MINIMAL MAHLER MEASURE IN REAL QUADRATIC FIELDS

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Abstract. We consider upper and lower bounds on the minimal height of an irrational number lying in a particular real quadratic field.

1. Introduction

For a polynomial \( F(x) = a_n \prod_{i=1}^{n} (x - \alpha_i) \) in \( \mathbb{C}[x] \) one defines its Mahler measure \( M(F) \) as
\[
M(F) = |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}.
\]

For an algebraic number \( \alpha \) we use \( M(\alpha) \) to denote the Mahler measure of an irreducible integer polynomial with root \( \alpha \). Thus the logarithmic Weil height of \( \alpha \) can be written
\[
h(\alpha) = \log M(\alpha) \frac{[\mathbb{Q}(\alpha) : \mathbb{Q}]}{[\mathbb{Q}]}.
\]

Of course \( M(\alpha) = 1 \) iff \( \alpha \) is a root of unity and the well known problem of Lehmer [3] is to determine whether there is a constant \( C > 1 \) such that \( M(\alpha) > C \) otherwise. Schinzel [4] showed that for \( \alpha \) in a Kroneckerian field (a totally real field or a quadratic extension of such a field) the value of \( M(\alpha) \) must in fact grow with its degree, with the absolute minimum \( M(\alpha) > 1 \) achieved for the golden ratio
\[
M \left( \frac{1 + \sqrt{5}}{2} \right) = \frac{1 + \sqrt{5}}{2}.
\]

Amoroso & Dvornicich [1] further extended this to cyclotomic fields. These of course include the quadratic fields \( \mathbb{Q}(\sqrt{d}) \), where \( d \) is a square-free positive integer. Since the golden ratio is not in all these fields we are interested in how
\[
L(d) := \min \left\{ M(\alpha) : \alpha \in \mathbb{Q}(\sqrt{d}) \setminus \mathbb{Q} \right\}
\]
varies with \( d \). We recall the discriminant of the field \( \mathbb{Q}(\sqrt{d}) \)
\[
D := \begin{cases} 
  d, & \text{if } d \equiv 1 \pmod{4}, \\
  4d, & \text{if } d \equiv 2 \text{ or } 3 \pmod{4}.
\end{cases}
\]

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Since $M(\alpha) = M(-\alpha) = M(\overline{\alpha})$ we assume that our $\alpha \in \mathbb{Q}(\sqrt{d}) \setminus \mathbb{Q}$ takes the form

(1) \[ \alpha = \frac{a + b\sqrt{d}}{c}, \quad a, b, c \in \mathbb{Z}, \quad a \geq 0, \ b > 0, \ c > 0, \ \gcd(a, b, c) = 1, \]

with conjugate

\[ \overline{\alpha} = \frac{a - b\sqrt{d}}{c}, \]

and

(2) \[ M(\alpha) = k \max\{1, |\alpha|\} \max\{1, |\overline{\alpha}|\} \]

where $k$ is the smallest positive integer such that

(3) \[ k(x - \alpha)(x - \overline{\alpha}) = k \left( x^2 - \frac{2a}{c}x + \frac{a^2 - b^2d}{c^2} \right) \in \mathbb{Z}[x]. \]

We show that the minimal measure must grow with $d$:

**Theorem 1.1.** For square-free $d \in \mathbb{N}$

\[ \frac{1}{2} \sqrt{D} < L(d) < \sqrt{D}. \]

2. **Proof of Theorem 1.1**

The upper bound follows at once from the following constructive examples:

**Lemma 2.1.** Suppose that $d \geq 2$ is a square-free positive integer and let $m$ be the integer in $(\sqrt{d} - 2, \sqrt{d})$ with the same parity as $d$. Then

\[ M\left( \frac{m + \sqrt{d}}{2} \right) = \begin{cases} 2, & \text{if } d = 2, \\ \frac{1}{2}(\sqrt{d} + m), & \text{if } d \equiv 1 \mod{4}, \\ \sqrt{d} + m, & \text{otherwise}. \end{cases} \]

**Proof.** Observe that $\alpha = (m + \sqrt{d})/2$ has $-1 < \overline{\alpha} < 0$, with $\alpha > 1$ for $d \geq 3$ (and $0 < \alpha < 1$ for $d = 2$). The minimal $k$ to make $k(x - \alpha)(x - \overline{\alpha}) = k \left( x^2 - mx + \frac{1}{4}(m^2 - d) \right)$ an integer polynomial is plainly $k = 1$ if $\alpha \equiv 1 \mod{4}$ and $k = 2$ for $d = 2$ or $3 \mod{4}$, and the claim is clear from (2).

For the lower bound we first observe that $c$ or $c/2$ must divide the lead coefficient $k$.

**Lemma 2.2.** Suppose that $d \geq 2$ is squarefree and $\alpha$ is of the form (1). Suppose that $k(x - \alpha)(x - \overline{\alpha})$ is in $\mathbb{Z}[x]$.

If $c$ is even and $d \equiv 1 \mod{4}$ then $c/2 | k$ with $k = c/2$ iff $a, b$ are odd with $2c | a^2 - db^2$. If $c$ is odd or $d \equiv 2$ or $3 \mod{4}$ then $c | k$ with $k = c$ iff $c | a^2 - db^2$.

