THE INTEGER GROUP DETERMINANTS FOR THE
SYMMETRIC GROUP OF DEGREE FOUR

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Abstract. For the symmetric group $S_4$ we determine all the integer values
taken by its group determinant when the matrix entries are integers.

1. Introduction

Suppose that $G = \{g_1, \ldots , g_n\}$ is a finite group. Assigning variables $x_g$, $g \in G$, one defines the group determinant, $D_G(x_{g_1}, \ldots , x_{g_n})$, to be the determinant of the
$n \times n$ group matrix whose $ij$th entry is $x_{g_i}^{-1}$. Plainly $D_G(x_{g_1}, \ldots , x_{g_n})$ will be a homogeneous polynomial of degree $n$ in the $x_g$.

For $G = \mathbb{Z}_n$, the cyclic group of order $n$, this is a circulant determinant, where
the next row in the group matrix is obtained from the previous row by a cyclic shift
of one to the right. An old problem of Olga Taussky-Todd is to determine which
integers can be achieved as an $n \times n$ circulant determinant when the entries in the
matrix are integers. We can of course ask the same question for an arbitrary finite
group $G$ and define $\mathcal{S}(G)$ to be this set of integers:

$\mathcal{S}(G) := \{D_G(x_{g_1}, \ldots , x_{g_n}) : x_{g_1}, \ldots , x_{g_n} \in \mathbb{Z}\}$.

Notice that $\mathcal{S}(G)$ is closed under multiplication

$D_G(a_{g_1}, \ldots , a_{g_n})D_G(b_{g_1}, \ldots , b_{g_n}) = D_G(c_{g_1}, \ldots , c_{g_n}), \quad c_g := \sum_{u, v \in G} a_ub_v,$

corresponding to multiplication $\left(\sum_{g \in G} a_g g\right) \left(\sum_{g \in G} b_g g\right) = \sum_{g \in G} c_g g$ in the group
ring. While it has been proved [6] that the group determinant polynomial deter-
mines the group it is not known whether the set of integer values $\mathcal{S}(G)$ determines
the group.

For $G = \mathbb{Z}_n$, Newman [12] and Laquer[10] gave divisibility restrictions on the
elements in $\mathcal{S}(G)$ and some attainable values. For example, $n^2$ and any $m$ with
gcd$(m, n) = 1$ will be in $\mathcal{S}(\mathbb{Z}_n)$, and if $m$ is in $\mathcal{S}(\mathbb{Z}_n)$ then so is $-m$ and if $p \mid m$ and $p^\alpha \parallel n$ then $p^{\alpha+1} \mid m$. Here and throughout $p$ denotes a prime. Newman and
Laquer both obtained a complete description for $G = \mathbb{Z}_p$ when $p \geq 2$, and Laquer
for $G = \mathbb{Z}_{2p}$ when $p \geq 3$:

$$S(\mathbb{Z}_p) = \{p^am : \gcd(m, p) = 1, \ a = 0 \text{ or } a \geq 2\},$$

$$S(\mathbb{Z}_{2p}) = \{2^ap^bm : \gcd(m, 2p) = 1, \ a = 0 \text{ or } a \geq 2, \ b = 0 \text{ or } b \geq 2\},$$

cases where the divisibility conditions are necessary and sufficient. Newman [13] similarly showed that

$$S(\mathbb{Z}_3) = \{3^am : \gcd(m, 3) = 1, \ a = 0 \text{ or } a \geq 3\},$$

but that $p^3$ is not in $S(\mathbb{Z}_{p^2})$ for any $p \geq 5$ (in particular the basic divisibility conditions are not sufficient). Currently no $S(\mathbb{Z}_{p^2})$ with $p \geq 5$ has been fully determined, though there are upper and lower set inclusions.

As shown by Vipismakul [18] there is a very close relationship between the group determinant for a finite abelian group and Lind’s generalization [11] of the Mahler measure for that group; in particular the corresponding Lind-Lehmer problem determines, though there are upper and lower set inclusions.

Similarly showed that

$$S(\mathbb{Z}_9) = \{3^am : \gcd(m, 3) = 1, \ a = 0 \text{ or } a \geq 3\},$$

but that $p^3$ is not in $S(\mathbb{Z}_{p^2})$ for any $p \geq 5$ (in particular the basic divisibility conditions are not sufficient). Currently no $S(\mathbb{Z}_{p^2})$ with $p \geq 5$ has been fully determined, though there are upper and lower set inclusions.

