Lecture 21

4. The concept of measure

Definition. Let \( X \) be a non-empty set, and let \( \mathcal{E} \) be an arbitrary collection of subsets of \( X \). Assume \( \emptyset \in \mathcal{E} \). A measure on \( \mathcal{E} \) is a map \( \mu : \mathcal{E} \to [0,1] \) with the following properties

\[ (0) \quad \mu(\emptyset) = 0. \]

\[ \text{(ADD}_\sigma\text{)} \quad \text{Whenever } (E_n)_{n=1}^\infty \subset \mathcal{E} \text{ is a pair-wise disjoint sequence, with } \bigcup_{n=1}^\infty E_n \in \mathcal{E}, \text{ it follows that we have the equality} \]

\[ \mu\left( \bigcup_{n=1}^\infty E_n \right) = \sum_{n=1}^\infty \mu(E_n). \]

Property \( \text{(ADD}_\sigma\text{)} \) is called \( \sigma\)-additivity.

Convention. For a sequence \((\alpha_n)_{n=1}^\infty \subset [0, \infty)\) we define

\[ \sum_{n=1}^\infty \alpha_n = \begin{cases} \sum_{n=1}^\infty \alpha_n & \text{if } \alpha_n \in [0, \infty), \forall n \in \mathbb{N} \\ \infty & \text{if there exists } n \in \mathbb{N} \text{ with } \alpha_n = \infty. \end{cases} \]

(Of course, in the first case, it is still possible to have \( \sum_{n=1}^\infty \alpha_n = \infty \).)

Remark 4.1. If \( \mu \) is a measure on \( \mathcal{E} \), then \( \mu \) is additive, i.e.

\[ \text{(ADD)} \quad \text{Whenever } (E_n)_{n=1}^N \subset \mathcal{E} \text{ is a finite pair-wise disjoint system, such that } E_1 \cup \cdots \cup E_N \in \mathcal{E}, \text{ it follows that we have the equality} \]

\[ \mu(E_1 \cup \cdots \cup E_N) = \mu(E_1) + \cdots + \mu(E_N). \]

This follows from \( \text{(ADD}_\sigma\text{)} \) \((0)\), after completing the sequence \( E_1, \ldots, E_N \) to an infinite sequence by taking \( E_n = \emptyset, \forall n > N \).

Comment. The most natural setting for measures is the one when \( \mathcal{E} \) is a \( \sigma\)-ring. In this case, the stipulation that \( \bigcup_{n=1}^\infty E_n \in \mathcal{E} \), which appears in the definition, is superfluous.

The purpose of this section is to study measures on more rudimentary collections.

Examples 4.1. Let \( X \) be a non-empty set.

A. If we take \( \mathcal{E} = \{\emptyset, X\} \) and we define \( \mu(\emptyset) = 0 \) and \( \mu(X) \) to be any element in \([0,\infty]\), then \( \mu \) is obviously a measure on \( \{\emptyset, X\} \).

B. If we take \( \mathcal{E} = \mathcal{P}(X) \) and we define

\[ \mu(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ \infty & \text{if } E \neq \emptyset \end{cases} \]

then \( \mu \) is a measure on \( \mathcal{P}(X) \).
C. If we take $\mathcal{E} = \mathcal{P}(X)$ and we define

$$\mu(E) = \left\{ \begin{array}{ll} \text{card } E & \text{if } E \text{ is finite} \\ \infty & \text{if } E \text{ is infinite} \end{array} \right.$$ 

then $\mu$ is a measure on $\mathcal{P}(X)$. This is called the counting measure.

**Exercise 1.** Let $X_1, X_2$ be non-empty spaces, let $\mathcal{E}_k \subset \mathcal{P}(X_k)$ be arbitrary collections with $\emptyset \in \mathcal{E}_k$, $k = 1, 2$. Let $\mu_1$ be a measure on $\mathcal{E}_1$ and $\mu_2$ be a measure on $\mathcal{E}_2$. Consider the collections

- $f_*\mathcal{E}_1 = \{ A \subset X_2 : f^{-1}(A) \in \mathcal{E}_1 \} \subset \mathcal{P}(X_2)$;
- $f^*\mathcal{E}_2 = \{ f^{-1}(A) : A \in \mathcal{E}_2 \} \subset \mathcal{P}(X_1)$.

A. Prove that the map $f_*\mu_1 : f_*\mathcal{E}_1 \to [0, \infty]$, defined by

$$(f_*\mu_1)(A) = \mu_1(f^{-1}(A)), \quad \forall A \in f_*\mathcal{E}_1,$$

is a measure on $f_*\mathcal{E}_1$.

B. If $f$ is surjective, prove that the map $f^*\mu_2 : f^*\mathcal{E}_2 \to [0, \infty]$, defined by

$$(f^*\mu_2)(B) = \mu_2(f(B)), \quad \forall B \in f^*\mathcal{E}_2,$$

is a measure on $f^*\mathcal{E}_2$.