**Proof.** Suppose that $p^t | c$ with $t \geq 1$.

For $k(a^2 - db^2)/c^2$ to be in $\mathbb{Z}$ we must have $p^{t+1} | k$ unless $p^t | a^2 - db^2$ and $p^t | k$ unless $p^{t+1} | a^2 - db^2$.

Hence we can assume that $p^{t+1} | a^2 - db^2$. Notice that in this case $p \nmid a$; since $p | a$ and $p^2 | a^2 - db^2$ would imply $p^2 | db^2$, but $\gcd(a, b, c) = 1$ means $p \nmid b$ and $d$ is squarefree. In particular this case can not happen when $p = 2$ and $d \equiv 2$ or $3 \mod{4}$ (since $a^2 - db^2 \not\equiv 0 \mod{4}$), and $a, b$ must be odd if $d \equiv 1 \mod{4}$. Hence $2kx/c$ in $\mathbb{Z}$ forces $p^t | k$ when $p$ is odd and $2^{t-1} | k$ when $p = 2$ and $d \equiv 1 \mod{4}$.

\[ \square \]
The following lemma completes the proof of the lower bound:

**Lemma 2.3.** Suppose that $d \geq 2$ is squarefree and $\alpha$ is of the form (1). Then

$$M(\alpha) > \frac{1}{2} \sqrt{D}.$$ 

Moreover

$$M(\alpha) > \sqrt{D}$$

unless $b = 1$ and $a < \sqrt{d}$, with $c \mid a^2 - d$ if $d \equiv 2$ or $3 \mod 4$ and with $c$ even and $2c \mid a^2 - d$ if $d \equiv 1 \mod 4$.

**Proof.** Observing that

$$\frac{2b\sqrt{d}}{c} = \alpha - \overline{\alpha} \leq \alpha + |\overline{\alpha}| < 2\alpha,$$

we have

$$M(\alpha) \geq k\alpha > k\frac{b\sqrt{d}}{c},$$

and the bound follows from $k \geq c$ if $c$ is odd or $d \equiv 2$ or $3 \mod 4$ (with $k \geq 2c$ if $c \nmid a^2 - db^2$), and $k \geq c/2$ if $c$ is even and $d \equiv 1 \mod 4$ (with $k \geq c$ if $2c \nmid a^2 - db^2$).

If $a \geq \sqrt{d}$ then $M(\alpha) \geq k\alpha > \sqrt{D}$.  

\[\square\]

3. Computations

Hence $\frac{1}{2} \sqrt{D} < L(d) < \sqrt{D}$, and an $\alpha$ of the form (1) with $\frac{1}{2} \sqrt{D} < M(\alpha) < \sqrt{D}$ must be of the form

$$\alpha = \frac{a + \sqrt{d}}{c}, \quad a < \sqrt{d},$$

with $c \mid a^2 - d$ if $d \equiv 2$ or $3 \mod 4$, and $c$ even with $2c \mid a^2 - d$ if $d \equiv 1 \mod 4$.

Since $|\overline{\alpha}| \leq \alpha$ we have $M(\alpha) = k \max\{1, \alpha, \alpha|\overline{\alpha}\}$, and in these cases we have

$$M(\alpha) = \varepsilon \max\left\{ c, a + \sqrt{d}, \frac{d - a^2}{c} \right\},$$

where

$$\varepsilon := \begin{cases} 1, & \text{if } d \equiv 2 \text{ or } 3 \mod 4, \\ \frac{1}{2}, & \text{if } d \equiv 1 \mod 4. \end{cases}$$
Figure 1. $L(d)/\sqrt{D}$ for $d \equiv 1 \mod 4$ less than five thousand.

Figure 2. $L(d)/\sqrt{D}$ for $d \equiv 2 \mod 4$ less than five thousand.
Figure 3. $L(d)/\sqrt{D}$ for $d \equiv 3 \mod 4$ less than five thousand.

Figure 4. $L(d)/\sqrt{D}$ for $d \equiv 1 \mod 4$ between five thousand and one million.
Figure 5. $L(d)/\sqrt{D}$ for $d \equiv 2 \mod 4$ between five thousand and one million.

Figure 6. $L(d)/\sqrt{D}$ for $d \equiv 3 \mod 4$ between five thousand and one million.
Figure 7. $L(d)/\sqrt{D}$ for $d$ between one billion and one billion five thousand.

Figure 8. $L(d)/\sqrt{D}$ for $d \equiv 2 \mod 4$ between one billion and one billion five thousand.
4. HOW GOOD ARE OUR BOUNDS?