As shown by Vipismakul [18] there is a very close relationship between the group determinant for a finite abelian group and Lind’s generalization [11] of the Mahler measure for that group; in particular the corresponding Lind-Lehmer problem of finding the minimal positive logarithmic Lind-Mahler measure for the group, corresponds to finding the smallest non-trivial group determinant

$$\lambda(G) := \min\{|m| : m \in S(G), \ |m| \geq 2\}.$$  

Kaiblinger [9] used the Lind measure approach to find

$$S(\mathbb{Z}_4) = \{2m + 1, \ 2^4m : m \in \mathbb{Z}\},$$

and $S(\mathbb{Z}_8)$, and obtain upper and lower inclusions for other $S(\mathbb{Z}_{2^k})$. The Lind-Lehmer constant $\lambda(G)$ is known for a number of groups, see for example [8, 14, 5, 15], including the cyclic groups $G = \mathbb{Z}_n$ with 892371480 $\mid n$. Writing $D_{2n}$ for the dihedral group of order $2n$, the sets $S(D_{2p}), S(D_{4p})$,

$$S(D_4) = \{4m + 1, \ 2^4m : m \in \mathbb{Z}\},$$

and $S(D_{16})$ were determined in [1], and $\lambda(D_{2n})$ found for $n < 1.89 \times 10^{47}$. In particular the cases $p = 2$ and 3 give us the Klein 4-group

$$S(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{4m + 1, \ 2^4(2m + 1), \ 2^6m : m \in \mathbb{Z}\},$$

and the symmetric group of degree 3

$$S(S_3) = \{2a^3b^m : \gcd(m, 6) = 1, \ a = 0 \text{ or } a \geq 2, \ b = 0 \text{ or } b \geq 3\}.$$  

In [16] a complete description of the integer group determinants was obtained for the remaining groups of order at most 14, including the alternating group $A_4$:

$$S(A_4) = S(A_4)_{odd} \cup S(A_4)_{even},$$

with

$$S(A_4)_{odd} = \{m \equiv 1 \text{ mod } 4 : 3 \nmid m \text{ or } 3^2 \mid m\},$$

$$S(A_4)_{even} = \{2^a3^b m : \gcd(m, 6) = 1, \ a = 4 \text{ or } a \geq 8, \ b = 0 \text{ or } b \geq 2\}.$$  

Although $S(S_3)$ and $S(A_4)$ were obtained without too much difficulty, other small groups in [16], for example $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, were considerably more complicated; making it clear that obtaining a description for general $S(G)$ is probably not feasible. Indeed, even in the case of circulant determinants, we are yet to obtain a complete description of $S(\mathbb{Z}_{15})$ and $S(\mathbb{Z}_{16})$. Our goal here is to show that $G = S_4$ is one of those rare cases where we can completely determine $S(G)$:
Theorem 1.1. For $G = S_n$ the odd group determinants in $S(G)$ are the integers $m \equiv 1 \mod 4$ with the property that $3 \nmid m$ or $3^3 \mid m$.

The even determinants are the integers of the form

$2^8m$, $m \equiv 1 \mod 4$, or $2^{16}m$, $m \equiv -1 \mod 4$, or $2^{12}m$,

where $m$ is an integer with $3 \nmid m$ or $3^3 \mid m$.

Comparing $S(S_4)$ to $S(G)$ for $G = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, S_3, A_4$ it is tempting to ask whether in general $S(G) \subseteq S(H)$ whenever $H \leq G$.

2. Factoring the group determinant for $S_4$

Dedekind [3] observed that for a finite abelian group the group characters $\hat{G}$ could be used to factor the group determinant into linear factors

$\varrho_G(x_{g_1}, \ldots, x_{g_n}) = \prod_{\chi \in \hat{G}} (\chi(g_1)x_{g_1} + \cdots + \chi(g_n)x_{g_n}).$ (1)

For non-abelian groups the group determinant will contain non-linear factors and, as shown by Frobenius [7], the counterpart to (1) is to use the set $\hat{G}$ of irreducible group representations for $G$,

$\varrho_G(x_{g_1}, \ldots, x_{g_n}) = \prod_{\rho \in \hat{G}} \det(x_{g_1}\rho(g_1) + \cdots + x_{g_n}\rho(g_n))^{\deg(\rho)},$

see Conrad [2] for a historical survey.

