phi(a) *' (\phi(b) *' \phi(c)) \quad \phi \text{ is a homomorphism} \]
\[ = a *' (b *' c) \quad \text{since *' is associative in } S' \]

5. (6 points) Let \(< G, * >\) be an abelian group. Define \(S = \{x \in G \mid x * x = e\}\).

Prove that \(S\) is closed under *.

Suppose \(a, b \in S\) we need to show that \(a * b \in S\)

\[ (a * b) * (a * b) = a * (b * a) \quad \text{G. commutative} \]
\[ = a * (a * b) * b \]
\[ = (a * a) * (b * b) \quad \text{G. commutative} \]
\[ = e * e \quad \text{G. commutative} \]
\[ = e \quad \text{G. identity} \]

Plainly \(a * b \in S\) by closure.
6. (20 points) Circle True (T) or False (F).

(a) $\mathbb{Q}^+ \times \mathbb{Q}^+$ is a group. **F**

(b) $\mathbb{Q}^+ \cdot \mathbb{Q}^+$ is a group. **T**

(c) $\langle \mathbb{Z}_4, +_4 \rangle$ is a subgroup of $\langle \mathbb{Z}_6, +_6 \rangle$. **T**

(d) Suppose that $a \sim b$ if $ab$ is even, then $\sim$ is an equivalence relation on $\mathbb{Z}$. **F**

(e) If $G, \ast >$ is a group with $a^{-1} = a$ for all $a$ in $G$ then $G$ is abelian. **T**

(f) The set $\{0, 4, 5\}$ is closed under addition mod 9. **F**

(g) In a group $\langle G, \ast >$ the equation $x \ast x = x$ has exactly one solution. **T**

(h) $\phi(x) = -x$ is an automorphism of $\langle \mathbb{Z}, + \rangle$. **T**

(i) If $\phi$ is an homomorphism from $\langle U_5, \cdot \rangle$ to $\langle \mathbb{Z}_5, +_5 \rangle$ with $\phi(w^4) = 3$ then $\phi(w^3) = 1$. [Here $U_5 = \{1, w, w^2, w^3, w^4\}$ denotes the fifth roots of unity]. **F**

(j) If $G$ is a group of order 3 then $G \cong \mathbb{Z}_3$. **F**

7. (10 points) $G = \{x, 1-x, \frac{1}{x}, \frac{1}{1-x}, \frac{x-1}{x}, \frac{x}{x-1} \}$ forms a group of order six under composition.

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(a) Is $G$ abelian? **F**

(b) A group of order 6 is isomorphic to $\mathbb{Z}_6$ or $D_3$. How can you tell that $G \not\cong \mathbb{Z}_6$?

$\mathbb{Z}_6$ is abelian or $x+x=x$ has 4 solutions in $G$ but 2 in $\mathbb{Z}_6$

(c) The symmetries of the equilateral triangle $D_3 = \{I, R, R^2, F, FR, FR^2\}$ where $R$ denotes a 120°-rotation and $F$ a flip. Suggest values

$\phi(R) = \frac{1}{1-x}$ or $\frac{x-1}{x}$, \hspace{1cm} \phi(F) = \frac{1}{x}$ or $\frac{1-x}{x}$ or $\frac{x}{1-x}$

likely to produce an isomorphism $\phi : D_3 \rightarrow G$.

[No need to check it, any sensible choice works and there are many right answers!].