Theorem 1.1 tells us that
\[ \frac{1}{2} < \frac{L(d)}{\sqrt{D}} < 1. \]

Figures 7, 8 & 9 make it seem reasonable to make the following conjecture:

**Conjecture 4.1.**
\[ \lim_{d \to \infty} \frac{L(d)}{\sqrt{D}} = \frac{1}{2}. \]

In view of (4) this can be equivalently written:

**Conjecture 4.2.** *For any square-free positive integer $d$ there exists an $a$ and $c$ with*
\[ a = o(\sqrt{d}), \quad c = (1 + o(1))\sqrt{d}, \]
*and $c \mid d - a^2$ when $d \equiv 2$ or $3 \mod 4$, $c$ even and $2c \mid d - a^2$ when $d \equiv 1 \mod 4$.*

Checking computationally, pairs $a$ and $c$ satisfying
\[ a < d^{2/5}, \quad d^{1/2} - d^{2/5} < c < d^{1/2} + d^{2/5} \]
and $c \mid d - a^2$ exist for all $827 < d < 2,000,000,000$, and even $c$ with $2c \mid d - a^2$ for all $d \equiv 1 \mod 4$ with $1,902,773 < d < 2,000,000,000$.

The $\frac{1}{2}$ in the lower bound is the optimal absolute constant.

**Theorem 4.1.**
\[ \liminf_{d \to \infty} \frac{L(d)}{\sqrt{D}} = \frac{1}{2}. \]
This follows at once from the following examples:

**Small Examples.** If \( d = m^2 + 1 \) then

\[
M\left(\frac{\sqrt{d} + 1}{m}\right) = \begin{cases} 
\sqrt{d} + 1, & \text{if } m \text{ is odd}, \\
\frac{1}{2} \left(\sqrt{d} + 1\right), & \text{if } m \text{ is even}.
\end{cases}
\]

It seems likely that the upper bound can be slightly reduced. The computations suggest that the largest value occurs at \( d = 293 \).

**Conjecture 4.3.**

\[
\sup_d \frac{L(d)}{\sqrt{D}} = \frac{L(293)}{\sqrt{293}} = \frac{M\left(\sqrt{293} + \frac{15}{2}\right)}{\sqrt{293}} = \frac{17}{\sqrt{293}} = 0.993150 \ldots.
\]

If we separate out the residue classes mod 4:

**Conjecture 4.4.**

\[
\begin{align*}
\sup_{d \equiv 2 \pmod{4}} \frac{L(d)}{\sqrt{D}} &= \frac{L(398)}{2\sqrt{398}} = \frac{M\left(\sqrt{398} + \frac{18}{2}\right)}{2\sqrt{398}} = \frac{\sqrt{398} + 18}{2\sqrt{398}} = 0.951129 \ldots, \\
\sup_{d \equiv 3 \pmod{4}} \frac{L(d)}{\sqrt{D}} &= \frac{L(227)}{2\sqrt{227}} = \frac{M\left(\sqrt{227} + \frac{13}{2}\right)}{2\sqrt{227}} = \frac{\sqrt{227} + 13}{2\sqrt{227}} = 0.962398 \ldots.
\end{align*}
\]

We found only ten values of \( d \), namely \( d = 293, 173, 227, 53, 437, 398, 83, 29, 167, 1077 \), with \( L(d)/\sqrt{D} > 0.9 \). As can be seen in the Appendix, the large values on each of the Figures 1, 2 & 3 noticeably seem to correspond to \( d \) with the property that \( d \) is a quadratic non-residues for all small primes \( p \nmid d \) (specifically all \( p < \sqrt{d} \) for \( d \equiv 2 \) or 3 mod 4 and \( p < \sqrt{d}/2 \) for \( d = 1 \) mod 4). Most of these \( d \) (with the exception of 437) are of the form \( d = \ell p \) with \( p \) prime and \( \ell \) small, and all have \( d \not\equiv 1 \pmod{8} \).

The following lemma shows why such \( d \) have large \( L(d) \) values.

**Lemma 4.1.** Suppose that \( d \) is a squarefree positive integer with \( d \equiv 2, 3 \pmod{4} \) or \( 5 \pmod{8} \), and that \( \left(\frac{d}{p}\right) = -1 \) for all primes \( p \nmid d \) with

\[
p < \begin{cases} 
\sqrt{d}, & \text{if } d \equiv 2 \text{ or } 3 \pmod{4}, \\
\sqrt{d}/2, & \text{if } d \equiv 5 \pmod{8}.
\end{cases}
\]

For each odd \( A \mid d \), with \( A < \sqrt{d} \), let \( m_A \) denote the integer in \( (\sqrt{d}/A - 2, \sqrt{d}/A) \) with the same parity as \( d \). Then, with \( \varepsilon \) as in (5),

\[
L(d) = \varepsilon \min_{A \mid d, A < \sqrt{d} \text{ odd}} \min \left\{ M\left(\frac{\sqrt{d} + m_A A}{2A}\right), M\left(\frac{\sqrt{d} + (m_A - 2)A}{2A}\right) \right\}
\]

\[
= \varepsilon \min_{A \mid d, A < \sqrt{d} \text{ odd}} \min \left\{ \sqrt{d} + m_A A, (d - (m_A - 2)^2 A^2)/2A \right\}
\]

\[
\geq \sqrt{D} - 2\varepsilon \max_{A \mid d, A < \sqrt{d} \text{ odd}} A.
\]