The irreducible representations for $S_4$ are discussed in Serre [17, §5.8]. One can also use GAP or a similar computer algebra system to generate them (although we have reduced the four generators used there to two and also reindexed).

We take as our two generators for $S_4$

$\alpha = (1234), \beta = (12),$

and order the even permutations, with the coefficients $a_1, \ldots, a_{12}$, by

$\{1, (13)(24), (14)(23), (12)(34), (134), (243), (142), (123), (143), (132), (124), (234)\}$

$= \{1, \alpha^2, \beta \alpha^2 \beta, \alpha \beta \alpha^2 \beta, \alpha \beta \alpha^3 \beta, \alpha \beta \alpha^2, \alpha^2 \beta \alpha, \alpha^2 \beta \alpha^3, \beta \alpha \}.$

and the odd permutations, with coefficients $b_1, \ldots, b_{12}$, by

$\{(1234), (1432), (24), (13), (14), (23), (1243), (1342), (12), (34), (1324), (1423)\}$

$= \{\alpha, \alpha^3, \alpha \beta \alpha^2 \beta, \alpha \beta \alpha^3 \beta, \beta \alpha \beta \alpha^2, \alpha^2 \beta \alpha, \beta, \alpha \beta \alpha, \alpha^3 \beta \alpha \}.$

For $S_4$ we have two linear representations, $\chi_0(x) = 1$ and $\chi_1(x) = \text{sgn}(x)$, giving two linear factors

$\ell_1 := (a_1 + \cdots + a_{12}) + (b_1 + \cdots + b_{12}),$

$\ell_2 := (a_1 + \cdots + a_{12}) - (b_1 + \cdots + b_{12}).$

We have one degree two representation with

$\rho_1(\beta) = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \quad \rho_1(\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \omega := e^{2\pi i/3},$
giving

\[ \rho_1(1) = \rho_1(\alpha^2) = \rho_1(\beta \alpha^2 \beta) = \rho_1(\alpha^2 \beta \alpha^2 \beta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ \rho_1(\alpha \beta) = \rho_1(\alpha^3 \beta) = \rho_1(\alpha \beta \alpha^2) = \rho_1(\alpha^3 \beta \alpha^2) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \]

\[ \rho_1(\beta \alpha^3) = \rho_1(\alpha^2 \beta \alpha) = \rho_1(\alpha^2 \beta \alpha^3) = \rho_1(\beta \alpha) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \]

\[ \rho_1(\alpha) = \rho_1(\alpha^3) = \rho_1(\alpha \beta \alpha^2 \beta) = \rho_1(\alpha^2 \beta \alpha^2 \beta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

\[ \rho_1(\alpha^3 \beta \alpha) = \rho_1(\alpha \beta \alpha^3) = \rho_1(\alpha \beta \alpha) = \rho_1(\alpha^3 \beta \alpha^3) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \]

\[ \rho_1(\beta) = \rho_1(\alpha^2 \beta \alpha^2) = \rho_1(\beta \alpha^2) = \rho_1(\alpha^2 \beta) = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \]

and, writing

\[ u_1 = a_1 + a_2 + a_3 + a_4, \quad u_2 = a_5 + a_6 + a_7 + a_8, \quad u_3 = a_9 + a_{10} + a_{11} + a_{12}, \]

\[ v_1 = b_1 + b_2 + b_3 + b_4, \quad v_2 = b_5 + b_6 + b_7 + b_8, \quad v_3 = b_9 + b_{10} + b_{11} + b_{12}, \]

the quadratic factor

\[ q_1 := \det \begin{pmatrix} u_1 + u_2 \omega + u_3 \omega^2 & v_1 + v_2 \omega + v_3 \omega^2 \\ v_1 + v_2 \omega^2 + v_3 \omega & u_1 + u_2 \omega^2 + u_3 \omega \end{pmatrix} = N(u_1 + u_2 \omega + u_3 \omega^2) - N(v_1 + v_2 \omega + v_3 \omega^2), \]

where \( N(x) = |x|^2 \).

