1. \((a) \Rightarrow (b)\) Since \(G\) is solvable we start with a sequence \(1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_s = G\) where \(G_{i+1}/G_i\) is abelian. Now suppose that for some \(i\), \(G_{i+1}/G_i\) is not cyclic. Fix some \(\overline{y} \in G_{i+1}/G_i\) which does not generate the whole quotient group (which exists since not cyclic). Then by the 4th Isom Theorem, the subgroup generated by \(\overline{y}\) lifts to a subgroup of \(G_{i+1}\), call it \(H_i\), which is a proper subgroup of \(G_{i+1}\) containing \(G_i\). Again by the 4th Isom Theorem \(H_i \trianglelefteq G_{i+1}\) and \(G_i \trianglelefteq H_i\) (since in the quotient group \(G_i\) corresponds to the identity element which is normal in any other subgroup). Also \(H_i \cong H_i/G_i\) again by 4th Isom. So \(H_i\) is abelian since every subgroup of this quotient group is abelian. And \(G_{i+1}/H_i\) is the quotient of an abelian group so abelian. We have successfully inserted a subgroup between \(G_i\) and \(G_{i+1}\) so now relabel the sequence: \(1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_s \trianglelefteq G_{s+1} = G\). Then repeat this argument for each \(G_{i+1}/G_i\) which is not cyclic. Since \(G\) is finite this will eventually end with cyclic factors.

\((b) \Rightarrow (c)\) This is similar to the previous proof. Start with a chain where the factors are cyclic and work in the quotient to find a subgroup there that can be lifted by the 4th Isom Theorem to a subgroup properly between \(G_{i+1}\) and \(G_i\). Relabel and continue until you have prime order factors.

\((c) \Rightarrow (d)\) Fix \(M\) a minimal nontrivial normal subgroup of \(G\). Assume the hint for a moment. Then \(1 \leq M \leq G\) satisfies \(M/1\) is abelian. But \(G/M\) might not be. Find the minimal nontrivial normal subgroup of \(G/M\). Lift it to a subgroup of \(G\) by 4th Isom Theorem, call it \(N_1\) then we have \(1 \leq M \leq N_1 \leq G\). Repeat for \(G/N_1\) and continue until \(G/N_i\) is prime which will happen eventually since \(G\) is a finite group.

To find the \(N\) in their hint, start with a composition series where each factor is prime:

\[1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_k = G.\]

We intersect this with \(M\) to get

\[1 = N_0 \cap M \trianglelefteq N_1 \cap M \leq N_2 \cap M \leq \cdots \leq N_k \cap M = M.\]

Suppose \(M \cap N_i\) is the first time (reading from the right) where the intersection is not all of \(M\). Prop III.6 (and proof of 3rd Isom) tells us that \(|M : M \cap N_i| = |MN_i : N_i|\) but the right side is less than or equal to \(|N_{i+1} : N_i|\) which is prime by (c). So let \(N\) in hint be \(M \cap N_i\).

Since \(M/N\) is prime, it is cyclic so abelian. For any \(x, y \in M\), \(xN\) and \(yN\) commute in the quotient which means \(xyN = yxN\) and so \(x^{-1}y^{-1}xy \in N\) by Prop III.3. To show \(x^{-1}y^{-1}xy \in gNg^{-1}\), we see that \(g^{-1}x^{-1}y^{-1}xyg \in N\) since \(g^{-1}x^{-1}y^{-1}xyg = (g^{-1}x^{-1}y^{-1}g)(g^{-1}y^{-1}g)(g^{-1}xg)(y^{-1}yg) = x^{-1}y^{-1}x_1y_1 \in N\). But the intersection of all \(G\) conjugates must be normal in \(G\) contradicting \(M\)’s minimality.

\((d) \Rightarrow (a)\). This follows immediately since (d) is a stronger statement than (a).
2. Any element of order 3 cannot be an element of a subgroup of order 4 (Lagrange for instance). This rules out 8 elements of $A_4$ (the three cycles) and leaves only the identity, $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, and $(1\ 4)(2\ 3)$. This is our unique subgroup, we will call it $H$. Verify that it is a subgroup and of order 4. To see it is isomorphic to $V_4 = \{a, b, ab, 1_{V_4}\}$ map any one of these order 2 elements to $a$, any other to $b$ and the third (which will be the product of the other two) to $ab$.

To show $H$ is normal we must show that for any $g \in A_4$ and $h \in H$, $ghg^{-1} \in H$. By Proposition 5 we know that if we conjugate any element of $S_n$ by another element, we will end up with something of the same cycle type. (This proof follows through for $A_n$ as well). So in $S_4$ the three nonidentity elements of type 2, 2 are all conjugates of each other and conjugating anything else by them still gives us an element of type 2, 2. Again the same holds true in $A_4$ and so for any $g \in G$, since the elements of $H$ are of 2, 2 type and all elements of this type are in $H$, $ghg^{-1} \in H$ for all $h$ and so $H$ is normal.

Be careful with this sort of argument, though. Some statements about $S_n$ and conjugation do not follow through to $A_n$. For example, see our proof that $A_5$ is simple.