**Proof.** Suppose that \( d \equiv 2 \) or 3 (mod 4). Since \( \left(\frac{d}{p}\right) = -1 \) for all \( p < \sqrt{d}, p \nmid d \) we have \( d - a^2 = A_1 \) or \( 2A_1 \) or \( A_1 p \) or \( 2A_1 p \) with \( A_1 \mid d \) odd and \( p > \sqrt{d} \) prime. Hence
we can assume that $c \mid d - a^2$ is of the form $c = A_2$ or $2A_2$ or $A_2p$ or $2A_2p$ where $A_2 \mid A_1$ and $0 \leq a < \sqrt{d}$, and

$$M \left( \frac{a + \sqrt{d}}{c} \right) = \max \left\{ c, \frac{a + \sqrt{d}}{c}, \frac{d - a^2}{c} \right\}.$$ 

Hence with $A = A_1$ or $A_1/A_2$ we can assume $A \mid d$ odd, $a = kA$, and it is enough to consider

$$M \left( \frac{\sqrt{d} + kA}{A} \right) = \max \left\{ A, \frac{\sqrt{d} + kA}{A}, \frac{d - k^2A^2}{A/A} \right\}$$

with $A < \sqrt{d}$ or

$$M \left( \frac{\sqrt{d} + kA}{2A} \right) = \max \left\{ 2A, \frac{\sqrt{d} + kA}{2A}, \frac{d - k^2A^2}{2A} \right\}$$

with $2A < \sqrt{d}$ and $k$ and $d$ the same parity. Hence

(6) $$M \left( \frac{\sqrt{d} + kA}{A} \right) = \begin{cases} \sqrt{d} + kA, & \text{if } \sqrt{d}/A < k < \sqrt{d}/A, \\ (d - k^2A^2)/A, & \text{if } k < \sqrt{d}/A, \\ \geq 2\sqrt{d} - A, & \text{if } k > \sqrt{d}/A. \\ \end{cases}$$

and for $k$ and $d$ the same parity

(7) $$M \left( \frac{\sqrt{d} + kA}{2A} \right) = \begin{cases} \sqrt{d} + kA, & \text{if } \sqrt{d}/A - 2 < k < \sqrt{d}/A, \\ (d - k^2A^2)/2A, & \text{if } k < \sqrt{d}/A - 2, \\ \geq 2\sqrt{d} - 2A. & \end{cases}$$

For $k \geq m_A$ the minimum of both is plainly

$$M \left( \frac{\sqrt{d} + m_A A}{2A} \right) = \sqrt{d} + m_A A.$$ 

In (7) the smallest for $k \leq m_A - 2$ is

$$M \left( \frac{\sqrt{d} + (m_A - 2)A}{2A} \right) = (d - (m_A - 2)^2A^2)/2A,$$

and for (6) the smallest for $k \leq m_A - 1$ is

$$M \left( \frac{\sqrt{d} + (m_A - 1)A}{A} \right) = (d - (m_A - 1)^2A^2)/A.$$ 

Writing $m_A = \sqrt{d}/A - \delta$, $0 < \delta < 2$ and observing that

$$M \left( \frac{\sqrt{d} + (m_A - 1)A}{A} \right) - M \left( \frac{\sqrt{d} + (m_A - 2)A}{2A} \right) = \delta \sqrt{d} + A \left( 1 - \frac{\delta^2}{2} \right) > \delta \left( \sqrt{d} - \frac{1}{2} A \right) > 0$$

the result follows.

Similarly for $d \equiv 5 \mod 8$ we must have $d - a^2 = 2^2 A_1$ or $2^2 A_1 p$, and our even $c$ with $2c \mid d - a^2$ must be of the form $c = 2A_2$ or $2A_2p$. Thus we again reduce to

$$M \left( \frac{\sqrt{d} + kA}{2A} \right) = \frac{1}{2} \max \left\{ 2A, \sqrt{d} + kA, \frac{d - k^2A^2}{2A} \right\}.$$
with $A \mid d$ odd, $2A < \sqrt{d}$, $k$ odd, and the minimum occurs for $k = m_A$ or $m_A - 2$ as before.

Plainly $d \equiv 2$ or $3$ (mod 4) or $5$ (mod 8) with no divisors in $(\sqrt{d}, \sqrt{d})$ that are quadratic non-residues for all $p < \sqrt{d}$ would have $L(d) \geq \sqrt{D} - o(\sqrt{d})$. In particular infinitely many would immediately give

$$\limsup_{d \to \infty} \frac{L(d)}{\sqrt{D}} = 1$$

in contradiction to Conjecture 4.1, but this seems unlikely:

**Conjecture 4.5.** All but finitely many squarefree $d$ have $\left(\frac{d}{p}\right) = 1$ for some odd prime $p < \frac{1}{2}\sqrt{D}$.

Assuming GRH for the mod 4$d$ character

$$\chi(n) := \begin{cases} \left(\frac{d}{n}\right), & \text{if } \gcd(n, 4d) = 1, \\ 0, & \text{otherwise}, \end{cases}$$

(where $\left(\frac{d}{n}\right)$ denotes the Jacobi symbol), we have the bound

$$(8) \quad \left| \sum_{n \leq x} \chi(n)\Lambda(n) \right| \ll x^{\frac{1}{4}} \log^2(Dx)$$

(see, for example, [2, Chapter 20]), and so we should in fact have $\left(\frac{d}{p}\right) = 1$ for some prime $p \ll \log^4 D$.

