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  \item [(SG1)] $\varphi_0$ is the identity, i.e. $\varphi_0(z) = z$, $z \in \mathbb{D}$;
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Let \( \mathbb{D} \) denote the unit disk \( \{z : |z| < 1\} \) and \( H(\mathbb{D}) \) the set of analytic functions on \( \mathbb{D} \). A one-parameter semigroup \( \{\varphi_t\}_{t \geq 0} \) of analytic functions on \( \mathbb{D} \) is a family of analytic functions \( \varphi_t : \mathbb{D} \to \mathbb{D} \) that satisfies the following three conditions:

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The trivial case is that $\varphi_t(z) = z$ for all $t \geq 0$. Otherwise, we say that $\{\varphi_t\}$ is nontrivial.
Examples of semigroups of analytic functions

\[ \phi_t(z) = e^{-ct}z, \text{ where } \Re c \geq 0. \]

Rotation and Shrinking, common fixed point at 0:

\[ \phi_t(z) = 1 + e^{-t}(z - 1). \]

Shrinking disks all tangent to unit circle at 1, common fixed point 1:

An unlimited variety of such examples is easily constructed:
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\[ \begin{align*}
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\end{array}
\end{align*} \]

Shrinking disks all tangent to unit circle at 1, common fixed point 1:

An unlimited variety of such examples is easily constructed:
$\Omega$ spiral-like:

$\exists c, R, c > 0$ s.t.

$\{w = c t; t \geq 0\} \subset \Omega$

for each $w \in \Omega$. 
\[ \exists c \in C \text{ s.t.} \]
\[ \{ w + ct : t > 0 \} \subset \Omega \]
for each \( \omega \)
Every nontrivial semigroup of analytic functions $\{\varphi_t\}_{t \geq 0}$ has a unique common fixed point $b$ with $|\varphi'_t(b)| \leq 1$ for all $t \geq 0$, called the Denjoy-Wolff point of the semigroup. Under a normalization, the Denjoy-Wolff point $b$ may be assumed to be 0 or 1. If $b = 0$, then $\varphi_t(z) = h^{-1}(e^{-ct}h(z))$, where $h$ is a univalent function from $D$ onto a spirallike domain $\Omega$, $h(0) = 0$, $\Re c \geq 0$, and $w - ct \in \Omega$ for each $w \in \Omega$, $t \geq 0$. If $b = 1$, then $\varphi_t(z) = h^{-1}(h(z) + ct)$, where $h : D \to \Omega$ is a Riemann map, $\Omega$ is close-to-convex, $h(0) = 0$, $\Re c \geq 0$, and $w + ct \in \Omega$ for each $w \in \Omega$, $t \geq 0$. 
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If \( b = 0 \), then

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where \( h \) is a univalent function from \( \mathbb{D} \) onto a spirallike domain \( \Omega \), \( h(0) = 0 \), \( \Re c \geq 0 \), and \( we^{-ct} \in \Omega \) for each \( w \in \Omega, t \geq 0 \).
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Composition semigroups

Associated with the semigroup $\{\varphi_t\}$ is the composition semigroup of linear operators $\{C_t\}$, where $C_t(f) = f \circ \varphi_t$ for $f \in H(D)$.

If $C_t$ is a bounded operator on some Banach space $X \subset H(D)$ for all $t \geq 0$, we say that the semigroup $\{\varphi_t\}$ acts on $X$.

If in addition the strong continuity condition $\lim_{t \to 0^+} \|f \circ \varphi_t - f\|_X = 0$ holds for all $f \in X$, then it is said that $\{\varphi_t\}$ is strongly continuous on $X$.
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Denote by $\{\varphi_t, X\}$ the maximal closed subspace of $X$ on which $\{C_t\}$ is strongly continuous.

Theorem (O. Blasco, M. Contreras, S. Díaz-Madrigal, J. Martínez, M. Papadimitrakis, and A. Siskakis)

Let $\{\varphi_t\}_{t \geq 0}$ be a semigroup with generator $G$ and $X$ a Banach space of analytic functions which contains the constant functions and such that $\sup_{0 \leq t \leq 1} \|C_t\| < \infty$. Then $\{\varphi_t, X\} = \{f \in X : Gf' \in X\}$.

Here $G(z) = \lim_{t \to 0^+} \varphi_t(z) - z$ is the infinitesimal generator of $\{\varphi_t\}_{t \geq 0}$. This convergence holds uniformly on compact subsets on $D$ so $G \in H(D)$. $G$ has a representation $G(z) = (bz - 1)(z - b)P(z)$, $z \in D$, where $b$ is the Denjoy-Wolff point of $\{\varphi_t\}_{t \geq 0}$, $P \in H(D)$ with $\Re P(z) \geq 0$ for all $z \in D$. 

