Lecture 12

3. Banach spaces

Definition. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. A Banach space over $\mathbb{K}$ is a normed $\mathbb{K}$-vector space $(X, \| \cdot \|)$, which is complete with respect to the metric
\[ d(x, y) = \| x - y \|, \quad x, y \in X. \]

Example 3.1. The field $\mathbb{K}$, equipped with the absolute value norm, is a Banach space. More generally, the vector space $\mathbb{K}^n$, equipped with any of the norms
\[ \| (\lambda_1, \ldots, \lambda_n) \|_\infty = \max \{ |\lambda_1|, \ldots, |\lambda_n| \}, \]
\[ \| (\lambda_1, \ldots, \lambda_n) \|_p = \left[ |\lambda_1|^p + \cdots + |\lambda_n|^p \right]^{1/p}, \quad p \geq 1, \]
is a Banach space.

Remark 3.1. Using the facts from the general theory of metric spaces, we know that for a normed vector space $(X, \| \cdot \|)$, the following are equivalent:

(i) $X$ is a Banach space;

(ii) given any sequence $(x_n)_{n \geq 1} \subset X$ with $\sum_{n=1}^{\infty} \| x_n \| < \infty$, the sequence $(y_n)_{n \geq 1}$ of partial sums, defined by $y_n = \sum_{k=1}^{n} x_k$, is convergent;

(iii) every Cauchy sequence in $X$ has a convergent subsequence.

This is pretty obvious, since the sequence of partial sums has the property that
\[ d(y_{n+1}, y_n) = \| y_{n+1} - y_n \| = \| x_{n+1} \|, \quad \forall n \geq 1. \]

Exercise 1*. Let $X$ be a finite dimensional normed vector space. Prove that $X$ is a Banach space.

Hints: Use induction on $\dim X$. The case $\dim X = 1$ is trivial. Assume the statement is true for all normed vector spaces of dimension $d$, and let us prove it for a normed vector space of dimension $d+1$. Fix such an $X$, and a linear basis $\{e_1, e_2, \ldots, e_n, e_{d+1}\}$ for $X$. Start with a Cauchy sequence $(x_n)_{n \geq 1} \subset X$. Write each term as
\[ x_n = \sum_{k=1}^{d+1} \alpha_n(k) e_k. \]
Prove first that $(\alpha_n(d+1))_{n \geq 1} \subset \mathbb{K}$ is bounded. Then extract a subsequence $(x_{n_p})_{p \geq 1}$ such that $(\alpha_{n_p}(d+1))_{p \geq 1}$ is convergent. If we take $\alpha(d+1) = \lim_{p \to \infty} \alpha_{n_p}(d+1)$, then prove that the sequence $(x_{n_p} - \alpha_{n_p}(d+1)e_{d+1})_{p \geq 1}$ is Cauchy in the space $\text{Span}\{e_1, \ldots, e_d\}$. Using the inductive hypothesis, conclude that $(x_{n_p})_{p \geq 1}$ is convergent in $X$. Thus, every Cauchy sequence in $X$ has a convergent subsequence, hence $X$ is Banach.

Exercise 2*. Let $n \geq 1$ be an integer, and let $\| \cdot \|$ be a norm on $\mathbb{K}^n$. Prove that there exist constants $C, D > 0$, such that
\[ C\| x \|_\infty \leq \| x \| \leq D\| x \|_\infty, \quad \forall x \in \mathbb{K}^n. \]
HINT: Let \( e_1, \ldots, e_n \) be the standard basis vectors for \( \mathbb{K}^n \), so that
\[
\alpha_1 e_1 + \cdots + \alpha_n e_n = (\alpha_1, \ldots, \alpha_n), \quad \forall (\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n.
\]
Define \( D = \|e_1\| + \cdots + \|e_n\| \). The existence of \( C \) is equivalent to the existence of some \( C' > 0 \) such that
\[
\|x\|_{\infty} \leq C' \|x\|, \quad \forall x \in \mathbb{K}^n.
\]
(If such a \( C' \) exists, then we take \( C = 1/C' \).) To prove the existence of \( C' \) as above, we consider the set \( T = \{ x \in \mathbb{K}^n : \|x\| \leq 1 \} \), and we need to prove that
\[
\sup_{x \in T} \|x\|_{\infty} < \infty.
\]
Argue by contradiction (see also the hint from the preceding exercise).

Exercise 3. Let \( X \) and \( Y \) be normed vector spaces. Consider the product \( X \times Y \), equipped with the natural vector space structure.

