MATH 510 Discrete Math – Final Exam
Tuesday, December 12, 2006

Check that that you have all five pages. Show all your work and reasoning.

1. (15 points) Find the generating function \( g(x) = \sum_{n=0}^{\infty} h_n x^n \) for the recurrence

\[
h_n - h_{n-1} - 6h_{n-2} = 0 \quad h_n = h_{n-1} + 6h_{n-2}, \quad h_0 = 8, \quad h_1 = 9.
\]

\[
g(x) = h_0 + h_1 x + h_2 x^2 + \cdots
\]

\[
-xg(x) = -h_0 x - h_1 x^2 - \cdots
\]

\[
-6x^2g(x) = -6h_0 x^2 - \cdots
\]

\[
g(x) = \frac{8 + x}{1 - x - 6x^2}
\]

(b) Use this generating function to solve the recurrence: \( h_n = \frac{5 \cdot 3^n + 3(-2)^n}{1 - 3x} \).

2. (14 points) \( G \) is a connected general planar graph of order \( n \) with degree sequence \( (d_1, \ldots, d_n) \).

(a) Explain how you would find \( e, \) the number of edges of \( G \). Find \( e \) for degree sequence \((5,5,2)\).

\[
e = \frac{1}{2} (d_1 + d_2 + \cdots + d_n)
\]

For \((5,5,2)\) \( e = \frac{1}{2} (5 + 5 + 2) = 6 \)

(b) Explain how you would find \( r, \) the number of regions of \( G \). Find \( r \) for degree sequence \((5,5,2)\).

\[
r = \frac{n + e - 2}{2} \quad \text{(Euler’s Formula)}
\]

For \((5,5,2)\) \( r = \frac{2 + 6 - 2}{2} = 3 \)

(c) Determine four non-isomorphic, general, connected, planar graphs with degree sequence \((5,5,2)\).

[Diagrams of four different planar graphs are shown, each with degree sequence \((5,5,2)\).]
3. (7 points) Give a combinatorial explanation of the identity \( \sum_{j=0}^{m} \binom{2m}{j} \binom{m}{m-j} = \binom{3m}{m} \).

\[ \binom{3m}{m} = \text{# ways of choosing a team of size } m \text{ from a pool of size } 3m. \]

Suppose the \( 3m \) consist of \( 2m \) men and \( m \) women and divide according to the number \( j \) of men in the team \( j = 0, 1, 2, \ldots, m \).

\[ \text{# ways to choose a team of size } m \text{ with } j \text{ men, } m-j \text{ women} = \binom{2m}{j} \times \binom{m}{m-j} = \text{choose } j \text{ men } \times \text{choose } m-j \text{ women}. \]

4. (14 points) (a) A classroom has three rows of seats, with ten seats in each row. How many different ways can 12 students be seated if 4 always sit in the back row, 2 always sit in the front row and the remaining 6 can sit anywhere?

\[ 10 \times 9 \times 7 \times 10 \times 9 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 \]

Seat back 6 students
Seat front 2 students
Seat remaining 4 students anywhere left

(b) A five character password is to contain three digits and two capital letters with no character repeated eg 7A3B5, 234DE, E178F, ... but not 7A3A5, 7231B, ... . How many passwords are there?

\[ \binom{10}{3} \times \binom{26}{2} \times 5! = 4,680,000 \text{ or } \binom{5}{2} \times 26 \times 25 \times 10 \times 9 \times 8 \]

5. (7 points) Five men and five women are to be seated around a circular table so that no two men (and no two women) end up sitting together. How many circular arrangements are possible if Alison refuses to sit next to her ex Bill?

\[ 1 \times 3 \times 4! \times 4! = 1728 \]

Seat Alison, seat Bill, seat remaining men, seat remaining women

\[ 1 \times 5! \times 4! - 2 \times 4! \times 4! = \text{total - Bill and Alison together} \]

\[ 1 \times 5! \times 4! - 2 \times 4! \times 4! = \text{total - Bill and Alison (Bill excl.)} \]

6. (7 points) Use the binomial theorem \((1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j\) to evaluate the sum \( \sum_{j=0}^{99} \binom{99}{j} \frac{(-1)^{j+1}}{j+1} \)

\[ \sum_{j=0}^{99} \binom{99}{j} \frac{(-1)^{j+1}}{j+1} = \int_{-1}^{1} \left( \sum_{j=0}^{99} \binom{99}{j} x^j \right) \frac{dx}{j+1} = \int_{-1}^{1} \left( \frac{1+x)^{99}}{100} \right)^{-1} \]

\[ = \int_{-1}^{1} \frac{1}{100} \left( \frac{1}{100} \right) dx = \left[ \frac{x}{100} \right]_{-1}^{1} = \frac{1}{100} - \frac{1}{100} = 0 \]
7. (10 points) Use the deferred acceptance algorithm to find the men-optimal stable complete marriage for the preferential ranking matrix

\[
\begin{array}{cccc}
    & a & b & c & d \\
A & 1,3 & 2,2 & 3,1 & 4,1 \\
B & 4,1 & 2,3 & 1,4 & 3,2 \\
C & 3,4 & 2,4 & 4,3 & 1,4 \\
D & 1,2 & 2,1 & 3,2 & 4,3 \\
\end{array}
\]

Here rows A, B, C, D are the women and columns a, b, c, d are the men.