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Composition Semigroups on BMOA and $H^\infty$
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Some known results:

(i) If $X \in \{H^p \ (1 \leq p < \infty), A^p \ (1 \leq p < \infty), \mathcal{D}, B_0, \text{VMOA}\}$ and $\{\varphi_t\}$ is any semigroup, then $[\varphi_t, X] = X$;
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The space $\text{BMOA}$ does not have the Dunford-Pettis property, and the corresponding statement had remained open.

Theorem (Anderson, Jovovic, S)

Suppose $H^\infty \subseteq X \subseteq B$. Then $[\varphi_t, X] \subseteq X$. In particular, $[\varphi_t, \text{BMOA}] \subseteq \text{BMOA}$.
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Theorem (Anderson, Jovovic, S)

Suppose $H^\infty \subseteq X \subseteq B$. Then $[\varphi_t, X] \not\subseteq X$. In particular, $[\varphi_t, BMOA] \not\subseteq BMOA.$
The theorem is an easy consequence of Proposition. Given any nontrivial semigroup \( \{ \phi_t \} \), there exists \( f \in H^\infty \) such that
\[
\lim \inf_{t \to 0} \| f \circ \phi_t - f \|_{B} \geq 1.
\]

Proof of theorem:
Each test function \( f \) in the proposition is in \( H^\infty \), and hence in \( X \) from the hypothesis that \( H^\infty \subseteq X \).
Since \( X \subseteq B \), the Closed Graph Theorem shows that \( \| \cdot \|_{B} \lesssim \| \cdot \|_{X} \) and bounding the Bloch norm away from 0 bounds the \( X \) norm as well. Thus it follows from the proposition that \( f \notin [\phi_t, X] \), and so \( [\phi_t, X] \not\subseteq X \).
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Let \( \{ \phi_t \} \) be a nontrivial semigroup, and consider the case that the corresponding Denjoy-Wolff point is 0. Then \( \phi_t(z) = h^{-1}(e^{-ct}h(z)) \), where \( h : D \to \Omega \), \( h(0) = 0 \), \( \text{Re} c \geq 0 \), and \( \Omega \) is spiral-like. Consider the case that \( \text{Re} c > 0 \). Choose \( w_0 \in \partial \Omega \) such that \( |w_0| = \inf \{|w| : w \in \partial \Omega\} \). Then \([0, w_0) \subset \Omega\), so \( w_0 \) is the principal point of an accessible prime end, and hence there is \( \gamma_0 \in \partial D \) such that \( \lim_{r \to 1^-} h(r \gamma_0) = w_0 \). Thus, \( \lim_{r \to 1^-} \phi_t(r \gamma_0) = h^{-1}(e^{-ct}w_0) \in D \), \( t > 0 \).
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Thus, \( \lim_{r \to 1^-} \varphi_t(r \gamma_0) = h^{-1}(e^{-ct} w_0) \in \mathbb{D}, \ t > 0. \)
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Then $\varphi_t(z) = h^{-1}(e^{-ct} h(z))$, where $h : \mathbb{D} \to \Omega$, $h(0) = 0$, $\text{Re} c \geq 0$, and $\Omega$ is spiral-like. Consider the case that $\text{Re} c > 0$. Choose $w_0 \in \partial \Omega$ such that $|w_0| = \inf\{|w| : w \in \partial \Omega\}$.
Then $[0, w_0) \subset \Omega$, so $w_0$ is the principal point of an accessible prime end, and hence there is $\gamma_0 \in \partial \mathbb{D}$ such that
$$\lim_{r \to 1^-} h(r \gamma_0) = w_0.$$ Thus, $\lim_{r \to 1^-} \varphi_t(r \gamma_0) = h^{-1}(e^{-ct} w_0) \in \mathbb{D}, \quad t > 0.$
Since $\varphi_t$ is univalent and bounded, $\varphi_t \in D \subset B_0$. 

Hence \[ \lim_{r \to 1^-} |\varphi_t'(r \gamma_0)| (1 - r) = 0. \]

Let $f$ be an infinite interpolating Blaschke product with zeros all on the radius $\{r \gamma_0 : 0 < r < 1\}$.

Then \[ \limsup_{r \to 1^-} |f'(r \gamma_0)| (1 - r) \geq \delta, \]
for some $\delta > 0$.