Note, a squarefree $d \not\equiv 1$ mod 8 with $\left(\frac{d}{p}\right) = -1$ for all odd primes $p \nmid d$ with $p < \frac{1}{2}\sqrt{D}$ must be of the form $d = (kA)^2 \pm 2A$ or $(2k - 1)A^2 \pm 4A$, for some $k$ and squarefree odd $A \mid d$ with $A < \frac{1}{2}\sqrt{D}$. To see this, write $d = N^2 + r$, $N = \lfloor \sqrt{d} \rfloor$, $1 \leq r \leq 2N$. If $r$ is even then $A = r/2$ is odd if $d \equiv 2, 3$ mod 4 and $A = r/4$ is odd if $d \equiv 5$ mod 8. Since $d$ is a square mod $A$ we must have $p \mid A \Rightarrow p \mid d$. As $d$ is squarefree, $A < \frac{1}{2}\sqrt{D}$ must be squarefree with $A \mid d$, giving $d = N^2 + 2A$ or $N^2 + 4A$ with $A \mid N$. Similarly for $r$ odd

$$d = \left(\frac{r + 1}{2}\right)^2 + \left(\frac{N - 1}{2}(r - 1)\right)\left(\frac{N + 1}{2}(r - 1)\right),$$

with $A = N - \frac{1}{2}(r - 1)$ odd for $d = 2$ or 3 mod 4 and $A = \frac{1}{2} (N - \frac{1}{2}(r - 1))$ odd for $d \equiv 5$ mod 8. Since $d$ is a square mod $A$, $A < \frac{1}{2}\sqrt{D}$ is squarefree with $A \mid d$, giving $d = (N + 1)^2 - 2A$ or $(N + 1)^2 - 4A$ with $A \mid N + 1$.

Conversely if $d$ is a quadratic residue mod $p$ for a suitably sized $p$ or if $d \equiv 1$ mod 8 then we can obtain a bound less than one for $L(d)/\sqrt{D}$:

**Lemma 4.2.** Suppose that $d$ is a square mod $q$, where $q$ is odd or 4 \mid q and $\lambda$ defined by

$$\lambda \sqrt{d} = \begin{cases} q, & \text{if } q \text{ is odd}, \\ \frac{1}{4}q, & \text{if } q \text{ is even}, \end{cases}$$
has $0 < \lambda < 1$. Then

$$\frac{L(d)}{\sqrt{D}} \leq \begin{cases} \frac{1}{2} (1 + \lambda + \sqrt{(1 - \lambda)^2 - 4\lambda^2}), & \text{if } 0 < \lambda < \frac{1}{4} (\sqrt{5} - 1), \\ \frac{1}{2\lambda}, & \text{if } \frac{1}{4} (\sqrt{5} - 1) < \lambda < \frac{1}{2} (\sqrt{3} - 1), \\ \frac{1}{2} (1 + \lambda), & \text{if } \frac{1}{2} (\sqrt{3} - 1) < \lambda < 1. \end{cases}$$

Notice that if we assume GRH then estimate (8), with

$$\sum_{x-y \leq n \leq x} \Lambda(n) = y + O(x^{\frac{1}{2}} \log^2 x)$$

from assuming RH, guarantees that \( \left( \frac{d}{p} \right) = 1 \) for some prime \( p \) in

$$\left( \frac{1}{2} (\sqrt{3} - 1)d^2, \frac{1}{2} (\sqrt{3} - 1)d^2 + cd^2 \log^2 d \right)$$

for suitably large \( c \), and Lemma 4.2 gives

$$\frac{L(d)}{\sqrt{D}} \leq \frac{1}{4} (\sqrt{3} + 1) + O \left( \frac{\log^2 d}{d^4} \right) = 0.683012 \ldots + o(1).$$

**Proof.** Suppose that \( r_0 \) has \( r_0^2 \equiv d \mod q \). If \( q \) is odd we take \( r \) to be the integer in \( (\sqrt{d} - 2q, \sqrt{d}) \) with the same parity as \( d \) and \( r \equiv r_0 \mod q \), write \( r = \sqrt{d} - \delta q \), and set

$$\alpha_1 = \frac{\sqrt{d} + r}{2q}, \quad \alpha_2 = \frac{\sqrt{d} + r - 2q}{2q}.$$

Notice that \( c = 2q \) and \( a = r \) or \( r - 2q \) will have \( c \mid (d - a^2) \) with \( 2c \mid d - a^2 \) when \( d \equiv 1 \mod 4 \).

