Mean value sets for general divergence form uniformly elliptic operators: What we know so far.

Niles Armstrong

Kansas State University

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1 Introduction/Motivation

2 Generalized MVT

3 Properties of Mean Value Sets

4 An Operator with Nonconvex Mean Value Sets
Theorem (Mean Value Theorem)

Let $u \in L^1_{loc}(\Omega)$ satisfy $\Delta u \leq 0$ in $\Omega$. Then for any $0 < r < s$ such that $B_s(x_0) \subset \Omega$, we have

$$u(x_0) \geq \int_{B_r(x_0)} u \, dx \geq \int_{B_s(x_0)} u \, dx$$

$$u(x_0) \geq \int_{\partial B_r(x_0)} u \, d\mathcal{H}^{n-1} \geq \int_{\partial B_s(x_0)} u \, d\mathcal{H}^{n-1}.$$
Theorem (Mean Value Theorem)

Let \( u \in L^1_{\text{loc}}(\Omega) \) satisfy \( \Delta u \leq 0 \) in \( \Omega \). Then for any \( 0 < r < s \) such that \( B_s(x_0) \subset \Omega \), we have

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\begin{align*}
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    u(x_0) & \geq \int_{\partial B_r(x_0)} u \, d\mathcal{H}^{n-1} \geq \int_{\partial B_s(x_0)} u \, d\mathcal{H}^{n-1}.
\end{align*}
\]

Here \( \Delta u \leq 0 \) in \( \Omega \) is meant in the weak sense, as in

\[
\int_{\Omega} u \Delta \phi \leq 0 \quad \text{for all non-negative } \phi \in C^{1,1}_0(\Omega).
\]
Remarks on the MVT

- Can be used to prove the following for sub/super-harmonic functions.
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  - The first theorem in Gilbarg and Trudinger’s text, the bible of elliptic PDEs.
The standard proof of the MVT can be a bit obnoxious for a few reasons:

- Relies on rather annoying changes of variables.
- Requires the harmonic function to be in $C^2(\Omega)$.
- Is not constructive in the sense of the mean value sets.

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Next define $\psi_r(x)$, to be equal to the paraboloid inside the sphere of radius $r$ and $\Gamma(x)$ outside. Finally, the desired test function is given by $\psi_s(x) - \psi_r(x)$ for $0 < s < r$. 
Idea in the proof
Touching $\Gamma$ from below: The Definition of $\Psi_s$
Benefits of Caffarelli’s Proof

It only assumes the function is weakly harmonic, no $C^2$ assumption. You can see where the solid mean value sets come from. This proof can be generalized by solving an obstacle problem!

We will consider operators of the form $L : \mathbb{R}^n \ni x \mapsto D_i (a_{ij}(x) D_j)$. Where $a_{ij}(x)$'s are bounded and measurable functions which obey:

$$a_{ij}(x) \equiv a_{ji}(x)$$

and

$$0 < \lambda / \xi_i \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda / \xi_i \xi_j$$

for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$.

Here, $Lu \leq 0$ in $\Omega$ means $\int_\Omega a_{ij}(x) D_j D_i \phi \, dx \geq 0$ for all non-negative $\phi \in W^{1,2}_0(\Omega)$. 

Niles Armstrong (KSU)
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Here, $Lu \leq 0$ in $\Omega$ means $\int_{\Omega} a^{ij}(x) D_j u D_i \phi \, dx \geq 0$ for all non-negative $\phi \in W^{1,2}_0(\Omega)$. 
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Littman, Stampacchia, and Weinberger (in 1963) showed the existence of Green’s functions on Euclidean balls for these operators. Moreover, they proved that these Green’s functions would be comparable to that for the Laplacian.
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Hence, one need only solve the appropriate obstacle problem and show that the solution works as a replacement for $\psi_r$ in the general setting. The details of this were shown by Blank and Hao in 2014.
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$$\begin{cases} \quad Lu = -\chi_{\{u < G\}} r^{-n} & \text{in } B_M(x_0), \text{ and} \\ \quad u = G(\cdot, x_0) & \text{on } \partial B_M(x_0) \end{cases}$$

where $G(x, x_0)$ is the Green’s function on $\mathbb{R}^n$ and $M > 0$ is sufficiently large.
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Note there is a small technicality with demanding $u = G(\cdot, x_0)$ on $\partial B_M(x_0)$. 
Theorem (Generalized MVT, Blank-Hao and Caffarelli)