Finally we have two degree three representations

\[ \rho_2(\alpha) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho_2(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \]

and \( \rho_3(x) = \text{sgn}(x) \rho_2(x) \). Here \( \rho_2 \) comes from the natural representation of \( S_4 \) in \( \mathbb{C}^3 \). That is, we take the 3-dimensional subspace

\[ V = \{ x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 : x_1 + x_2 + x_3 + x_4 = 0 \} \]

of a 4-dimensional vector space and let \( S_4 \) permute its basis vectors \( e_1, e_2, e_3, e_4 \), although to produce our \( \rho_2 \) we take a less obvious basis for \( V \):

\[ e_1 + e_2 - e_3 - e_4, \quad -e_1 + e_2 + e_3 - e_4, \quad e_1 - e_2 + e_3 - e_4. \]
For the even permutations we have
\[
\rho_2(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_2((13)(24)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_2((14)(23)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]
\[
\rho_2((12)(34)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho_2((134)) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \rho_2((243)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]
\[
\rho_2((142)) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_2((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_2((143)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},
\]
\[
\rho_2((132)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho_2((124)) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \rho_2((234)) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},
\]
and for the odd permutations
\[
\rho_2((1234)) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho_2((1432)) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho_2((24)) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
\[
\rho_2((13)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_2((14)) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \rho_2((23)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]
\[
\rho_2((1243)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \rho_2((1342)) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho_2((12)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]
\[
\rho_2((34)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \rho_2((1324)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \rho_2((1423)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Thus we have two cubic factors
\[
d_1 := \det (A + B), \quad d_2 := \det (A - B),
\]
where
\[
A = \begin{pmatrix} a_1 - a_2 - a_3 + a_4 & a_9 + a_{10} - a_{11} - a_{12} & -a_5 + a_6 - a_7 + a_8 \\ a_5 - a_6 - a_7 + a_8 & a_1 - a_2 + a_3 - a_4 & -a_9 + a_{10} + a_{11} - a_{12} \\ -a_9 + a_{10} - a_{11} + a_{12} & -a_5 - a_6 + a_7 + a_8 & a_1 + a_2 - a_3 - a_4 \end{pmatrix}
\]
and
\[
B = \begin{pmatrix} b_9 + b_{10} - b_{11} - b_{12} & -b_1 + b_2 - b_3 + b_4 & -b_5 + b_6 + b_7 - b_8 \\ b_1 - b_2 - b_3 + b_4 & b_5 + b_6 - b_7 + b_8 & b_9 - b_{10} + b_{11} - b_{12} \\ -b_5 + b_6 - b_7 + b_8 & b_9 - b_{10} - b_{11} + b_{12} & -b_1 - b_2 + b_3 + b_4 \end{pmatrix}.
\]

So for \( G = S_4 \) the group determinant takes the form
\[
\rho_G \left( a_1, \ldots, a_{12}, b_1, \ldots, b_{12} \right) = \ell_1 \ell_2 q_1^2 d_1^3 d_2^3.
\]
We can think of the determinant as the unormalized Lind measure of the ‘polynomial’ (really an element in the group ring \( \mathbb{Z}[S_4] \))

\[
\begin{align*}
a_1 + a_2 x^2 + a_3 y x^2 y + a_4 x^2 y^2 x + a_5 y x + a_6 x y + a_7 x y x^2 + a_8 x^3 y x^2 \\
+ a_9 y x^3 + a_{10} y x^2 y + a_{11} x y x^3 + a_{12} y x + b_1 x + b_2 x^3 + b_3 y x^2 y + b_4 y x^2 y x \\
+ b_5 x^3 y x + b_6 x y x^3 + b_7 y x x + b_8 x^3 y x^3 + b_9 y + b_{10} x^2 y x^2 + b_{11} y x^2 + b_{12} x^2 y,
\end{align*}
\]

where monomials do not commute but we can reduce a polynomial in \( \mathbb{Z}[x,y] \) to this form using the group relations \( y^2 = 1, x^4 = 1, y x y = x^3 y x, y x^3 y = x y x, y x^2 y x^2 = x^2 y x^2 y, x^3 y x^2 y x = x y x^2 y \) etc.

### 3. Proof of Theorem 1.1

**Proof.** We first show that we can achieve the stated values.

We begin with the values coprime to 3 that are 1 mod 4.