Also, by continuity of $f'$ on $D$,
\[ \lim_{r \to 1^-} |f'(\varphi_t(r \gamma_0))| (1 - r) = |f'(h - 1)(e^{-ct}w_0)| < \infty. \]

Thus, for all fixed $t > 0$,
\[ \|f \circ \varphi_t - f\|_{B_0} \geq \limsup_{r \to 1^-} |f'(\varphi_t(r \gamma_0))\varphi_t'(r \gamma_0) - f'(r \gamma_0)| (1 - r) \geq \delta. \]

Replacing $f$ by $f/\delta$ gives \[ \|f \circ \varphi_t - f\|_{B_0} \geq 1. \]
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Composition Semigroups on BMOA and $H^\infty$
Since $\varphi_t$ is univalent and bounded, $\varphi_t \in \mathcal{D} \subset \mathcal{B}_0$.

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\[ \geq \delta. \]

Replacing $f$ by $f/\delta$ gives $\|f \circ \varphi_t - f\|_B \geq 1$. 
Uniform convergence of \( \{ \varphi_t \} \)

From (SG1) and (SG3) we have the pointwise convergence \( \varphi_t(z) \to z \) as \( t \to 0^+ \). This is easily be extended to uniform convergence on compact subsets of \( D \).

It was recently observed by P. Gumenyuk that this extends to uniform convergence on all of \( D \) for every semigroup \( \{ \varphi_t \} \).

Theorem (Gumenyuk; Anderson, Jovovic, S)

For every semigroup \( \{ \varphi_t \} \),

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\lim_{t \to 0^+} \| \varphi_t(z) - z \|_{H^\infty} = 0
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For every semigroup $\{\varphi_t\}$,

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Consequences

Let $X$ be a Banach space that contains $H^\infty$, and let $X_P$ be the closure of the polynomials in $X$. For all semigroups $\{\phi_t\}$, $X_P \subset [\phi_t, X]$.

This provides a unified proof of some of the known results mentioned above: $[\phi_t, H_p] = H_p$ and $[\phi_t, A_p] = A_p$, all $\{\phi_t\}$.

$VMOA \subseteq [\phi_t, BMOA]$ and $B_0 \subseteq [\phi_t, B]$, all $\{\phi_t\}$.

And hence also that $VMOA = [\phi_t, VMOA]$ and $B_0 = [\phi_t, B_0]$, all $\{\phi_t\}$.

It also establishes that the natural extension to $H^\infty$ is valid: The disk algebra $A$ satisfies $A \subset [\phi_t, H^\infty]$, all $\{\phi_t\}$.
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It also establishes that the natural extension to $H^\infty$ is valid:

The disk algebra $A$ satisfies $A \subset [\varphi_t, H^\infty]$, all $\{\varphi_t\}$.
proof of uniform convergence

This time we consider the case that the Denjoy-Wolff point of \( \{ \phi_t \} \) is the point 1 on the unit circle. Then there is \( c \in \mathbb{C} \) with \( \Re c \geq 0 \) and univalent \( h : D \rightarrow \Omega \), where \( \Omega \) close-to-convex, such that \( \phi_t(z) = h^{-1}(h(z) + ct) \).

If \( c = 0 \), the result is trivial. So assume \( c \neq 0 \).

Suppose \( \phi_t(z) \) does not converge uniformly to \( z \) in \( D \).

Then there exist some \( \delta > 0 \) and infinite sequences \( \{ t_n \} \), \( t_n \to 0^+ \) and \( \{ z_n \} \subset D \) such that \( \delta \leq |\phi_{t_n}(z_n) - z_n| \), \( n \geq 1 \).

Letting \( w_n = h(z_n) \in \Omega \),

\[
|\phi_{t_n}(z_n) - z_n| = |h^{-1}(h(z_n) + ct_n) - h^{-1}(h(z_n))| = |h^{-1}(w_n + ct_n) - h^{-1}(w_n)|.
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The points $w_n$ and $w_n + ct_n$ are endpoints of a line segment in $\Omega$ which pulls back to the Jordan arc $\eta_n = \{h^{-1}(w_n + ct_n) : 0 \leq t \leq t_n\} \subset D$.

Since $t_n \to 0$ and $\Omega$ is compact in the Riemann sphere, we may pass to a subsequence of $\{w_n\}$ and assume the line segment $[w_n, w_n + ct_n] = h(\eta_n) \to w_0 \in \Omega \cup \{\infty\}$.

However, $\text{diam} \eta_n \geq |h^{-1}(w_n + ct_n) - h^{-1}(w_n)| = |\phi_{t_n}(z_n) - z_n| \geq \delta$, contradicting the fact that univalent functions do not have Koebe arcs.

Therefore, $|\phi_{t_n}(z_n) - z_n| \to 0$ uniformly in $D$ as $t \to 0^+$. 

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Therefore, $|\varphi_t(z) - z| \to 0$ uniformly in $\mathbb{D}$ as $t \to 0^+$. 

Mahalo!