Let $L$ be any divergence form elliptic operator with ellipticity $\lambda$, $\Lambda$. For any $x_0 \in \Omega$, there exists an increasing family $D_r(x_0)$ which satisfies the following:

1. for $0 < r < s$, $D_r(x_0) \subset D_s(x_0)$.
2. $B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0)$, with $c$, $C$ depending only on $n$, $\lambda$ and $\Lambda$.
3. For any $v$ satisfying $Lv \leq 0$ in $\Omega$ and $0 < r < s$, we have

$$v(x_0) \geq \int_{D_r(x_0)} v \geq \int_{D_s(x_0)} v. \quad (1)$$
About the sets $D_r(x_0)$

Although the $D_r(x_0)$ given above are nested and comparable to balls in the sense that:

$$B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0),$$

the theorem does not give much information about their geometry or topology. For instance, it is unknown whether these sets are star-shaped w.r.t. the origin and/or if they are convex (as we will see they need not be) and/or if they are each homeomorphic to a ball.

Aryal and Blank continued to study these mean value sets and proved three nice properties of them in general in 2017.
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Lemma (Aryal-Blank)

For any $x_0 \in \Omega$ and for any $r > 0$ such that $B_{Cr}(x_0) \subset \Omega$, the set $D_r(x_0)$ has exactly one component and it contains $x_0$.
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Properties of $D_r(x_0)$

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**Theorem (Aryal-Blank)**

Fix $x_0, y_0 \in \Omega$ and assume that there exists $0 < s < t$ so that $y_0$ is not contained in $D_s(x_0)$, and is compactly contained within $D_t(x_0)$. Then there exists a smallest $r \in (s, t)$ such that $y_0 \in \partial D_r(x_0)$. 
Assume \( y_0 \in \partial D_r(x_0) \), and assume that \( c \) and \( C \) are the constants given in Theorem (2). Fix \( h \in (0, 1/2] \). There exists a positive constant \( \tau \) such that

\[
\frac{|B_{chr}(y_0) \cap D_r(x_0)|}{|B_{chr}(y_0)|} \geq \tau.
\]

Although \( \tau \) depends on dimension and ellipticity, it has no dependence on \( x_0, y_0, h, \) or \( r \).
Properties of $D_r(x_0)$ cont.

**Theorem (Aryal-Blank)**

Assume $y_0 \in \partial D_r(x_0)$, and assume that $c$ and $C$ are the constants given in Theorem (2). Fix $h \in (0, 1/2]$. There exists a positive constant $\tau$ such that

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This statement concerns the geometry of the Mean Value Sets... See the picture on the next slide!
Example of what cannot happen

In words, the set $D_r(x_0)$ cannot be too thin near its boundary points, so the following picture is ruled out.
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is still possible. Obviously this still leaves a lot of open questions.
Instead of looking at mean value sets in general my recent work has been to look at the mean values sets for a particular operator.

\[ a_{ij}(x) = f(x) \delta_{ij}(x) \]

where 

\[ f(x) = \begin{cases} \alpha x^n > 0 \\ \beta x^n < 0 \end{cases} \]

for some fixed \( \alpha, \beta > 0 \). Note this operator has independent interest in composite material problems. See Li-Vogelius and Li-Nirenberg.
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Given such an operator what can we say about the mean value sets?
We expect the most interesting properties of the mean value sets for such an operator to appear on or near the interface.
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Thus we aim to do a blow-up analysis at such free boundary points. Where one does a quadratic rescaling of a solution centered at the free boundary point in question, as in

\[ u_\epsilon(x) := \frac{u(x_0 + \epsilon x)}{\epsilon^2}. \]
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In order to make use out of such rescalings we will use a Weiss type monotonicity formula. Such formulas often tell you that convergent blowup limits are homogeneous of degree two.
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It is worth noting that similar quasi-monotonicity formulas have been developed recently.