Taking \( a_1 = 1 + k \) and the remaining values equal to \( k \) gives

\[
\mathcal{H}_G = (1 + 24k) \cdot 1 \cdot (1 - 0)^2 \cdot |I_3|^3 \cdot |I_5|^3 = 1 + 24k.
\]

Taking \( a_2 = a_5 = a_9 = 1 + k, b_3 = b_5 = 1 + k \) and the remaining values \( k \) we get

\[
\mathcal{H}_G = (5 + 24k) \cdot 1 \cdot (0 - 1)^2 \cdot \begin{vmatrix}
-1 & 0 & -2 & 3 \\
0 & 0 & -1 & 2 \\
-2 & -1 & 2 & 0 \\
0 & 0 & -1 & 0
\end{vmatrix} = 5 + 24k.
\]

With \( a_1 = a_3 = 1 + k, a_5 = a_6 = a_7 = 1 + k, a_9 = a_{10} = a_{11} = 1 + k, b_1 = b_3 = b_4 = 1 + k, b_5 = b_6 = 1 + k, b_{11} = b_{12} = 1 + k \) and the other values \( k \) we obtain

\[
\mathcal{H}_G = (13 + 24k) \cdot 1 \cdot (1 - 0)^2 \cdot \begin{vmatrix}
-2 & 0 & -1 & 3 \\
-1 & 4 & 0 & 2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0
\end{vmatrix} = 13 + 24k.
\]

From \( a_1 = a_2 = a_3 = 1 + k, a_6 = a_7 = a_8 = 1 + k, a_9 = a_{10} = a_{11} = 1 + k \) and \( b_1 = b_3 = b_4 = 1 + k, b_5 = b_7 = b_8 = 1 + k, b_9 = b_{10} = 1 + k \), the others \( k \), we get

\[
\mathcal{H}_G = (17 + 24k) \cdot 1 \cdot (0 - 1)^2 \cdot \begin{vmatrix}
1 & 0 & 0 & 3 \\
0 & 0 & 1 & 2 \\
0 & 1 & 1 & 0
\end{vmatrix} = 17 + 24k.
\]

Next we get the powers of three, \( \pm 3^j \) with \( j \geq 3 \), that are 1 mod 4.

With \( a_1 = a_3 = 1 + k, b_3 = 1 + k \) and the others \( k \) we have

\[
\mathcal{H}_G = (3 + 24k) \cdot 1 \cdot (4 - 1)^2 \cdot \begin{vmatrix}
0 & -1 & 0 & 3 \\
-1 & 2 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1
\end{vmatrix} = -3^3(1 + 8k),
\]

and with \( a_1 = a_2 = 1 + k \) and \( a_5 = 1 + k \) and the other values \( k \)

\[
\mathcal{H}_G = (3 + 24k) \cdot 3 \cdot (3 - 0)^2 \cdot \begin{vmatrix}
0 & 0 & -1 & 6 \\
1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0
\end{vmatrix} = 3^4(1 + 8k).
\]

Finally we deal with the powers of two.
Taking $a_1, a_5 = 1$ and the others zero gives

$$
\mathcal{D}_G = 2 \cdot 2 \cdot (1 - 0)^2 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}^6 = 2^8,
$$

while $a_1 = -1, a_5 = a_6 = 1$ and $b_5 = 1, b_{10} = -1$ with the others zero has

$$
\mathcal{D}_G = 1 \cdot 1 \cdot (7 - 3)^2 \cdot \begin{vmatrix} -2 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{vmatrix}^3 = 2^{10},
$$

and $a_2 = a_5 = 1, a_9 = 1$ and $b_{11} = 1$ the rest zero

$$
\mathcal{D}_G = 4 \cdot 2 \cdot (0 - 1)^2 \cdot \begin{vmatrix} -2 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & -2 & 1 \end{vmatrix}^3 = 2^{12},
$$

and $a_1 = a_2 = 1, a_5 = -1, a_9 = -1, b_1 = 1$ and the rest zero

$$
\mathcal{D}_G = 1 \cdot -1 \cdot (9 - 1)^2 \cdot \begin{vmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}^3 = -2^{12}.
$$

Taking $a_1 = a_2 = 1 + k, a_6 = 1 + k, a_{10} = a_{11} = 1 + k, b_4 = 1 + k, b_6 = 1 + k, b_{10} = 1 + k$ and the rest $k$

$$
\mathcal{D}_G = (8 + 24k) \cdot 2 \cdot (1 - 0)^2 \cdot \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}^3 = 2^{13}(1 + 3k),
$$

and $a_2 = a_3 = a_4 = 1 + k, a_5 = 1 + k, a_9 = 1 + k, b_4 = 1 + k, b_5 = b_6 = 1 + k$, and the rest $k$

$$
\mathcal{D}_G = (8 + 24k) \cdot 2 \cdot (4 - 3)^2 \cdot \begin{vmatrix} -1 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 0 \end{vmatrix}^3 = -2^{13}(1 + 3k),
$$

giving the remaining powers of two. Products of these achieve the stated values.

