1. The torsion set is nonempty since the identity has finite order. Homework 1, problem 1 showed that the order of $x$ and $x^{-1}$ are the same so if $x \in G$ is in the torsion set, then $x^{-1}$ has finite order too so is also in the torsion subgroup. Now suppose $g$ and $h$ are in the torsion set so $g$ has order $n$ and $h$ has order $m$ for finite $n, m$. Homework 1, problem 2 showed that if $g, h \in G$ commute then $(gh)^n = g^n h^n$. Since $G$ is abelian in our example $(gh)^{nm} = g^{nm} h^{nm} = 1_G \cdot 1_G$ so $gh$ has order dividing $nm$, in particular finite order so $gh$ is in the torsion set. Since this set is closed under inverses and the group operation, it is indeed a subgroup.

In class we mentioned the following example. Let $G = \text{GL}_2(\mathbb{R})$ with $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 2 \\ 1/2 & 0 \end{pmatrix}$. Both have order 2 so are in the torsion set but $ab = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$, infinite order so this set is not closed under the group operation.

2. We know determinants (over any field $F$) satisfy the following two properties: $\text{det}(A^{-1}) = \text{det}(A)^{-1}$ and $\text{det}(AB) = \text{det}(A) \text{det}(B)$. So any matrix of determinant 1 will have inverse also of determinant 1 (and hence in $\text{SL}_n(F)$) while if $A$ and $B$ both have determinant 1 then $AB$ also has determinant 1 so is in $\text{SL}_n(F)$ as well. Therefore the subset is closed under inverses and the group operation so is a subgroup.

3. (a) Let $S = \{(a, 1) \mid a \in A\}$. $S$ is nonempty since $(1_A, 1_B) \in S$, $(a^{-1}, 1)$ is in $S$ and $(a, 1) \cdot (a^{-1}, 1) = 1_{A \times B} = (a^{-1}, 1) \cdot (a, 1)$ since the group operation on direct products is componentwise. Also if $(a_1, 1)$ and $(a_2, 1)$ are in $S$ then their product $(a_1 a_2, 1)$ is also in $S$ since $a_1 a_2$ is in $A$. So this is a subgroup.

(b) This is identical to (a) except use the second component and the fact that $B$ is a group.

(c) Let $S = \{(a, a) \mid a \in A\}$. This set is nonempty since $(1_A, 1_A) \in S$. It is closed under inverses since $(a^{-1}, a^{-1}) \in S$ and is the inverse of $(a, a)$. And if $(a_1, a_1), (a_2, a_2) \in S$ then their product $(a_1 a_2, a_1 a_2)$ is in $S$ as well since $A$ is a group so $a_1 a_2 \in A$.

4. If $a \in Z(G)$ then $ga = ag$ for any $g \in G$ or $gag^{-1} = a$ for any $g \in G$. Thus $\{g \in G \mid gag^{-1} = a \text{ for all } a \in Z(G)\} = C_G(Z(G))$ contains every element of $G$ and since the centralizer is a subgroup of $G$ this means $C_G(Z(G)) = G$. The centralizer of a set is a subgroup of the normalizer of that set which is in turn a subgroup of $G$ so if $C_G(A) = G$, $N_G(A)$ must also be $G$ or in this particular case $N_G(Z(G)) = G$.

5. We do the computations for (a). The other two parts follow similarly. $C_G(A) = \{g \in G \mid
$g a g^{-1} = a \}$. So we just check the conjugate of each element of $A$ by each non-identity element in $G$.

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(1 2)(1 2 3)(1 2) = (1 3 2) \quad (1 2)(1 3 2)(1 2) = (1 2 3)
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We can see that only the elements of $A$ fix the elements of $A$ under conjugation.

For $N_G(A) = \{ g \in G \mid g a g^{-1} = A \}$, we know $A \leq N_G(A)$ since $A$ is a subgroup. So we need to check the three elements not in $A$, $(1 2), (1 3), (2 3)$, send elements in $A$ to elements in $A$ under conjugation. By the calculations above, we see that is all of the elements.

6. $\mathbb{Z}/48\mathbb{Z}$ is a cyclic group with one generator $\bar{1}$ (thinking of this as a quotient group with group operation addition). Then we know from Proposition 5 from class that the other generators will be $\bar{a}$ where $(a, 48) = 1$:

\[
\{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47\}
\]

7. Let $\phi : \mathbb{Z} \rightarrow H$ sending $a$ to $h^a$ (suppose $H$ is a multiplicative group). Now suppose $\psi$ is another homom sending 1 to $h$. Then for $b > 0$, $\psi(b) = \psi(b \cdot 1) = \psi(1 + \cdots + 1) = \psi(1)^b$ (by simple induction and then $\psi(-b) = \psi(b)^{-1} = (\psi(1)^b)^{-1} = \psi(1)^{-b}$.) So $\psi(a) = \psi(1)^a = h^a = \phi(a)$ for all $a$ so these homomorphisms are the same.

8. (b) Since $\text{SL}_2(\mathbb{F}_3)$ is generated by two elements of order 3 each, if it was isomorphic to $S_4$, then $S_4$ would also have to be generated by two elements of order 3. If we take any two order 3 elements of $S_4$ (not inverses of each other) and compute the subgroup they generate, (insert computations here) we see that they only generate a special subgroup called $A_4$ (which we will talk about soon). In particular they do not generate elements of the form $(1 2)$ or $(1 2 3 4)$.

9. As an example, take $H$ to be the set of rational numbers of the form $a/2^n$ for $n \geq 0$ and $a \in \mathbb{Z}$ with group operation $+$. It is easy to check this is a subgroup. To see that this is not cyclic, suppose by way of contradiction that $H = \langle a/2^n \rangle$. Then $1/2^{n+1} = k \cdot (a/2^n)$ for some $k \in \mathbb{Z}$. But this would mean $2ak = 1$ which cannot be true for $a, k \in \mathbb{Z}$. Thus $H$ cannot be cyclic.