1. (10 points) Let $a = 2^413^217$, $b = 2^35^413$. Find the following:
   (a) The prime factorization of $(a, b) =$
   (b) The prime factorization of $[a, b] =$
   (c) The value of $e$ if $2^e || (a^2 + b^3)$

2. (6 points) What orders are possible modulo 23? Find the order of 2 mod 23?

3. (7 points) Prove that $\sqrt[3]{36}$ is irrational.

4. (6 points) Give (with justification) a factor of $2^{88} + 1$.

5. (5 points) You have chosen to do RSA cryptography with modulus $n = 101 \cdot 127$ and encode exponent $e = 17$. Give (but do not solve) the congruence that you would use to find a decode exponent $d$.

6. (5 points) Find the quadratic $\alpha$ with continued fraction expansion $\alpha = [3; 3, 3, 3, ...]$. 
7. (12 points) (a) Find the prime factorization of $375 =$

(b) $\tau(n)$ is the number of positive divisors: $\tau(375) =$

(c) $\sigma(n)$ is the sum of the positive divisors: $\sigma(375) =$

(d) If $F(n) = \sum_{d|n} \tau(d)$ then $F(375) =$

8. (8 points) Use induction to prove that for all positive integers $n$

$$5^n \equiv 4n + 1 \pmod{16}.$$ 

9. (6 points) Suppose that $g(n)$ is a multiplicative function satisfying $n\tau(n) = \sum_{d|n} g(d)$.

(a) From the Möbius Inversion Formula $g(n) = \sum_{d|n} \overset{\text{}}{\text{}}$.

(b) Evaluate $g(375) =$

10. (12 points) Find all right-angled triangles with coprime integer sides and one side of length 85.
11. (12 points) (a) For which odd primes $p$ does the Legendre symbol satisfy \( \left( \frac{2}{p} \right) = 1 \)?

(b) For which distinct odd primes $p, q$ is \( \left( \frac{p}{q} \right) = - \left( \frac{q}{p} \right) \)?

(c) Evaluate the Legendre symbol \( \left( \frac{293}{331} \right) \)

12. (24 points) Circle True (T) or False (F).

- T F (a) If $15 | a^2$ then $15 | a$.
- T F (b) If $x^2 \equiv 1 \pmod{35}$ then $x \equiv \pm 1 \pmod{35}$.
- T F (c) \{21, −3, 13, −15, −4\} is a complete residue system mod 5.
- T F (d) $\text{ord}_{75}a | 20$ for all $(a, 75) = 1$.
- T F (e) $7^{53} \equiv 2 \pmod{11}$.
- T F (f) \{1, 3, −3, 9\} is a reduced residue system mod 10.
- T F (g) The Fibonacci numbers satisfy $f_{2n+3} - f_{2n+2} = f_{2n+1}$.
- T F (h) $\underbrace{72727272727272727272}_{10 \text{ times}} \equiv 6 \pmod{11}$.
- T F (i) If $p$ is an odd prime then $2^p \equiv 2 \pmod{p}$.
- T F (j) 6601 = 7 · 23 · 41 is a Carmichael number.
- T F (k) The composition $\tau(\tau(n))$ is multiplicative.
- T F (l) The simultaneous congruences $x \equiv 8 \pmod{15}$ and $x \equiv -1 \pmod{18}$ have no solution.

13. (7 points) Find the continued fraction expansion of $\sqrt{7}$. 
14. (10 points) (a) Use the Euclidean algorithm to compute the greatest common divisor (217, 161)
(b) Solve the linear equation $217x - 161y = 21$ or explain why there are no solutions.

15. (8 points) Use the Chinese Remainder Theorem to solve the simultaneous congruences
   
   \[
   x \equiv 3 \pmod{5} \\
   x \equiv 2 \pmod{7} \\
   x \equiv -1 \pmod{11}
   \]

16. (12 points) (a) Calculate the continued fraction expansion of $\frac{4169}{3864}$

(b) Calculate the continued fraction convergents

(c) A swimming contest is to be held in two pools. Contestants will swim $x$ laps of a pool of length 38.64m or $y$ laps of a pool of length 41.69m respectively. What would be sensible choices of whole numbers $x$ and $y$ so that the distance is approximately the same in both cases (assume that they want to swim less than 3km)?