We now concentrate on the most rudimentary types of collections $\mathcal{E}$ on which measures can be somehow easily defined. Actually, what we have in mind is a set of easy conditions on a map $\mu : \mathcal{E} \to [0, \infty]$ which would guarantee that $\mu$ is a measure.

**Definition.** Let $X$ be a non-empty set. A collection $\mathcal{J} \subset \mathcal{P}(X)$ is called a semiring, if it satisfies the following properties:

- $\emptyset \in \mathcal{J}$;
- if $A, B \in \mathcal{J}$, then $A \cap B \in \mathcal{J}$;
- if $A, B \in \mathcal{J}$ and $A \subset B$, then there exists an integer $n \geq 1$, and sets $D_0, D_1, \ldots, D_n \in \mathcal{J}$, such that $A = D_0 \subset D_1 \subset \cdots \subset D_n = B$, and $D_k \setminus D_{k-1} \in \mathcal{J}$, $\forall k \in \{1, \ldots, n\}$.

Remark that every ring is a semiring.

**Exercise 2.** Prove that the semiring type is not consistent. Give an example of two semirings $\mathcal{J}_1, \mathcal{J}_2 \subset \mathcal{P}(X)$, such that $\mathcal{J}_1 \cap \mathcal{J}_2$ is not a semiring.

**Hint:** Use the set $X = \{1, 2, 3\}$.

**Exercise 3.** Let $X_1, \ldots, X_n$ be non-empty sets, and let $\mathcal{J}_k \subset \mathcal{P}(X_k)$, $k = 1, \ldots, n$, be semirings. Prove that

$$\mathcal{J} = \{ A_1 \times \cdots \times A_n : A_1 \in \mathcal{J}_1, \ldots, A_n \in \mathcal{J}_n \} \subset \mathcal{P}(X_1 \times \cdots \times X_n)$$

is a semiring.

**Hint:** First prove the case $n = 2$, and then use induction.

**Example 4.2.** Take $X = \mathbb{R}$. The collection

$$\mathcal{J} = \{ \emptyset \} \cup \{(a, b) : a, b \in \mathbb{R}, \ a < b \} \subset \mathcal{P}(\mathbb{R})$$

is a semiring.

Indeed, the first two axioms are pretty clear. To prove the third axiom, we start with two intervals $A = [a, b)$ and $B = [c, d)$ with $A \subset B$. This means that $a \geq c$ and $b \leq d$. If $a = c$ or $b = d$, we set $D_0 = A$ and $D_1 = B$. If $a > c$ and $b < d$, we set $D_0 = A$, $D_1 = [a, d)$ and $D_2 = B$. 

More generally, by Exercise 3, the collection of "half-open boxes"

\[ \mathcal{J}_n = \{ \emptyset \} \cup \left\{ \prod_{j=1}^{n} (a_j, b_j) : a_1 < b_1, \ldots, a_n < b_n \} \subset \mathcal{P}(\mathbb{R}^n) \]

is a semiring.

Exercise 4. Let \( \mathcal{J}_n \subset \mathcal{P}(\mathbb{R}^n) \) be the semiring defined above. Prove that the \( \sigma \)-ring \( \mathcal{S}(\mathcal{J}) \) generated by \( \mathcal{J}_n \) coincides with \( \text{Bor}(\mathbb{R}^n) \).

The ring generated by a semiring has a particularly nice description (compare to Proposition 2.1):

**Proposition 4.1.** Let \( \mathcal{J} \) be a semiring on \( X \). For a subset \( A \subset X \), the following are equivalent:

(i) \( A \) belongs to \( \mathcal{R}(\mathcal{J}) \), the ring generated by \( \mathcal{J} \);

(ii) There exists an integer \( n \geq 1 \), and a pair-wise disjoint system \( (A_j)^n_{j=1} \subset \mathcal{J} \), such that \( A = A_1 \cup \cdots \cup A_n \).

**Proof.** Denote by \( \mathcal{R} \) the collection of all subsets \( A \subset X \) that satisfy condition (ii). It is obvious that

\[ \mathcal{J} \subset \mathcal{R} \subset \mathcal{R}(\mathcal{J}) \],

so (see Section III.2) we only need to prove that \( \mathcal{R} \) is a ring.

Let us first remark that we obviously have the property:

(i) if \( A, B \in \mathcal{R} \), and \( A \cap B = \emptyset \), then \( A \cup B \in \mathcal{R} \).

Secondly, we remark that we have have the implication:

(ii) \( A, B \in \mathcal{J} \Rightarrow A \setminus B \in \mathcal{R} \).

Indeed, since \( A \cap B \in \mathcal{J} \), by the definition of a semiring, there exist \( D_0, D_1, \ldots, D_n \in \mathcal{J} \) with \( A \cap B = D_0 \subset D_1 \subset \cdots \subset D_n = A \), and \( D_k \setminus D_{k-1} \in \mathcal{J} \), \( \forall k \in \{1, \ldots, n\} \). Then the equality

\[ A \setminus B = \bigcup_{k=1}^{n} (D_k \setminus D_{k-1}) \]

shows that \( A \setminus B \) indeed belongs to \( \mathcal{R} \).