3. First suppose $g_1$ in $G_b$. By definition this means $g_1 \cdot b = b$. We show that $g^{-1}g_1g \cdot a = a$ which will mean $g^1G_bg \in G_a$. Now by one of the axioms of group action $g^{-1}g_1g \cdot a = g^{-1}g_1 \cdot (g \cdot a) = g^{-1}g_1 \cdot b = g^{-1} \cdot b$ since $g_1 \in G_b$. And finally $g^{-1} \cdot b = a$ since $g \cdot a = b$.

Conversely, let $g_2 \in G_a$. We show $gg_2g^{-1} \cdot b = b$. Again one of the axioms of group action gives us $gg_2 \cdot (g^{-1} \cdot b) = gg_2 \cdot a = g \cdot a = b$.

4. The left regular representation of $S_3$ is the permutation representation we get from $S_3 \to S_{|S_3|} = S_6$ from the action of left multiplication of $S_3$ on itself. The element $1_{S_3}$ acts trivially on every element of $S_3$ so its image under the representation is $1_{S_6}$. Now consider the element $(1\ 2)$ acting on each element of $S_3$ (so for $\sigma \in S_3$ we want to compute $(1\ 2) \circ \sigma$. The induced map on $S_3$ sends $1_{S_3}$ to $1_{S_3}$ and $(1\ 2)$ to $1_{S_3}$. It sends $(2\ 3)$ to $(1\ 2) \circ (2\ 3) = (1\ 2\ 3)$ and $(1\ 3)$ to $(1\ 3)$ as $(1\ 3) = (1\ 3\ 2)$. Finally it sends $(1\ 2\ 3)$ to $(1\ 3\ 2)$ and $(1\ 3\ 2)$ to $(1\ 3)$. Converting these into permutations using the numbering scheme of the problem gives us $(1\ 2) \to (1\ 2)(3\ 5)(4\ 6)$, $(2\ 3) \to (1\ 3)(2\ 6)(4\ 5)$, $(1\ 3) \to (1\ 4)(2\ 5)(3\ 6)$, $(1\ 2\ 3) \to (1\ 5\ 6)(2\ 4\ 3)$, and $(1\ 3\ 2) \to (1\ 6\ 5)(2\ 3\ 4)$.

5. Let $G$ act on the set of cosets of $G/H$ by sending $(g, g_1H)$ to $gg_1H$. This action induces a permutation representation from $G$ to $S_{G/H} = S_n$. The kernel of this map is a subgroup of $G$ and is normal (Prop III.5). Also by the 1st Isomorphism Theorem, the $G/\ker G$ is isomorphic to some subgroup of $S_n$. $|S_n| = n!$ so $|G : \ker G| \leq n!$.

6. (a) Let $Z_2 = \langle x \rangle$ and so elements of $Z_2 \times S_3$ are denoted $(a, b)$ with $a \in Z_2$ and $b \in S_3$.

There are six conjugacy classes: $\{(1_{Z_2}, 1_{S_3})\}$, $\{(x, 1_{S_3})\}$, $\{(1_{Z_2}, (1\ 2))\}$, $\{(1_{Z_2}, (1\ 3))\}$, $\{(1_{Z_2}, (2\ 3))\}$,
{(x, (1 2)), (x, (1 3)), (x, (2 3))}, {(1_{Z_2}, (1 2 3)), (1_{Z_2}, (1 3 2))}, {(x, (1 2 3)), (x, (1 3 2))}.

(b) Unlike in $S_n$, in $A_n$ all elements of the same cycle type may not be in the same conjugacy class. There are four conjugacy classes in $A_4$: $\{1_{A_4}\}$, $\{(1 2)(3 4), (1 3)(2 4),(1 4)(2 3)\}$, $\{(1 2 3), (1 4 2), (1 3 4), (2 3 4)\}$, $\{(1 3 2), (1 2 4), (1 4 3), (2 4 3)\}$.

7. For any group $G$, the identity element will be in its own conjugacy class since $g1_g = 1_g$ for any $g \in G$. Proposition 2 tells us that the size of a conjugacy class is $|G : C_G(s)|$ for some $s$ in the conjugacy class. This is the order of some subgroup of $G$ and so by Lagrange must divide the size of the group. If there were only two conjugacy classes, all other elements of the group would have to be in the other conjugacy class, meaning the size of this other conjugacy class would be $|G| - 1$. The only way this could also divide the size of $G$ is if $|G| = 2$.

8. Any element of an odd order group $G$ cannot have order 2 (Lagrange) and so any nonidentity $x \in G$ must have an inverse $x^{-1} \neq x$. Suppose $x$ and its inverse were in the same conjugacy class, call it $O_x$. Then for any $y \in O_x$ we claim $y^{-1}$ is in there. If $y = gxg^{-1}$ then $y^{-1} = (gxg^{-1})^{-1} = (g^{-1}x^{-1}g)$ and so $y^{-1}$ is in the conjugacy class. This implies there are an even number of elements in the conjugacy class which contradicts the fact that the size of any conjugacy class of $G$ must divide the order of $G$ (which is an odd number) by Prop IV.2.

9. Inner automorphisms are a subgroup of the automorphism group of a group $G$. We also know that subgroups of cyclic groups are all cyclic from Theorem II.2. If $\text{Aut}(G)$ is cyclic, so is $\text{Inn}(G)$. But we know from Corollary 5 that $\text{Inn}(G) \cong G/Z(G)$ which means $G/Z(G)$ is cyclic in this case. But homework 3 problem 5 tells us that if this quotient is cyclic, then the group is abelian.