Quasi-monotonicity for Lipschitz $a_{ij}(x)$. For Cardi, Gelli, and Spadaro in 2015. Quasi-monotonicity for $a_{ij}(x) \in W_{1,p}^s$. Geraci in 2017. While these formulas work for large classes of operators they obviously do not cover our case. Also, they need an error term to ensure the monotonicity of their formulas.
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Theorem (Weiss’ Original Monotonicity Formula)

Let \( u \in W^{1,2}(\Omega) \) be a solution to

\[
\begin{cases}
\Delta u &= \frac{1}{2} \mathcal{X}_{\{u > 0\}} \\
u &\geq 0
\end{cases}
\quad \text{in } \Omega.
\]

Define

\[
\Phi_{x_0}(r; u) := r^{-n-2} \int_{B_r(x_0)} (|\nabla u|^2 + u) \, dx - 2r^{-n-3} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}.
\]

Then, for \( 0 < \rho < \sigma \) we have

\[
\Phi_{x_0}(\sigma; u) - \Phi_{x_0}(\rho; u) = \int_{\rho}^{\sigma} r^{-n-2} \int_{\partial B_r(x_0)} \frac{2}{r} (\nabla u \cdot x - 2u)^2 \, d\mathcal{H}^{n-1} \, dr \geq 0.
\]
Theorem (Weiss Type Monotonicity Formula, A.)

Let $u \in W^{1,2}(\Omega)$ satisfy

\[
\begin{cases}
    Lu = \frac{1}{2} \chi_{\{u > 0\}} & \text{in } \Omega, \\
    u \geq 0 & \text{in } \Omega.
\end{cases}
\]

For $x_0 \in \{x_n = 0\}$ define

\[
\Phi_{x_0}(r; u) := r^{-n-2} \int_{B_r(x_0)} f \left( |\nabla u|^2 + u \right) dx - 2r^{-n-3} \int_{\partial B_r(x_0)} f u^2 d\mathcal{H}^{n-1}.
\]

Then, for $0 < \rho < \sigma$ we have

\[
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**Lemma (Transmission Condition, Li-Vogelius)**

Let $u \in W^{1,2}(B_1)$ satisfy $Lu = 0$ in $B_1$. Then $u \in C^\infty(\overline{B_1^+}) \cap C^\infty(\overline{B_1^-})$ and

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\alpha \lim_{s \downarrow 0} \frac{\partial}{\partial x_n} u(x', s) = \beta \lim_{t \uparrow 0} \frac{\partial}{\partial x_n} u(x', t).
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Note that for $u_\varepsilon(x) := \frac{u(\varepsilon x)}{\varepsilon^2}$ we have $\Phi_{x_0}(r; u_\varepsilon) = \Phi_{x_0}(\varepsilon r; u)$. 
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If \( x_0 \in \{u = 0\} \cap \{x_n = 0\} \) then \( \lim_{r \downarrow 0} \Phi_{x_0}(r; u) = \Phi_{x_0}(0^+; u) \).
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As an immediate consequence we get that convergent blow-up solutions at free boundary points on the interface are homogeneous of degree two.
Explicit Blow-Up Solutions

If we restrict ourselves to $\mathbb{IR}^2$ the blow-up solutions at free boundary points on the interface have the form $u = r^2 g(\theta)$.
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This fact, together with the transmission condition and some results from obstacle problems, allows us to turn our PDE into an ODE. Hence, we can compute the possible blow-up solutions explicitly.
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This fact, together with the transmission condition and some results from obstacle problems, allows us to turn our PDE into an ODE. Hence, we can compute the possible blow-up solutions explicitly.

In particular we get very strict condition on how the free boundary can cross the interface.
Pictures of Blow-Up Solutions

Niles Armstrong (KSU)
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Remarks on Blow-Up Solutions

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**Theorem (Measure Convergence, A.)**

Let $D_r(x_0)$ be the mean value sets corresponding to such an operator $L$. Then we have

$$|D_r(x_0) \Delta B_r(x_0)| \to 0 \text{ as } \alpha \to \beta.$$
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Nonconvexity of the Mean Value Sets

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By mollifying the discontinuous $a^{ij}(x)$’s and proving another measure type convergence result we again get nonconvex mean value sets but now for coefficients that are $C^\infty$. 
A Few Interesting Open Problems

Any type of regularity of $a_{ij}(x)$ implies topological/geometrical structure of $D_r(x_0)$.

$D_r(x_0) \subset D_s(x_0)$ for all $0 < r < s$.

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Thanks!


F. Geraci. The classical obstacle problem with coefficients in fractional Sobolev spaces. DOI: 10.1007/s10231-017-0692-x.