Thirdly, we prove the implication:

(iii) \( A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R} \).

Write \( A = A_1 \cup \cdots \cup A_m \) and \( B = B_1 \cup \cdots \cup B_n \), with \( (A_i)^m_{i=1} \), \( (B_k)^n_{k=1} \subset \mathcal{J} \) pair-wise disjoint systems. If we define the sets \( D_{ik} = A_j \cap B_k \in \mathcal{J} \), \( (i, k) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \) then it is obvious that

\[ A \cap B = \bigcup_{i=1}^{m} \bigcup_{k=1}^{n} D_{ik}, \]

and \( (D_{ik})_{1 \leq i \leq m, 1 \leq j \leq n} \subset \mathcal{J} \) is a pair-wise disjoint system, therefore \( A \cap B \) indeed belongs to \( \mathcal{R} \).

Finally, we show the implication:

(iv) if \( A, B \in \mathcal{R} \) and \( A \supset B \), then \( A \setminus B \in \mathcal{R} \).

Write \( A = A_1 \cup \cdots \cup A_m \), with \((A_i)^m_{i=1} \subset \mathcal{J} \) a pair-wise disjoint system. Notice that

\[ A \setminus B = \bigcup_{i=1}^{m} (A_i \setminus B), \]
with \((A_i \setminus B)^m_{i=1}\) a pair-wise disjoint system, so by (i) it suffices to show that
\[A_i \setminus B \in \mathcal{R}, \forall i \in \{1, \ldots, m\}.
\]
To prove this, we fix \(i\) and we write \(B = B_1 \cup \cdots \cup B_n\),
with \((B_k)_{k=1}^n \subset \mathcal{J}\) a pair-wise disjoint system. Then
\[A_i \setminus B = (A_i \setminus B_1) \cap \cdots \cap (A_i \setminus B_n),\]
and the fact that \(A_i \setminus B\) belongs to \(\mathcal{R}\) follows from (ii) and (iii).

Having proven (i)-(iv), it we now prove that \(\mathcal{R}\) is a ring. By (iii), we only need
to prove the implication
\[(*)\ A, B \in \mathcal{R} \Rightarrow A \Delta B \in \mathcal{R}.
\]
On the one hand, using (iv), it follows that the sets \(A \setminus B = A \setminus (A \cap B)\) and
\(B \setminus A = B \setminus (A \cap B)\) both belong to \(\mathcal{R}\). Since \(A \Delta B = (A \setminus B) \cup (B \setminus A)\), and
\((A \setminus B) \cap (B \setminus A) = \varnothing\), by (i) it follows that \(A \Delta B\) indeed belongs to \(\mathcal{R}\).

**Theorem 4.1 (Semiring-to-ring extension).** Let \(\mathcal{J}\) be a semiring on \(X\), and let
\(\mu: \mathcal{J} \rightarrow [0, \infty]\) be an additive map with \(\mu(\varnothing) = 0\).

(i) There exists a unique additive map \(\bar{\mu}: \mathcal{R}(\mathcal{J}) \rightarrow [0, \infty]\), such that \(\bar{\mu}|_{\mathcal{J}} = \mu\).

(ii) If \(\mu\) is \(\sigma\)-additive, then so is \(\bar{\mu}\).

**Proof.** The key step is contained in the following

Claim: If \((A_i)_{i=1}^m \subset \mathcal{J}\) and \((B_j)_{j=1}^n \subset \mathcal{J}\) are pair-wise disjoint systems, with
\[A_1 \cup \cdots \cup A_m = B_1 \cup \cdots \cup B_n,
\]
then \(\mu(A_1) + \cdots + \mu(A_m) = \mu(B_1) + \cdots + \mu(B_n)\).

To prove this fact, we define the pair-wise disjoint system \((D_{ij})_{1 \leq i \leq m} \subset \mathcal{J}\)
by \(D_{ij} = A_i \cap B_j, \forall (i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}\). Since
\[
\bigcup_{j=1}^n D_{ij} = A_i, \quad \forall i \in \{1, \ldots, m\},
\]
\[
\bigcup_{i=1}^m D_{ij} = B_j, \quad \forall j \in \{1, \ldots, n\},
\]
using additivity, we have the equalities
\[
\sum_{j=1}^n \mu(D_{ij}) = \mu(A_i), \quad \forall i \in \{1, \ldots, m\},
\]
\[
\sum_{i=1}^m \mu(D_{ij}) = \mu(B_j), \quad \forall j \in \{1, \ldots, n\},
\]
and then we get
\[
\sum_{i=1}^m \mu(A_i) = \sum_{i=1}^m \left[ \sum_{j=1}^n \mu(D_{ij}) \right] = \sum_{j=1}^n \left[ \sum_{i=1}^m \mu(D_{ij}) \right] = \sum_{j=1}^n \mu(B_j).
\]

