MATH 510  Discrete Math – Final Exam
Monday, December 10, 2007

Check that that you have all six pages. Show all your work and reasoning.

1. (15 points) Find the generating function \( g(x) = \sum_{n=0}^{\infty} h_n x^n \) for the recurrence

\[
h_n - 2h_{n-1} - 15h_{n-2} = 0 \quad h_n = 2h_{n-1} + 15h_{n-2}, \quad h_0 = 3, \quad h_1 = -1.
\]

\[
g(x) = \frac{h_0 + (h_1 - h_0)x}{1 - 2x - 15x^2}.
\]

(b) Use this generating function to solve the recurrence: \( h_n = 2(h_{n-1})^2 + 5^n \).

\[
g(x) = \frac{2 - 7x}{1 - 2x - 15x^2} = \frac{2 - 7x}{(1 - 2x)(1 - 5x)} = \frac{2 - 7x}{1 - 5x} + \frac{7x}{1 + 2x}
\]

2. (10 points) The Fibonacci sequence satisfies the recurrence

\[ f_n = \frac{f_{n-1} + f_{n-2}}{f_1 = 1, \quad f_2 = 1. \]

Use induction to prove that \( f_1 + f_3 + \cdots + f_{2n-1} = f_{2n} \).

**Base Step**: \( n = 1 \), \( f_1 = 1 = f_2 \).

**Inductive Step**: Assume \( f_1 + f_3 + \cdots + f_{2k-1} = f_{2k} \) (and thus \( f_1 + \cdots + f_{2k-1} = f_{2k} \)).

\[
f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} = f_{2k} + f_{2k+1} = f_{2k+2} = f_{2(k+1)} \quad \text{(Fibonacci recurrence)}.
\]

Hence, the result holds for all \( n \) by the principle of math induction.
3. (12 points) The inclusion-exclusion principle for three sets $A, B, C \subseteq S$ states that:

$$|A \cap B \cap C| = |A| + |B| + |C| - |A \cup B| - |A \cup C| - |B \cup C| + |A \cup B \cup C|$$

Use this to count the number of integers from 1 to 1000 which are not multiples of 10, 4 or 15.

- $A = \text{multiples of 10}$
- $B = \text{multiples of 4}$
- $C = \text{multiples of 15}$

$$|A \cap B \cap C| = 1000 - (|\text{1000}| + |\text{1000}| + |\text{1000}|) + (|\text{1000}| + |\text{1000}| + |\text{1000}|) - |\text{1000}|$$

$$= 1000 - (100 + 250 + 66) + (100 + 250 + 66) - 16$$

$$= 667$$

4. (12 points) Seven houses in a row are to be each painted green, yellow, blue, white or purple. How many ways if:

(a) Adjacent houses should not be the same color.

$$5 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 20,480$$

(b) At least one of the houses must be green.

$$5^7 - 4^7 = 61,741$$

(c) Exactly three houses should be blue, two green, and the remaining two not the same color.

$$\binom{7}{3} \cdot \binom{4}{2} = 1,260$$

5. (8 points) Two five member teams each with a specified team captain are to be chosen from 20 volunteers, 8 of them girls and 12 boys. One team is to be all girl, the other team is unrestricted except that the team captain must be a boy. How many different ways can the teams and captains be selected?

$$8 \times 12 \times \binom{7}{4} \times \binom{14}{4} = 8,363,360$$

6. (7 points) Ten people are to be seated around a circular table. How many circular arrangements are possible if Alison, Bob, Carol and Doug all refuse to sit next to or opposite from Eric?

$$5! \times 6 \times 5 \times 3 \times 2! = 4,320$$
7. (10 points) Use the deferred acceptance algorithm to find the **women-optimal** stable complete marriage for the preferential ranking matrix

\[
\begin{array}{c|cccc}
A & 1,3 & 3,2 & 2,1 & 4,1 \\
B & 4,1 & 3,3 & 1,4 & 2,2 \\
C & 2,4 & 3,4 & 4,3 & 1,4 \\
D & 1,2 & 2,1 & 3,2 & 4,3 \\
\end{array}
\]

Here rows A, B, C, D are the women and columns a, b, c, d are the men.

- A chooses a  
  - a engages D  
  - D rejects A  

- B chooses c  
  - c engages B  

- C chooses d  
  - d engages C  

- D chooses a  
  - a engages D  

- A chooses c  
  - c engages B  

- B chooses d  
  - d engages C  

- C chooses a  
  - a engages C  

- C chooses b  
  - b engages C

8. (15 points) (a) Give the rook bipartite graph for the 6-by-6 chessboard with forbidden positions shown.

(b) Find a max-matching. Run the matching algorithm to verify this and indicate the corresponding non-attacking rooks on the board.