It remains to show that the determinants can only take the values claimed. Plainly we have the congruences

$$
(2) \quad \ell_1 \equiv \ell_2 \mod 2, \quad d_1 \equiv d_2 \mod 2, \quad q_1 \equiv \ell_1 \ell_2 \mod 3,
$$

and, as can be checked on Maple,

$$
(3) \quad d_1 d_2 \equiv \ell_1 \ell_2 q_1^2 \mod 4
$$

and

$$
(4) \quad d_1 + d_2 \equiv (\ell_1 + \ell_2)q_1 \mod 4.
$$

Suppose that $3 \mid \mathcal{D}_G = \ell_1 \ell_2 q_1^2 (d_1 d_2)^3$. If $3 \mid d_1 d_2$ then $3^3 \mid (d_1 d_2)^3$. If $3 \mid \ell_1 \ell_2$ or $3 \mid q_1$ then $3$ divides both by (2) and $3^3 \mid \ell_1 \ell_2 q_2^3$. Hence $3 \mid \mathcal{D}_G$ or $3^3 \mid \mathcal{D}_G$.

If $\mathcal{D}_G$ is odd then by (3) we have $\mathcal{D}_G \equiv (\ell_1 \ell_2 q_1^2)^3 \equiv 1 \mod 4$.

Suppose that $\mathcal{D}_G$ is even. From (3) we have $2 \mid d_1 d_2$ and $2 \mid \ell_1 \ell_2 q_1$. Hence by (2) we have $2 \mid d_1$ and $2 \mid d_2$, and either $2 \mid q_1$ or $2 \mid \ell_1$ and $2 \mid \ell_2$. Hence $2^2 \mid (\ell_1 \ell_2) q_1^2$, $2^6 \mid (d_1 d_2)^3$ and $2^8 \mid \mathcal{D}_G$. It remains to rule out $2^9 \parallel \mathcal{D}_G$ and $2^{11} \parallel \mathcal{D}_G$, and show that the $2^8 \parallel \mathcal{D}_G$ or $2^{10} \parallel \mathcal{D}_G$ are of the stated forms.
Similarly, replacing the the $d_1$ gives
\begin{equation}
  d_1 = \ell_1 (q_1 + 2uv + 2w) + 4C(\tilde{a}, \tilde{b})
\end{equation}
where $C(\tilde{a}, \tilde{b})$ is a homogeneous cubic integer polynomial in the $a_1, \ldots, a_{12}, b_1, \ldots, b_{12}$,
\begin{align*}
  u &:= u_1 + u_2 + u_3, & v &:= v_1 + v_2 + v_3,
\end{align*}
and
\begin{align*}
  w &:= u_1 B_1 + u_2 B_2 + u_3 B_3 + v_1 A_1 + v_2 A_2 + v_3 A_3,
\end{align*}
where
\begin{align*}
  A_1 &= a_1 + a_2 + a_5 + a_8 + a_9 + a_{10}, & B_1 &= b_1 + b_2 + b_5 + b_8 + b_{11} + b_{12},
  A_2 &= a_1 + a_3 + a_6 + a_8 + a_{10} + a_{12}, & B_2 &= b_2 + b_3 + b_5 + b_8 + b_{10} + b_{11},
  A_3 &= a_1 + a_4 + a_7 + a_8 + a_{10} + a_{11}, & B_3 &= b_1 + b_3 + b_5 + b_7 + b_{10} + b_{12}.
\end{align*}
Similarly, replacing the the $b_i$ by $-b_i$
\begin{equation}
  d_2 = \ell_2 (q_1 - 2uv - 2w) + 4C(\tilde{a}, -\tilde{b})
\end{equation}
and we get
\begin{align}
  d_1 + d_2 &\equiv 2uq_1 + 4uv^2 + 4vw \pmod 8, \\
  d_1 - d_2 &\equiv 2vq_1 + 4u^2v + 4uw \pmod 8.
\end{align}