To prove (i), for any set \(A \in \mathcal{R}(\mathcal{J})\) we choose (use Proposition 4.1) a finite
pair-wise disjoint system \((A_i)_{i=1}^m \subset \mathcal{J}\), with \(A = A_1 \cup \cdots \cup A_n\), and we define
\[(1) \quad \bar{\mu}(A) = \mu(A_1) + \cdots + \mu(A_n).
\]
By the above Claim, the number \(\bar{\mu}(A)\) is independent of the particular choice of the
pair-wise disjoint system \((A_i)_{i=1}^m\). Also, it is clear that \(\bar{\mu}|_{\mathcal{J}} = \mu\), and \(\bar{\mu}\) is additive.
The uniqueness is also clear, because the equality $\bar{\mu}|_\mathcal{J} = \mu$ and additivity of $\bar{\mu}$ force (1).

(ii). Assume now that $\mu$ is $\sigma$-additive, and let us prove that $\bar{\mu}$ is again $\sigma$-additive. Start with a pair-wise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{R}(\mathcal{J})$, with $\bigcup_{n=1}^\infty A_n \in \mathcal{R}(\mathcal{J})$, and let us prove the equality

$$\bar{\mu}\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \bar{\mu}(A_n).$$

Since $\bigcup_{n=1}^\infty A_n \in \mathcal{R}$, there exists a finite pair-wise disjoint system $(B_i)_{i=1}^p \subset \mathcal{J}$, such that $\bigcup_{n=1}^\infty A_n = B_1 \cup \cdots \cup B_p$. With this choice we have

$$\bar{\mu}\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{i=1}^p \bar{\mu}(B_i).$$

For each $i \in \{1, \ldots, p\}$, we have $B_i = \bigcup_{n=1}^\infty (B_i \cap A_n)$. Fix for the moment a pair $(n, i) \in \mathbb{N} \times \{1, \ldots, p\}$. Since $B_i \cap A_n \in \mathcal{R}(\mathcal{J})$, it follows that there exist an integer $N_{ni} \geq 1$ and a finite pair-wise disjoint system $(C_{ki}^{ni})_{k=1}^{N_{ni}} \subset \mathcal{J}$, such that $B_i \cap A_n = \bigcup_{k=1}^{N_{ni}} C_{ki}^{ni}$.

Since, for each $i \in \{1, \ldots, p\}$, the countable system $(C_{ki}^{ni})_{n \in \mathbb{N}}_{1 \leq k \leq N_{ni}} \subset \mathcal{J}$ is pair-wise disjoint, and we have the equality

$$\bigcup_{n=1}^\infty (B_i \cap A_n) = B_i \in \mathcal{J},$$

by the $\sigma$-additivity of $\mu$, we have

$$\mu(B_i) = \sum_{n=1}^\infty \sum_{k=1}^{N_{ni}} \mu(C_{ki}^{ni}), \quad \forall i \in \{1, \ldots, p\}.$$

Since, for each $n \in \mathbb{N}$, the finite system $(C_{ki}^{ni})_{1 \leq k \leq N_{ni}} \subset \mathcal{J}$ is pair-wise disjoint, and we have the equality

$$\bigcup_{i=1}^p \bigcup_{k=1}^{N_{ni}} C_{ki}^{ni} = \bigcup_{i=1}^\infty (B_i \cap A_n) = A_n \in \mathcal{J},$$

by the definition of $\bar{\mu}$, we have

$$\bar{\mu}(A_n) = \sum_{i=1}^p \sum_{k=1}^{N_{ni}} \mu(C_{ki}^{ni}), \quad \forall i \in \{1, \ldots, p\}.$$ Combining this with (4) yields

$$\sum_{n=1}^\infty \bar{\mu}(A_n) = \sum_{n=1}^\infty \sum_{i=1}^p \sum_{k=1}^{N_{ni}} \mu(C_{ki}^{ni}) = \sum_{i=1}^p \mu(B_i),$$

and the equality (2) follows from (3).

□

\textbf{Definition.} Let $X$ be a non-empty set, and let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of sets. We say that a map $\mu : \mathcal{E} \to [0, \infty)$ is sub-additive, if

\textup{(ADD$-$)} whenever $A \in \mathcal{E}$, and $(A_n)_{n=1}^p$ is a finite sequence in $\mathcal{E}$ with $A \subset \bigcup_{k=1}^p A_k$, it follows that $\mu(A) \leq \sum_{k=1}^p \mu(A_k)$. 

Note that we do not require the $A_k$’s to be pair-wise disjoint. With this terminology, Theorem 4.1 has the following.

**Corollary 4.1.** Let $X$ be a non-empty set $X$, and let $\mathcal{J} \subset \mathcal{P}(X)$ be a semiring. Then any additive map $\mu : \mathcal{J} \to [0, \infty]$ is sub-additive.

