Appendix A

Zorn Lemma

In this Appendix we review basic set theoretical results, which are consequences of the following postulate:

**Axiom of Choice.** Given any non-empty collection \( \{X_i : i \in I\} \) of non-empty sets, the cartesian product

\[
\prod_{i \in I} X_i
\]

is non-empty.

Recall that the cartesian product is defined as

\[
\prod_{i \in I} = \{f : I \to \bigcup_{i \in I} X_i : f(i) \in X_i, \ \forall i \in I\}.
\]

In order to formulate several consequences of the Axion of Choice, we need several concepts.

**Definitions.** Given a set \( X \), by a relation on \( X \) one means simply as subset \( R \subset X \times X \). The standard notation for relations is:

\[
x \mathrel{R} y \iff (x, y) \in R.
\]

An order relation on \( X \) is a relation \( \prec \) with the following properties:

- \( x \prec x, \ \forall x \in X \);
- if \( x, y, z \in X \) satisfy \( x \prec y \) and \( y \prec z \), then \( x \prec z \);
- if \( x, y \in X \) satisfy \( x \prec y \) and \( y \prec x \), then \( x = y \).

In this case the pair \((X, \prec)\) is called an ordered set.

An ordered set \((X, \prec)\) is said to be totally ordered, if

- for any elements \( x, y \in X \) one has either \( x \prec y \) or \( y \prec x \).

More generally, given an (arbitrary) ordered set \((X, \prec)\), by a totally ordered subset of \((X, \prec)\), one means a subset \( T \subset X \), which becomes totally ordered with respect to the order relation \( \prec \mid_T \).

**Example.** Fix a set \( M \), and take \( X \) to be the collection of all subsets of \( M \). Then \( X \) carries a natural order relation defined by inclusion:

\[
A \prec B \iff A \subset B.
\]

A totally ordered subset \( T \) of \((X, \subset)\) is called a chain of subsets of \( M \). Two subset \( A, B \subset M \) will be said to be comparable, if either \( A \subset B \), or \( B \subset A \), i.e. the collection \( \{A, B\} \) is a chain of subsets of \( M \).

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\(^1\) By a “collection of sets” one simply means a set whose elements are sets themselves.
DEFINITION. Let $M$ be a set. A collection $\mathcal{F}$ of subsets of $M$ is said to have the chain property, if

(c) whenever $C \subset \mathcal{F}$ is a chain, it follows that the union $\bigcup_{C \in \mathcal{E}} C$ also belongs to $\mathcal{F}$.

**Lemma A.1.** Let $M$ be a set, let $\mathcal{F}$ be a collection of subsets of $M$ with the chain property. For every set $A \in \mathcal{F}$, the collection

$$\text{comp}(A; \mathcal{F}) = \{ B \in \mathcal{F} : B \text{ comparable to } A \}$$

has the chain property.

**Proof.** Let $C \subset \text{comp}(A; \mathcal{F})$ be a chain, and put $T = \bigcup_{C \in \mathcal{F}} C$. Since $\mathcal{F}$ has the chain property, we have $T \in \mathcal{F}$. To show that $T$ is comparable with $A$, we consider the two possibilities:

**Case 1:** $A \supset C$, for all $C \in \mathcal{F}$. In this case we have $A \supset \bigcup_{C \in \mathcal{F}} C = T$.

**Case 2:** There exists $C_0 \in \mathcal{F}$, such that $A \subset C_0$. In this case we have $A \subset C_0 \subset T$. □

**Lemma A.2.** Let $M$ be some non-empty set, let $\mathcal{F}$ be a non-empty collection of subsets of $M$, with the chain property. Suppose one has a map $\mathcal{F} \ni A \mapsto x_A \in M$, with the property that $A \cup \{ x_A \} \in \mathcal{F}, \forall A \in \mathcal{F}$. Then there exists $A \in \mathcal{F}$ such that $x_A \in A$.

**Proof.** For each $A \in \mathcal{F}$ we define $A^+ = A \cup \{ x_A \}$. Call a subset $\mathcal{G} \subset \mathcal{F}$ inductive, if it has the chain property, and

($+$) $A \in \mathcal{G} \Rightarrow A^+ \in \mathcal{G}$.

It is quite clear that if $\mathcal{G}_i, i \in I$ is a collection of inductive subsets of $\mathcal{F}$, then the intersection $\bigcap_{i \in I} \mathcal{G}_i$ is again an inductive subset of $\mathcal{F}$.

Fix now some subset $A_0 \in \mathcal{F}$, and define

$$\mathcal{G}_0 = \bigcap_{\text{inductive } A_0} \mathcal{G}.$$  

Note that the subset $\mathcal{G}_0 = \{ A \in \mathcal{F} : A \supset A_0 \}$ is an inductive subset of $\mathcal{F}$, so in particular, $\mathcal{G}_0$ is non-empty, and $\mathcal{G}_0 \subset \mathcal{F}_0$, i.e.

(1) $A \supset A_0$, $\forall A \in \mathcal{G}_0$.

Claim: The set $\mathcal{G}_0$ is a chain.