For \( 4 \mid q \) (which of course only occurs when \( d \equiv 1 \mod 4 \)) we write \( q = 2^l q_1 \) with \( q_1 \) odd and \( l \geq 2 \) and take \( r \) to be the integer in \( (\sqrt{d} - 2^{l-1}q_1, \sqrt{d}) \) with \( r \equiv r_0 \mod 2^{l-1}q_1 \) and set

$$\alpha_1 = \frac{\sqrt{d} + r}{2^{l-1}q_1}, \quad \alpha_2 = \frac{\sqrt{d} + r - 2^{l-1}q_1}{2^{l-1}q_1}.$$

Again \( c = 2^{l-1}q_1 \) and \( a = r \) or \( r - 2^{l-1}q_1 \) will have \( 2c \mid (d - a^2) \).

Writing \( r = \sqrt{d} - \delta q \) for \( q \) odd, and \( r = \sqrt{d} - 2^{l-2}q_1 \) for \( q \) even, we have \( r = (1 - \lambda \delta) \sqrt{d} \) with \( 0 < \delta < 2 \) and

$$\alpha_1 = \frac{(2 - \delta \lambda)}{2\lambda}, \quad \overline{\alpha}_1 = -\frac{\delta}{2}, \quad \alpha_2 = \frac{(2 - 2\lambda - \delta \lambda)}{2\lambda}, \quad \overline{\alpha}_2 = -\frac{\delta}{2} - 1.$$

For \( \alpha_1 \) and \( \alpha_2 \) we also plainly have \( k = ec = 2c\lambda \sqrt{d} = \lambda \sqrt{D} \).

Clearly \( \alpha_1 > 0, \alpha_2 < -1, -1 < \overline{\alpha}_1 < 0 \) and \( -2 < \overline{\alpha}_2 < -1 \).

If \( \alpha_1 < 1 \) then \( M(\alpha_1) = \lambda \sqrt{D} \overline{\alpha}_1 = \frac{1}{2} (1 + \lambda) \sqrt{D} \). Hence we can assume that \( \alpha_1 > 1 \) (this is automatic for \( \lambda < \frac{1}{2} \)).

So

$$M(\alpha_1) = \lambda \sqrt{D} \alpha_1 = \sqrt{D} \left( 1 - \frac{\delta}{2}\lambda \right).$$

If \( \alpha_2 < 1 \) then

$$M(\alpha_2) = \lambda \sqrt{D} |\overline{\alpha}_2| = \sqrt{D} \lambda \left( 1 + \frac{\delta}{2} \right),$$

and plainly

$$\min\{M(\alpha_1), M(\alpha_2)\} \leq \frac{1}{2} (M(\alpha_1) + M(\alpha_2)) = \frac{1}{2} (1 + \lambda) \sqrt{D}. $$
So we can assume that $\alpha_2 > 1$ and

$$M(\alpha_2) = \lambda \sqrt{D |\alpha_2|} \alpha_2 = \sqrt{D} \left( 1 + \frac{\delta}{2} \right) \left( 1 - \lambda - \frac{\delta}{2} \lambda \right).$$

Observing that the quadratic is maximized for $\frac{\delta}{2} = \frac{1}{2\lambda} - 1$ we plainly have

$$M(\alpha_2) \leq \sqrt{D} \frac{1}{4\lambda}$$

with this less than $\frac{1}{2}(1 + \lambda) \sqrt{D}$ for $\frac{1}{2}(\sqrt{3} - 1) < \lambda < 1$. For $\lambda < \frac{1}{4}(\sqrt{5} - 1)$ the value $\frac{\delta}{2} = \frac{1}{2\lambda} \left( 1 - \lambda - \sqrt{(1 - \lambda)^2 - 4\lambda^2} \right)$ equating $M(\alpha_1)$ and $M(\alpha_2)$ is less than $\frac{1}{2\lambda} - 1$ and the minimum of the two is at most the value at that point:

$$\min\{M(\alpha_1), M(\alpha_2)\} \leq \sqrt{D} \left( \frac{1}{2} (1 + \lambda) + \frac{1}{2} \sqrt{(1 - \lambda)^2 - 4\lambda^2} \right).$$

\[\square\]

In particular from the lemma we immediately obtain a bound away from 1 for the $d \equiv 1 \pmod{8}$.

**Corollary 4.1.** If $d \equiv 1 \pmod{8}$ then

$$L(d) \sqrt{D} \leq \frac{1}{4}(\sqrt{5} + 1) = 0.809016 \ldots.$$

If $d \equiv 1 \pmod{3}$ then

$$L(d) \sqrt{D} \leq \frac{1}{7}(2 + 3\sqrt{2}) = 0.891805 \ldots.$$

Computations indicate room for improvement in these bounds.