**Proof.** Let $\bar{\mu} : \mathcal{R}(\mathcal{J}) \to [0, \infty]$ be the additive extension of $\mu$ to the ring generated by $\mathcal{J}$. It suffices to prove that $\bar{\mu}$ is sub-additive. Start with sets $A, A_1, \ldots, A_n \in \mathcal{R}(\mathcal{J})$ such that $A \subset A_1 \cup \ldots \cup A_n$. Define the sets $B_k = A_k \setminus (A_1 \cup \ldots \cup A_{k-1})$, for all $k \in \{1, \ldots, n\}$, $k \geq 2$.

Since we work in a ring, the sets $B_k$, $B_k \cap A$, $B_k \setminus A$, and $A \setminus B_n$, $n \in \mathbb{N}$, all belong to $\mathcal{R}(\mathcal{J})$. Moreover, the sequence $(B_k)_{k=1}^n$ is pair-wise disjoint and it satisfies

- $B_k \subset A_k$, $\forall k \in \{1, \ldots, n\}$,
- $\bigcup_{k=1}^n B_k = \bigcup_{k=1}^n A_k \supseteq A,$

so by the additivity of $\bar{\mu}$, we get

$$\sum_{k=1}^n \bar{\mu}(A_k) = \sum_{k=1}^n \bar{\mu}((A_k \setminus B_k) \cup B_k) = \sum_{k=1}^n \left[\bar{\mu}(A_k \setminus B_k) + \bar{\mu}(B_k)\right] \geq \sum_{k=1}^n \bar{\mu}(B_k) = \sum_{k=1}^n \bar{\mu}((B_k \setminus A) \cup (B_k \cap A)) = \sum_{k=1}^n \left[\bar{\mu}(B_k \setminus A) + \bar{\mu}(B_k \cap A)\right] \geq \sum_{k=1}^n \bar{\mu}(B_k \cap A) = \bar{\mu}\left(\bigcup_{k=1}^n [B_k \cap A]\right) = \bar{\mu}(A). \quad \square$$

**Exercise 5.** Let $X_1, X_2$ be non-empty sets, let $\mathcal{J}_k \subset \mathcal{P}(X_k)$, $k = 1, 2$, be semirings, and let $\mu_k : \mathcal{J}_k \to [0, \infty]$ be additive maps. Consider the semiring (see Exercise 3)

$$\mathcal{J} = \{A_1 \times A_2 : A_1 \in \mathcal{J}_1, A_2 \in \mathcal{J}_2\} \subset \mathcal{P}(X_1 \times X_2).$$

Then the map $\mu : \mathcal{J} \to [0, \infty]$ defined by

$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_1)$$

is additive. Here we use the convention $0 \cdot \infty = \infty \cdot 0 = 0$.

**Hints:** One wants to show that, whenever $A_1 \times A_2 \in \mathcal{J}$ is written as union

$$A_1 \times A_2 = \bigcup_{k=1}^n (A_1^k \times A_2^k),$$

with $(A_1^k \times A_2^k)_{k=1}^n \subset \mathcal{J}$ pair-wise disjoint, it follows that

$$\mu_1(A_1^k) \cdot \mu_2(A_2) = \sum_{k=1}^n \mu_1(A_1^k) \cdot \mu_2(A_2^k).$$

Analyze first the case of “strips,” that is, when $A_1^1 = \cdots = A_n^1 = A_1$ or $A_1^2 = \cdots = A_n^2 = A_2$. In the general case, use induction, by picking some $k$ such that $A_1^k \subsetneq A_1$ and splitting $A_1 \times A_2$ into “strips” of the form $B_1 \times A_2$, where $B_1, \ldots, B_m \in \mathcal{J}_1$ are pairwise disjoint, with $B_1 = A_1^1$ and $B_1 \cup \cdots \cup B_m = A_1$.

**Comment.** In connection with the above exercise, one can as the following

**Question:** With the notations above, is it true that, if both $\mu_1$ and $\mu_2$ are measures, then $\mu$ is also a measure?

As we shall see a bit later in the course, that the answer is is “yes.”
Since we work in a ring, the sets \( B \) of the ring \( A \) form a sequence \( (\sigma) \) so we only need to prove
\[
\mu(\sigma)(\mu) \leq \sum_{n=1}^{\infty} \mu(A_n).
\]
Note that we do not require the \( A_n \)’s to be pair-wise disjoint.

**Proposition 4.2** (characterization of semiring measures). Let \( X \) be a non-empty set, let \( \mathcal{J} \subset \mathcal{P}(X) \) be a semiring, and let \( \mu : \mathcal{J} \to [0, \infty] \) be a map with \( \mu(\emptyset) = 0 \). The following are equivalent:

(i) \( \mu \) is a measure on \( \mathcal{J} \);

(ii) \( \mu \) is additive, and \( \sigma \)-sub-additive.