(c) A min-cover is \{ x_1, x_2, x_3, y_1, y_4, y_6 \}.

\begin{align*}
&x_1 & y_1 \\
x_2 & y_2 \\
x_3 & y_3 \\
x_4 & y_4 \\
x_5 & y_5 \\
x_6 & y_6
\end{align*}

\begin{align*}
&x_1 & y_1 \\
x_2 & y_2 \\
x_3 & y_3 \\
x_4 & y_4 \\
x_5 & y_5 \\
x_6 & y_6
\end{align*}

\begin{align*}
&x_1 & y_1 \\
x_2 & y_2 \\
x_3 & y_3 \\
x_4 & y_4 \\
x_5 & y_5 \\
x_6 & y_6
\end{align*}

**Answers may vary**
9. (20 points) (a) How many times do you need to take your pen off the paper to trace the complete graph $K_6$? Explain.

(b) Does the complete bipartite graph $K_{3,4}$ have a Hamilton path? If so draw one, if not say why not.

(c) Define what it means for a (simple) graph $G$ of order $n$ to have the Ore property.

(d) Prove that a (simple) graph of order $n$ with two connected components cannot have the Ore property.

(e) A (simple) graph $G$ has degree sequence $(4, 4, 4, 4, 4, 4)$. Can $G$ have a bridge? Explain.

10. (14 points) (a) Suppose that $h_n$ is the number of ways to choose $n$ pieces of fruit from an unlimited choice of apples, oranges, pears, pineapples and mangoes.

(b) Suppose that in (a) the apples are loose but the oranges are in packs of 3, the pears in bags of 5, there are only four pineapples and two mangoes.

(i) Find and simplify the generating function $g(x) = \sum_{n=0}^{\infty} h_n x^n$.

(ii) Give a formula for $h_n$.
11. (12 points) Suppose that $G$ is a tree of order 9 with a vertex of degree 6.
(a) How many edges does $G$ have? \( n-1 = 8 \)
(b) What degree sequences can $G$ have?
\[ d_1 + \ldots + d_9 = 2e \]
\[ d_1 + \ldots + d_9 + 6 = 1e \]
\[ d_1 + \ldots + d_9 + 10 = 0 \]
(c) Draw three non-isomorphic trees of order 9 with a vertex of degree 6.

12. (14 points) How many ways can you put 6 non-attacking rooks on the 6-by-6 chessboard with forbidden positions shown?
\[ r_1 = \text{try to count all non-attacking rooks on the whole board} \]
\[ r_2 = 2 \times 6 \times 6 \]
\[ r_3 = 3 \times 4 + 4 \times 4 = 23 \]
\[ r_4 = 3 \times 4 + 4 \times 4 + 1 = 29 \]
\[ r_5 = 3 \times 4 + 1 \times 4 = 16 \]
\[ r_6 = 3 \times 4 = 12 \]

13. (12 points) (a) Evaluate the coefficient of $x^3y^8w^3$ in the multinomial expansion of $(2x - y + 3z + 5w)^8$.
\[ \text{Coefficient} = \frac{8!}{3!3!1!1!} \times (2)^3 \times (-1)^8 \times y^8 \times z^3 \times w^3 \]

(b) Use the binomial theorem $(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$ to evaluate the sum $\sum_{j=0}^{n} j \binom{n}{j} q^{j-1}$.
14. (25 points) \( G \) is a connected general planar graph of order \( n \) with \( e \) edges and \( r \) regions.
(a) State Euler's formula relating \( n, e \) and \( r \\
\[ n - e + r = 2 \]
(b) Explain how to find \( e \), the number of edges of \( G \) from the degree sequence \((d_1, \ldots, d_n)\).
\[ e = \frac{1}{2}(d_1 + \cdots + d_n) \]

(c) Find \( e, n \) and \( r \) for degree sequence \((5,5,4)\).
\[ n = 3 \quad e = \frac{1}{2}(5+5+4) = 7 \quad r = 2 + e - n = 2 + 7 - 3 = 6 \]

(d) Draw five non-isomorphic, general, connected, planar graphs with degree sequence \((5,5,4)\).

(e) Are the complete bipartite graphs \( k_{2,3} \) or \( k_{3,2} \) planar? Explain.
\( k_{2,3} \) is planar:
\[ n = 4, e = 9 \quad r = 2 + e - n = 5 \quad \text{if planar} \]
\[ 12 = 2e - f - \text{faces} \geq 4 + \text{faces} \geq 20 \]

(because in a bipartite graph each vertex must have at least 2 edges). False as \( k_{2,3} \) is not planar.

15. (14 points) Bob picks 11 integers from 1 to 20. When using the box principle label your boxes clearly.
(a) Prove that his choice must include integers \( a, b \) with one a multiple of the other.

(b) Prove that his choice includes three numbers differing from each other by at most 3.