- a chooses B
- b chooses D
- c chooses A
- d chooses A

- A engages c, rejects d
- B engages a
- D engages b
- D engages a

8. (10 points) The inclusion-exclusion principle for three sets \( A, B, C \subseteq S \) states that:

\[
|A \cap B \cap C| = |S| - (|A| + |B| + |C|) + (|A \cap B| + |A \cap C| + |B \cap C|) - |A \cap B \cap C|
\]

Use this to count the number of integer solutions to

\[
x_1 + x_2 + x_3 + x_4 = 12, \quad 0 \leq x_1, \quad 0 \leq x_2 \leq 5, \quad 0 \leq x_3 \leq 4, \quad 0 \leq x_4 \leq 3.
\]

\[
A: \text{ solutions with } x_2 \geq 6 \\
B: \text{ solutions with } x_3 \geq 5 \\
C: \text{ solutions with } x_4 \geq 4
\]

\[
|A \cap B \cap C| = \binom{15}{3} - \left( \binom{9}{3} + \binom{10}{3} + \binom{3}{3} \right) + \left( \binom{4}{3} + \binom{5}{3} + \binom{2}{3} \right) - 0
\]

\[
= \binom{15}{3} - \binom{9}{3} - \binom{10}{3} - \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{2}{3} - 0
\]

= 455 - 84 - 120 - 165 + 4 + 10 + 20 - 0 = 120

9. (6 points) Evaluate the coefficient of \( x^2y^3z^2w \) in the multinomial expansion of \( (2x - y + 3z + 5w)^8 \).

\[
\begin{align*}
\frac{8!}{2!3!2!1!} & \cdot (2x)^2 (3z)^2 (5w)^3 (y)^3 \\
& = -202,400 \cdot x^2 y^3 z^2 w
\end{align*}
\]
10. (12 points) (i) Does $G$ have an open Euler trail? Explain No!

Degree vertices.

(ii) Does $G$ have a Hamilton cycle? Explain.

No! $G$ contains a bridge $x$.

(iii) Show that $G$ has a Hamilton path.

11. (10 points) Suppose that $h_n$ is the number of ways to choose $n$ pieces of fruit from an unlimited choice of apples, oranges, pears, bananas and mangos, where the apples, oranges and pears are loose, the bananas are in bunches of five, and you want at most four mangos.

(a) Find and simplify the generating function $g(x) = \sum_{n=0}^{\infty} h_n x^n$.

$$g(x) = (1+x+x^2+\ldots)(1+x+x^2+\ldots)(1+x^3+x^6+\ldots)(1+x+x^3+x^5+x^7)$$

$$= \frac{1}{1-x} \frac{1}{1-x} \frac{1}{1-x^3} \frac{1-x^5}{1-x}$$

$$= (1-x^{-1})^{-4}$$

(b) Give a formula for $h_n = \binom{n+3}{3}$.

12. (10 points) For the bipartite graph and matching $M$:

(a) Find an $M$-alternating chain and hence a new matching $M'$ with 5 edges.

(b) Show that it's a max matching by finding a cover $S$ with 5 vertices:

$$S = \{ x_1, x_2, x_3, y_4, y_6 \}$$
13. (12 points) Suppose that $G$ is a tree of order $n$.
(a) How many edges does $G$ have? \( n - 1 \)
(b) Prove that if $G$ has a vertex of degree 5 then it must have at least 5 pendent vertices.

\[
\text{If not then at most have degree 1 and the rest degree at least 2}
\]
\[
\text{But then}
\]
\[
2(n-1) = a_1 + \ldots + a_n \geq 5 + 1 + 1 + 1 + \frac{2n-5}{2} = 2n-1 > 2(n-1) \quad \text{ contradiction}
\]
\[
\text{of Since a tree contains no cycles paths heading out along each edge from the degree 5 vertex must each eventually terminate in a different pendent vertex.}
\]
(c) Draw the three non-isomorphic trees of degree 8 with a vertex of degree 5.

![Diagram of three non-isomorphic trees of degree 8 with a vertex of degree 5]

14. (10 points) How many ways can you put 6 non-attacking rooks on the 6-by-6 chessboard with forbidden positions shown?

\[
\Gamma_n = \text{number of ways to put } n \text{ non-attacking rooks on } n \times n \text{ forbidden squares}
\]

\[
\Gamma_1 = 9
\]
\[
\Gamma_2 = 2 + 4 + 4 \cdot 5 = 26
\]
\[
\Gamma_3 = 2 \cdot 5 + 4 \cdot 4 = 26
\]
\[
\Gamma_4 = 2 \cdot 4 = 8
\]
\[
\Gamma_5 = \Gamma_6 = 0
\]

\[
\# \text{ ways} = 6! - 9 \cdot 5! + 26 \cdot 4! - 26 \cdot 3! + 8 \cdot 2! = 124
\]

15. (6 points) Bob picks 25 integers from 51, 52, \ldots, 97, 98. Use the box principle to prove that he must have picked three whose digits sum up to the same amount (show clearly what boxes you are using).

Box according to the sum of the digits

\[
\begin{array}{cccccc}
6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}
\]

\[
25 = 2 \times 12 + 1 \quad \text{so one of the twelve boxes must receive at least 3 integers}
\]