**Conjecture 4.6.**

\[
\sup_{d \equiv 1 \pmod{8}} \frac{L(d)}{\sqrt{D}} = \frac{L(41)}{\sqrt{41}} = M(\frac{\sqrt{41} + 27}{4}) = \frac{\sqrt{41} + 3}{2\sqrt{41}} = 0.734261 \ldots,
\]

\[
\sup_{d \equiv 1 \pmod{3}} \frac{L(d)}{\sqrt{D}} = \frac{L(13)}{\sqrt{13}} = M(\frac{\sqrt{13} + 1}{2}) = \frac{4}{\sqrt{13}} = 0.832050 \ldots.
\]

**Proof.** If $d \equiv 1 \pmod{8}$ then we can solve $r^2 \equiv d \pmod{2^l}$ for any $l$. Hence if we pick $l$ such that $\frac{1}{2}(\sqrt{d} - 1) \sqrt{\lambda} \leq 2^{l-2} \leq \frac{1}{2}(\sqrt{d} - 1)$ and we can apply the lemma with $\frac{1}{2}(\sqrt{d} - 1) \leq \lambda \leq \frac{1}{2}(\sqrt{d} - 1)$. Likewise, for an odd prime $p$, if $p \nmid d$ and $\left( \frac{d}{p} \right) = 1$ then we can solve $r^2 \equiv d \pmod{p^l}$ for any $l$. Choosing $l$ so that

$$\frac{1}{1 + \sqrt{(p-1)^2 + 4}} \sqrt{\lambda} \leq p^l \leq \frac{p}{1 + \sqrt{(p-1)^2 + 4}} \sqrt{\lambda},$$

and applying the lemma with $q = p^l$ gives

$$L(d) \sqrt{D} \leq \frac{1}{2} \left( 1 + \frac{p}{1 + \sqrt{(p-1)^2 + 4}} \right).$$

Taking $p = 3$ gives the result claimed for $d \equiv 1 \pmod{3}$.

Likewise, for $d \equiv 1, 4 \pmod{5}$ we get the upper bound $0.956859 \ldots$ (from $d = 29$ we know $0.928476 \ldots$ would be best possible).
For $d \equiv 1, 2, 4 \mod 7$ we get $0.977844\ldots$ and for $d \equiv 1, 3, 4, 5, 9 \mod 11$ the bound $0.991157\ldots$ (from $d = 53$ these can not be reduced below $0.961523\ldots$).

For $d \equiv 1, 3, 4, 9, 10, 12 \mod 13$ we get the bound $0.993713\ldots$ (the optimal bound is likely $0.988371\ldots$ from $d = 173$).

For $d \equiv 1, 2, 4, 8, 9, 13, 15, 16 \mod 17$ our bound gives $0.996364\ldots$ (optimal is probably $0.993150\ldots$ at $d = 293$). □

5. Appendix of Large Values

We give the largest values found in Figure 1, Figure 2 & Figure 3 down to the first value not satisfying the quadratic non-residue conditions of Lemma 4.1.

**Largest values for $d \equiv 1 \mod 4$.**

\[
\begin{align*}
L(293) &= \frac{M\left(\sqrt{293^2+15}\right)}{\sqrt{293}} = \frac{17}{\sqrt{293}} = 0.993150\ldots, \quad \left(\frac{293}{p}\right) = -1, p = 3, 5, 7, 11, 13, \\
L(173) &= \frac{M\left(\sqrt{173^2+11}\right)}{\sqrt{173}} = \frac{13}{\sqrt{173}} = 0.988371\ldots, \quad \left(\frac{173}{p}\right) = -1, p = 3, 5, 7, 11, \\
L(53) &= \frac{M\left(\sqrt{53^2+5}\right)}{\sqrt{53}} = \frac{7}{\sqrt{53}} = 0.961523\ldots, \quad \left(\frac{53}{p}\right) = -1, p = 3, 5, \\
L(437) &= \frac{M\left(\sqrt{437^2+19}\right)}{\sqrt{437}} = \frac{\sqrt{437 + 19}}{2\sqrt{437}} = 0.954446\ldots, \quad \left(\frac{437}{p}\right) = -1, p = 3, 5, 7, 11, 13, 17, 29, \\
L(29) &= \frac{M\left(\sqrt{29^2+3}\right)}{\sqrt{29}} = \frac{5}{\sqrt{29}} = 0.928476\ldots, \quad \left(\frac{29}{p}\right) = -1, p = 3, \\
L(1077) &= \frac{M\left(\sqrt{1077^2+27}\right)}{\sqrt{1077}} = \frac{\sqrt{1077^2 + 27}}{2\sqrt{1077}} = 0.911363\ldots, \quad \left(\frac{1077}{p}\right) = -1, p = 5, 7, 11, 13, 17, 19, 23, \\
L(77) &= \frac{M\left(\sqrt{77^2+7}\right)}{\sqrt{77}} = \frac{\sqrt{77^2 + 7}}{\sqrt{77}} = 0.898862\ldots, \quad \left(\frac{77}{p}\right) = -1, p = 3, 5, \\
L(453) &= \frac{M\left(\sqrt{453^2+15}\right)}{\sqrt{453}} = \frac{19}{\sqrt{453}} = 0.892697\ldots, \quad \left(\frac{453}{p}\right) = -1, p = 5, 7, 11, 13, 17, \\
L(717) &= \frac{M\left(\sqrt{717^2+21}\right)}{\sqrt{717}} = \frac{\sqrt{717^2 + 21}}{\sqrt{717}} = 0.892129\ldots, \quad \left(\frac{717}{p}\right) = -1, p = 5, 7, 11, 13, 17, 19, \\
L(3053) &= \frac{M\left(\sqrt{3053^2+41}\right)}{\sqrt{3053}} = \frac{49}{\sqrt{3053}} = 0.886814\ldots, \quad \left(\frac{3053}{7}\right) = 1.
\end{align*}
\]

Note other $\alpha$ may achieve the minimum, for example $M\left(\frac{\sqrt{437^2+19}}{2}\right) = M\left(\frac{\sqrt{437^2+19}}{2}\right)$. 