What we need to prove is the fact that $\mathcal{G}_0$ is totally ordered by inclusion. Consider the set

$$\mathcal{F} = \{ T \in \mathcal{G} : T \text{ is comparable with every } A \in \mathcal{G}_0 \} = \bigcap_{A \in \mathcal{G}_0} \text{comp}(A; \mathcal{G}_0),$$

and we try to prove that $\mathcal{F} = \mathcal{G}_0$. By Lemma A.1 it is clear that $\mathcal{F}$ has the chain property. Using (1), it is clear that $A_0 \in \mathcal{G}_0$. Finally, we need to prove property ($+$). We prove this indirectly as follows. Fix $T \in \mathcal{F}$, consider the collection

$$\forall T = \text{comp}(T^+; \mathcal{G}_0) = \{ A \in \mathcal{G}_0 : A \text{ comparable with } T^+ \},$$
and let us prove that $V_T = S_0$, by showing that $V_T$ is an inductive set, and contains $A_0$. First of all, by Lemma A.1, it follows that $V_T$ has the chain property. Secondly, using (1) we have $A_0 \subset T \subset T^+$, so $A_0 \in V_T$. Finally, to check property (+), we start with some $V \in V_T$, and we show that $V^+ \in V_T$. In the case when $T^+ \subset V$, we are done, because we have $T^+ \subset V \subset V^+$. Assume $T^+ \not\subset V$, so that we have $V \subset T$. Since $T$ is comparable with $V^+$, we either have $V^+ \subset T$, in which case we are done, or we have $T \subset V^+$. In the latter case, we have $V \subset T \subset V^+$. Since $V^+ = V \cup \{x_V\}$, the above inclusions forces either $T = V$, which gives $T^+ = V^+$, or $T = V^+$. Clearly, either case gives $V^+ \in V_T$. Having shown that $V_T$ is inductive, the inclusion $V_T \subset S_0$ will force the equality $V_T = S_0$. In turn, the definition of $V_T$ proves that $T^+ \in \mathcal{T}$, so $\mathcal{T}$ is indeed inductive. Finally, the inclusion $\mathcal{T} \subset S_0$ then forces $\mathcal{T} = S_0$, and by the definition of $\mathcal{T}$, it follows that $S_0$ is indeed a chain.

Having proven the Claim, we now take $A = \bigcup_{G \in S_0} G$. Since $S_0$ has the chain property, it follows that $A \in S_0$. By construction we have $A \supset G$, $\forall G \in S_0$.

In particular we have $A \supset A^+$, which clearly forces $x_A \in A$. □

**Definitions.** Let $(X, \prec)$ be an ordered set. By a maximal element for $X$ one means an element $x \in X$ with the property:

$$\{y \in X : x \prec y\} = \{x\}.$$ 

In other words, this means that there is no element $y \in X$, with $x \prec y$ and $y \neq x$.

Given a subset $S \subset X$, an element $x \in X$ is said to be an upper bound for $S$, if $s \prec x$, $\forall s \in S$.

If such an $x$ exists, we say that $S$ has an upper bound. (It is not assumed that $x$ belongs to $S$!)

**Lemma A.3 ("Easy" Zorn Lemma).** Let $M$ be a set, and let $\mathcal{F}$ be a collection of subsets of $M$. Assume

- the Axiom of Choice is true;
- $\mathcal{F}$ has the chain property;
- $\mathcal{F}$ and is hereditary, in the sense that, whenever $A \in \mathcal{F}$, it follows that all subsets of $A$ belong to $\mathcal{F}$.

Then, when equipped with the inclusion relation, $(\mathcal{F}, \subset)$ has at least one maximal element.

**Proof.** The proof will be carried on by contradiction. Assume no $A \in \mathcal{F}$ is maximal. For each $A \in \mathcal{F}$, define

$$X_A = \{x \in M : A \cup \{x\} \in \mathcal{F}\}.$$ 

Claim: For every $A \in \mathcal{F}$, the set $X_A$ is non-empty.

Indeed, since $A$ is not maximal, there exists some $B \in \mathcal{F}$, with $A \subsetneq B$. In particular, there exists some $x \in B \setminus A$, and since $A \cup \{x\} \subset B$, by the hereditary property, it follows that $x \in X_A$.

Use now the Axiom of Choice, to find a map

$$\mathcal{F} \ni A \longrightarrow x_A \in M,$$
such that $x_A \in X_A$, $\forall A \in \mathcal{F}$. This means that $A \cup \{x_A\} \in \mathcal{F}$, and $x_A \not\in A$, for all $A \in \mathcal{F}$. By Lemma A.2 this is however impossible. □

**Theorem A.1** (Zorn Lemma). Assume the Axiom of Choice is true. Let $(X, \prec)$ be a non-empty ordered set, with the following property:

1. (z) every totally ordered subset $A \subset X$ has an upper bound.

Then $X$ has at least one maximal element.

**Proof.** Define the collection

$$\mathcal{F} = \{A \subset X : A \text{ totally ordered subset}\}.$$ 

Clearly $\mathcal{F}$ is non-empty (it contains, for instance, all singletons).

It is quite clear that $\mathcal{F}$ satisfies the hypothesis of Lemma A.3. So $(\mathcal{F}, \subset)$ has a maximal element $A$. Take now $x$ to be an upper bound for $A$, i.e. $a \prec x$, $\forall a \in A$.

Now we prove that $x$ is maximal for $(X, \prec)$. Suppose $y \in X$ satisfies $x \prec y$. Then clearly $A \cup \{y\}$ will still be a totally ordered subset of $X$, i.e. $A \cup \{y\} \in \mathcal{F}$. The maximality of $A$ in $(\mathcal{F}, \subset)$ will force $A \cup \{y\} = A$, so we get $y \in A$, hence $y \prec x$. Since we also have $x \prec y$, this forces $y = x$. □