**Proof.** (i) \( \Rightarrow \) (ii). Assume \( \mu \) is a measure on \( \mathcal{J} \). It is clear that \( \mu \) is additive, so we only need to prove \( \sigma \)-sub-additivity. Use Theorem 4.1 to find a measure \( \tilde{\mu} \) on the ring \( R(\mathcal{J}) \) generated by \( \mathcal{J} \), such that
\[
\tilde{\mu}(A) = \mu(A), \quad \forall A \in \mathcal{J}.
\]
Then it suffices to show that \( \tilde{\mu} \) is \( \sigma \)-sub-additive. Start with a set \( A \in R(\mathcal{J}) \), and a sequence \( (A_n)_{n=1}^{\infty} \subset R(\mathcal{J}) \), such that \( A \subset \bigcup_{n=1}^{\infty} A_n \). Define the sets \( B_1 = A_1 \), and
\[
B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1}), \quad \text{for all } n \geq 2.
\]
Since we work in a ring, the sets \( B_n \), \( B_n \cap A \), \( B_n \setminus A \), and \( A_n \setminus B_n \), \( n \in \mathbb{N} \), all belong to \( R(\mathcal{J}) \). Moreover, the sequence \( (B_n)_{n=1}^{\infty} \) is pair-wise disjoint and it satisfies
- \( B_n \subset A_n \), \( \forall n \in \mathbb{N} \),
- \( \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \cap A \),
so by \( \sigma \)-additivity of \( \tilde{\mu} \), we get
\[
\sum_{n=1}^{\infty} \tilde{\mu}(A_n) = \sum_{n=1}^{\infty} \tilde{\mu}((A_n \setminus B_n) \cup B_n) = \sum_{n=1}^{\infty} \tilde{\mu}(A_n \setminus B_n) + \tilde{\mu}(B_n) \geq \sum_{n=1}^{\infty} \tilde{\mu}(B_n) = \sum_{n=1}^{\infty} \tilde{\mu}((B_n \setminus A) \cup (B_n \cap A)) = \sum_{n=1}^{\infty} \tilde{\mu}(B_n \setminus A) + \tilde{\mu}(B_n \cap A) \geq \sum_{n=1}^{\infty} \tilde{\mu}(B_n \cap A) = \tilde{\mu}(\bigcup_{n=1}^{\infty} |B_n \cap A|) = \tilde{\mu}(A).
\]

(ii) \( \Rightarrow \) (i). Assume \( \mu : \mathcal{J} \to [0, \infty] \) is additive and \( \sigma \)-sub-additive, and let us show that \( \mu \) is \( \sigma \)-additive. We again use Theorem 4.1, to find an additive map \( \bar{\mu} : R(\mathcal{J}) \to [0, \infty] \), such that \( \bar{\mu}|_{\mathcal{J}} = \mu \). Start with a pair-wise disjoint sequence \( (A_n)_{n=1}^{\infty} \subset \mathcal{J} \), such that the union \( A = \bigcup_{n=1}^{\infty} A_n \) belongs to \( \mathcal{J} \). On the one hand, by \( \sigma \)-sub-additivity, we have the inequality
\[
\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n).
\]
On the other hand, for any integer \( N \geq 1 \), we have
\[
\mu(A) = \bar{\mu}(A) = \bar{\mu}\left( \bigcup_{n=1}^{N} A_n \cup (A \setminus \bigcup_{n=1}^{N} A_n) \right) \geq \\
\mu(\bigcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \bar{\mu}(A_n) = \sum_{n=1}^{N} \mu(A_n),
\]
which then gives
\[
\mu(A) \geq \sup_{N \in \mathbb{N}} \sum_{n=1}^{N} \mu(A_n) = \sum_{n=1}^{\infty} \mu(A_n),
\]
so using (5) we immediately get \( \mu(A) = \sum_{n=1}^{\infty} \mu(A_n) \).

The following technical result will be often employed in subsequent sections.

**Lemma 4.1 (Continuity).** Let \( \mathcal{J} \) be a semiring, and let \( \mu \) be a measure on \( \mathcal{J} \).

(i) If \( (A_n)_{n=1}^{\infty} \subset \mathcal{J} \) is a sequence of sets, with \( A_1 \subset A_2 \subset \ldots \), and \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{J} \), then
\[
\mu\left( \bigcup_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} \mu(A_n).
\]

(ii) If \( (B_n)_{n=1}^{\infty} \subset \mathcal{J} \) is a sequence of sets, with \( B_1 \supseteq B_2 \supseteq \ldots \), and \( \bigcap_{n=1}^{\infty} B_n \in \mathcal{J} \), and \( \mu(B_1) < \infty \), then
\[
\mu\left( \bigcap_{n=1}^{\infty} B_n \right) = \lim_{n \to \infty} \mu(B_n).
\]

**Proof.** Using Theorem 4.1, we can assume that \( \mathcal{J} \) is already a ring. (Otherwise we replace \( \mathcal{J} \) by \( \mathcal{R}(\mathcal{J}) \), and \( \mu \) by its extension \( \bar{\mu} \).)