Largest values for $d \equiv 2 \mod 4$.

\[
\begin{align*}
L(398) &= \frac{M\left(\frac{\sqrt{398} + 18}{2}\right)}{2\sqrt{398}} = \frac{\sqrt{398} + 18}{2\sqrt{398}} = 0.951129\ldots, \quad \left(\frac{398}{p}\right) = -1, \quad p = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \\
L(38) &= \frac{M\left(\frac{\sqrt{38} + 4}{2}\right)}{2\sqrt{38}} = \frac{11}{2\sqrt{38}} = 0.892217\ldots, \quad \left(\frac{38}{p}\right) = -1, \quad p = 3, 5, \\
L(62) &= \frac{M\left(\frac{\sqrt{62} + 6}{2}\right)}{2\sqrt{62}} = \frac{\sqrt{62} + 6}{2\sqrt{62}} = 0.881000\ldots, \quad \left(\frac{62}{p}\right) = -1, \quad p = 3, 5, 7, 11, \\
L(318) &= \frac{M\left(\frac{\sqrt{318} + 12}{6}\right)}{2\sqrt{318}} = \frac{\sqrt{318} + 12}{2\sqrt{318}} = 0.836463\ldots, \quad \left(\frac{318}{p}\right) = -1, \quad p = 5, 7, 11, 13, 17, 19, 23, \\
L(14) &= \frac{M\left(\frac{\sqrt{14} + 2}{2}\right)}{2\sqrt{14}} = \frac{\sqrt{14} + 2}{2\sqrt{14}} = 0.767261\ldots, \quad \left(\frac{14}{p}\right) = -1, \quad p = 3, \\
L(138) &= \frac{M\left(\frac{\sqrt{138} + 6}{6}\right)}{2\sqrt{138}} = \frac{\sqrt{138} + 6}{2\sqrt{138}} = 0.755376\ldots, \quad \left(\frac{138}{p}\right) = -1, \quad p = 5, 7, 11, 13, \\
L(22) &= \frac{M\left(\frac{\sqrt{22} + 2}{3}\right)}{2\sqrt{22}} = \frac{\sqrt{22} + 2}{2\sqrt{22}} = 0.713200\ldots, \quad \left(\frac{22}{3}\right) = 1.
\end{align*}
\]
Largest values for \( d \equiv 3 \mod 4 \).

\[
\frac{L(227)}{2\sqrt{227}} = \frac{M\left(\frac{\sqrt{227}+13}{2}\right)}{2\sqrt{227}} = \frac{29}{2\sqrt{227}} = 0.962398 \ldots, \quad \left(\frac{227}{p}\right) = -1, \quad p = 3, 5, 7, 11, 13, 17, 19, 23,
\]

\[
\frac{L(83)}{2\sqrt{83}} = \frac{M\left(\frac{\sqrt{83}+7}{2}\right)}{2\sqrt{83}} = \frac{17}{2\sqrt{83}} = 0.932966 \ldots, \quad \left(\frac{83}{p}\right) = -1, \quad p = 3, 5, 7, 11, 13,
\]

\[
\frac{L(167)}{2\sqrt{167}} = \frac{M\left(\frac{\sqrt{167}+11}{2}\right)}{2\sqrt{167}} = \frac{167}{2\sqrt{167}} = 0.925602 \ldots, \quad \left(\frac{167}{p}\right) = -1, \quad p = 3, 5, 7, 11, 13, 17, 19,
\]

\[
\frac{L(447)}{2\sqrt{447}} = \frac{M\left(\frac{\sqrt{447}+15}{6}\right)}{2\sqrt{447}} = \frac{41}{2\sqrt{447}} = 0.813517 \ldots, \quad \left(\frac{447}{p}\right) = -1, \quad p = 3, 7, 11, 13, 17, 19, 23, 29, 31, 37,
\]

\[
\frac{L(23)}{2\sqrt{23}} = \frac{M\left(\frac{\sqrt{23}+3}{2}\right)}{2\sqrt{23}} = \frac{23}{2\sqrt{23}} = 0.812771 \ldots, \quad \left(\frac{23}{p}\right) = -1, \quad p = 3, 5,
\]

\[
\frac{L(3)}{2\sqrt{3}} = \frac{M\left(\frac{\sqrt{3}+1}{2}\right)}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} = 0.788675 \ldots, \quad \left(\frac{3}{p}\right) = -1, \quad p = 3, 7, 11, 13, 17, 19, 23, 29, 31, 37,
\]

\[
\frac{L(827)}{2\sqrt{827}} = \frac{M\left(\frac{\sqrt{827}+15}{14}\right)}{2\sqrt{827}} = \frac{827}{2\sqrt{827}} = 0.760800 \ldots, \quad \left(\frac{827}{7}\right) = 1.
\]

References