Suppose that $2^2 \mid \ell_1$ or $2 \mid \ell_1$ and $2 \mid q_1$, then from (5) we have $d_1 \equiv 0 \pmod 4$ and and, since $2 \parallel \ell_2, d_2$, we get $2^{12} \mid D_G$. Similarly if $2^2 \mid \ell_2$ using $d_2$. Hence if $2^{12} \mid D_G$ we can assume that $2 \mid \ell_1 \ell_2$ and $2 \mid q_1$ or $2 \parallel \ell_1, \ell_2$ and $2 \mid q_1$. If $2 \parallel \ell_1, \ell_2$ then $4 \mid (\ell_1 + \ell_2)$ and if $2 \parallel \ell_1 \ell_2$ then $2 \mid q_1$ and $2 \parallel (\ell_1 + \ell_2)$. In either case (4) gives $d_1 \equiv d_2 \pmod 4$, and if $2^2 \mid d_1$ or $d_2$ then $2^2$ divides both and $2^{14} \mid D_G$. Hence we have $2 \parallel d_1, d_2$ and so an even number of $2$'s divide $(d_1d_2)^2$, $q_1^2$ and $\ell_1\ell_2$, ruling out $2^9$ or $2^{11} \parallel D_G$.

Suppose that $2^8$ or $2^{10} \parallel D_G$. So $d_1 = 2d_1$, $d_2 = 2\delta_2$ with $\delta_1, \delta_2$ odd, and for $2^{10} \parallel D_G$ we must have $2 \parallel \ell_1 \ell_2$ and $2 \parallel q_1$, and for $2^8 \parallel D_G$ we have either $2 \parallel \ell_1, \ell_2$ and $2 \parallel q_1$, or $2 \parallel \ell_1, \ell_2$ and $2 \parallel q_1$. Suppose that $2 \parallel \ell_1, \ell_2$. If $\ell_1\ell_2 = u^2 - v^2 \equiv 1 \pmod 4$ then $2 \parallel u$, $2 \parallel v$ and from (6) we have $\delta_1 + \delta_2 \equiv uq_1 \pmod 4$. For $D_G = 2^{10}m$ we have $q_1 = 2^2\rho$ with $\rho$ odd, and $\delta_1 \equiv -\delta_2 \pmod 4$ and $m = \ell_1\ell_2\rho^2(\delta_1\delta_2)^3 \equiv 1 \cdot 1 \cdot (-1)^3 \equiv -1 \pmod 4$. For $D_G = 2^8m$ we have $q_1 = 2^2\rho$ with $\rho$ odd and $\delta_1 + \delta_2 \equiv 2 \pmod 4$ and $\delta_1 \equiv \delta_2 \pmod 4$ and $m = \ell_1\ell_2\rho^2(\delta_1\delta_2)^3 \equiv 1 \cdot 1 \cdot 1^3 \equiv 1 \pmod 4$. Similarly when $\ell_1\ell_2 \equiv -1 \pmod 4$ we have $2 \parallel u$, $2 \parallel v$, and use (7) to get $\delta_1 \equiv \delta_2 \pmod 4$ for $2^{10}$ and $\delta_1 \equiv -\delta_2 \mod 4$ for $2^8$, and $m \equiv -1$ and $1 \pmod 4$ respectively as before.

This leaves the case $D_G = 2^8m$ where $\ell_1 = 2\xi_1$, $\ell_2 = 2\xi_2$ with $\xi_1, \xi_2$ odd. If $\xi_1\xi_2 \equiv 1 \pmod 4$ then $4 \parallel v = (\xi_1 - \xi_2)$ and $2 \parallel u = (\xi_1 + \xi_2)$ and from (7) we get $\delta_1 \equiv \delta_2 \pmod 4$ and $m = \xi_1\xi_2q_1^2(\delta_1\delta_2)^3 \equiv 1 \pmod 4$. Likewise if $\xi_1\xi_2 \equiv -1 \pmod 4$ then $4 \parallel u$ and $2 \parallel v$ and from (6) we have $\delta_1 \equiv -\delta_2$ and again $m \equiv 1 \pmod 4$. □

References


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