(i). Consider the sets \( D_1 = A_1 \), and \( D_k = A_n \setminus A_{k-1}, \forall k \geq 2 \). It is clear that \( (D_k)_{k=1}^{\infty} \) is a pairwise disjoint sequence in \( \mathcal{J} \), and we have the equality
\[
\bigcup_{k=1}^{n} D_k = A_n, \; \forall \; n \geq 1.
\]
This gives of course the equality
\[
\bigcup_{k=1}^{\infty} D_k = \bigcup_{n=1}^{\infty} A_n \in \mathcal{J}.
\]

Using this equality, combined with the \((\sigma-)\)-additivity of \( \mu \), and with (6), we get
\[
\mu\left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{k=1}^{\infty} \mu(D_k) = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \mu(D_k) \right] = \lim_{n \to \infty} \mu\left( \bigcup_{k=1}^{n} D_k \right) = \lim_{n \to \infty} \mu(A_n).
\]

(ii). Consider the sets \( B = \bigcap_{n=1}^{\infty} B_n \), and \( A_n = B_1 \setminus B_n \), \( \forall \; n \geq 1 \). It is clear that \( (A_n)_{n=1}^{\infty} \subset \mathcal{J} \), and we have \( A_1 \subset A_2 \subset \ldots \). Moreover, we have \( \bigcup_{n=1}^{\infty} A_n = B_1 \setminus B \), so by part (i), we get
\[
\mu(B_1 \setminus B) = \lim_{n \to \infty} \mu(B_1 \setminus B_n).
\]
Using the fact that \( \mu(B_1) < \infty \), it follows that
\[
\mu(B) \leq \mu(B_n) \leq \mu(B_1) < \infty, \; \forall \; n \geq 1.
\]
This gives then the equalities
\[ \mu(B_1 \triangle B) = \mu(B_1) - \mu(B) \text{ and } \mu(B_1 \triangle B_n) = \mu(B_1) - \mu(B_n), \quad \forall n \geq 1, \]
so the equality (7) immediately gives \( \mu(B) = \lim_{n \to \infty} \mu(B_n) \).

The above result has a (minor) generalization, which we record for future use. To formulate it we introduce the following.

Notation. Let \( R \) be a ring, and let \( \mu \) be a measure on \( R \). For two sets \( A, B \in R \), we write \( A \subset_{\mu} B \) if \( \mu(A \setminus B) = 0 \).

Using this notation, we have the following generalization of Lemma 4.1.

**Proposition 4.3.** Let \( R \) be a ring, and let \( \mu \) be a measure on \( R \).

(i) If \( (A_n)_{n=1}^{\infty} \subset R \) is a sequence of sets, with \( A_1 \subset_{\mu} A_2 \subset_{\mu} \ldots \) and \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{R} \), then

\[ \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n). \]

(ii) If \( (B_n)_{n=1}^{\infty} \subset R \) is a sequence of sets, with \( B_1 \supset_{\mu} B_2 \supset_{\mu} \ldots \) and \( \bigcap_{n=1}^{\infty} B_n \in \mathcal{R} \), and \( \mu(B_1) < \infty \), then

\[ \mu\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} \mu(B_n). \]

**Proof.** (i). Define the sequence of sets \( (E_n)_{n=1}^{\infty} \subset R \), by \( E_n = \bigcup_{k=1}^{n} A_k \), \( \forall n \geq 1 \). Notice that, \( A_1 = E_1 \), and for each \( n \geq 2 \), we have \( A_n \subset E_n \), as well as the equality

\[ E_n \setminus A_n = \bigcup_{k=1}^{n-1} [A_n \setminus A_k]. \]

Using sub-additivity, it follows that

\[ \mu(E_n \setminus A_n) \leq \sum_{k=1}^{n-1} \mu(A_n \setminus A_k), \]

which forces \( \mu(E_n \setminus A_n) = 0 \). This gives

\[ \mu(E_n) = \mu(A_n) + \mu(E_n \setminus A_n) = \mu(A_n), \quad \forall n \geq 1. \]

Since \( \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \), and we have the inclusions \( E_1 \subset E_2 \subset \ldots \), by Lemma 4.1, combined with (8), we get

\[ \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n) = \lim_{n \to \infty} \mu(A_n). \]

Part (ii) is proven exactly as part (ii) from Lemma 4.1.

**Exercise 6.** Let \( \mu \) be a measure on a ring \( \mathcal{R} \). Prove that, for \( A, B \in \mathcal{R} \), one has the implication

\[ A \subset_{\mu} B \Rightarrow \mu(A) \leq \mu(B). \]
Example 4.3. Fix some integer \( n \geq 1 \). Consider the semiring of “half-open boxes” in \( \mathbb{R}^n \)
\[
\mathcal{J}_n = \{ \emptyset \} \cup \{ \prod_{j=1}^{n} (a_j, b_j) : a_1 < b_1, \ldots, a_n < b_n \} \subset \mathcal{P}(\mathbb{R}^n).
\]
For a non-empty box \( A = [a_1, b_1) \times \cdots \times [a_n, b_n) \in \mathcal{J}_n \), we define
\[
\text{vol}_n(A) = \prod_{k=1}^{n} (b_k - a_k).
\]
We also define \( \text{vol}_n(\emptyset) = 0 \).

Theorem 4.2. With the above notations, the map \( \text{vol}_n : \mathcal{J} \to [0, \infty) \) is a measure on \( \mathcal{J}_n \).

Proof. First we prove additivity. Using Exercise ?? (and induction on \( n \)) it suffices to analyze only the case \( n = 1 \), i.e. the case of half-open intervals in \( \mathbb{R} \). We need to show the implication
\[
[a, b) = \bigcup_{k=1}^{p} [a_k, b_k) \text{ pair-wise disjoint} \implies b - a = \sum_{k=1}^{p} (b_k - a_k).
\]
We can prove this using induction on \( p \). The case \( p = 1 \) is trivial. Assuming that the above fact holds for \( p = N \), let us prove it for \( p = N + 1 \). Pick \( k_1 \in \{1, \ldots, N + 1\} \) such that \( a_{k_1} = a \). Then we clearly have
\[
\bigcup_{1 \leq k \leq N+1 \atop k \neq k_1} [a_k, b_k) = [b_{k_1}, b),
\]
so by the inductive hypothesis we get
\[
b - b_{k_1} = \sum_{1 \leq k \leq N+1 \atop k \neq k_1} (b_k - a_k),
\]
so we get
\[
\sum_{k=1}^{N+1} (b_k - a_k) = (b_{k_1} - a_{k_1}) + (b - b_{k_1}) = b - a_{k_1} = b - a,
\]
and we are done.

We now prove that \( \text{vol}_n \) is \( \sigma \)-sub-additive. Suppose we have \( A \in \mathcal{J}_n \) and a sequence \( (A_k)_{k=1}^{\infty} \subset \mathcal{J}_n \), such that \( A \subset \bigcup_{k=1}^{\infty} A_k \), and let us prove the inequality
\[
\text{vol}_n(A) \leq \sum_{k=1}^{\infty} \text{vol}_n(A_k).
\]
It will be helpful to introduce the following notations. For every half-open box
\[
B = [x_1, y_1) \times \cdots \times [x_n, y_n),
\]
and every \( \delta > 0 \), we define the boxes boxes
\[
B^\delta = [x_1 - \delta, y_1) \times \cdots \times [x_n - \delta, y_n) \text{ and } B_{\delta} = [x_1, y_1 - \delta) \times \cdots \times [x_n, y_n - \delta).
\]
It is clear that, for any box $B \in \mathcal{J}_n$ we have
\begin{equation}
\mathcal{B}_\delta \subset B \subset \text{Int}(B^\delta),
\end{equation}
\begin{equation}
\text{vol}_n(B) = \lim_{\delta \to 0^+} \text{vol}_n(B^\delta) = \lim_{\delta \to 0^+} \text{vol}_n(B_\delta).
\end{equation}

To prove (10), we fix some $\varepsilon > 0$, and we choose positive numbers $\delta$ and $(\delta_k)_{k=1}^{\infty}$, such that
\begin{equation}
\text{vol}_n(A_\delta) > \text{vol}_n(A) - \varepsilon, \text{ and } \text{vol}_n((A_k)^{\delta_n}) < \frac{\varepsilon}{2^k} + \text{vol}_n(A_k), \ \forall k \in \mathbb{N}.
\end{equation}

Notice now that, using (11), we have the inclusions
\begin{equation}
\overline{A_\delta} \subset A \subset \bigcup_{k=1}^{\infty} A_k \subset \text{Int}((A_k)^{\delta_n}),
\end{equation}
and using the compactness of $\overline{A_\delta}$, there exists some $N \geq 1$, such that
\begin{equation}
\overline{A_\delta} \subset \bigcup_{k=1}^{N} \text{Int}((A_k)^{\delta_n}).
\end{equation}

This immediately gives the inclusion
\begin{equation}
A_\delta \subset \bigcup_{k=1}^{N} (A_k)^{\delta_n}.
\end{equation}

Using sub-additivity (see Corollary 4.1) we now get
\begin{equation}
\text{vol}_n(A_\delta) \leq \sum_{k=1}^{N} \text{vol}_n((A_k)^{\delta_n}),
\end{equation}
and using (13) we have
\begin{equation}
\text{vol}_n(A) - \varepsilon \leq \sum_{k=1}^{N} \left[ \frac{\varepsilon}{2^k} + \text{vol}_n(A_k) \right] \leq \varepsilon + \sum_{k=1}^{N} \text{vol}_n(A_k) \leq \varepsilon + \sum_{k=1}^{\infty} \text{vol}_n(A_k).
\end{equation}

This gives
\begin{equation}
\text{vol}_n(A) - 2\varepsilon \leq \sum_{k=1}^{\infty} \text{vol}_n(A_k).
\end{equation}

But since this inequality holds for all $\varepsilon > 0$, the inequality (10) immediately follows. \qed