(i) Prove that \( \|(x, y)\| = \|x\| + \|y\|, \quad (x, y) \in X \times Y \) defines a norm on \( X \times Y \).

(ii) Prove that, when equipped with the above norm, \( X \times Y \) is a Banach space, if and only if both \( X \) and \( Y \) are Banach spaces.

There are two key constructions which enable one to construct new Banach space out of old ones.

**Proposition 3.1.** Let \( X \) be a normed vector space, and let \( Y \) be a Banach space. Then \( \mathcal{L}(X, Y) \) is a Banach space, when equipped with the operator norm.

**Proof.** Start with a Cauchy sequence \( (T_n)_{n \geq 1} \subset \mathcal{L}(X, Y) \). This means that for every \( \varepsilon > 0 \), there exists some \( N_\varepsilon \) such that
\[
\|T_m - T_n\| < \varepsilon, \quad \forall m, n \geq N_\varepsilon.
\]
Notice that, if one takes for example \( \varepsilon = 1 \), and we define
\[
C = 1 + \max\{\|T_1\|, \|T_2\|, \ldots, \|T_{N_1}\|\},
\]
then we clearly have
\[
\|T_n\| \leq C, \quad \forall n \geq 1.
\]
Notice that, using (1), we have
\[
\|T_m x - T_n x\| \leq \varepsilon \|x\|, \quad \forall m, n \geq N_\varepsilon, \quad x \in X,
\]
which proves that

- **for every** \( x \in X \), **the sequence** \( (T_n x)_{n \geq 1} \subset Y \) **is Cauchy.**

Since \( Y \) is a Banach space, for each \( x \in X \), the sequence \( (T_n)_{n \geq 1} \) will be convergent. We define the map \( T : X \to Y \) by
\[
T x = \lim_{n \to \infty} T_n x, \quad x \in X.
\]
Using (2) we immediately get
\[
\|T x\| \leq C \|x\|, \quad \forall x \in X.
\]
Since \( T \) is obviously linear, this prove that \( T \) is continuous. Finally, if we fix \( n \geq N_\varepsilon \) and we take \( \lim_{m \to \infty} \) in (3), we get
\[
\|T_n x - T x\| \leq \varepsilon \|x\|, \quad \forall n \geq N_\varepsilon, \quad x \in X,
\]
which proves precisely that we have the inequality
\[
\|T_n - T\| \leq \varepsilon, \quad \forall n \geq N_\varepsilon,
\]
hence \( (T_n)_{n \geq 1} \) is convergent to \( T \) in the norm topology. \( \square \)
COROLLARY 3.1. If $X$ is a normed vector space, then its topological dual $X^* = \mathcal{L}(X, \mathbb{K})$ is a Banach space.

**Proof.** Immediate from the fact that $\mathbb{K}$ is a Banach space. \qed

As a direct application of the above result we get

COROLLARY 3.2. If $I$ is a non-empty set, if $p \in [1, \infty]$, then $\ell^p(I)$ is a Banach space.

**Proof.** For $p = 1$ we know that $\ell^1 \simeq (c_0)^*$. For $p \in (1, \infty]$, we know that $\ell^p \simeq (\ell^q)^*$, where $q$ is Hölder conjugate to $p$. \qed

**Proposition 3.2.** Let $X$ be a Banach space, and let $Z \subset X$ be a linear subspace. The following are equivalent:

(i) $Z$ is a Banach space, when equipped with the norm from $X$;
(ii) $Z$ is closed in $X$, in the norm topology.

**Proof.** This is a particular case of a general result from the theory of complete metric spaces. \qed

COROLLARY 3.3. Let $I$ be a non-empty set, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Then $c_0^\mathbb{K}(I)$ is a Banach space.

**Proof.** Use the fact that $c_0^\mathbb{K}(I)$ is closed in $\ell^\infty(I)$. \qed

**Exercise 4.** Let $X$ be an infinite dimensional Banach space, and let $B$ be a linear basis for $X$. Prove that $B$ is uncountable.

**Hint:** If $B$ is countable, say $B = \{b_n : n \in \mathbb{N}\}$, then

$$X = \bigcup_{n=1}^{\infty} F_n,$$

where $F_n = \text{Span}(b_1, b_2, \ldots, b_n)$. Since the $F_n$'s are finite dimensional linear subspaces, they will be closed. Use Baire's Theorem to get a contradiction.

**Comments.** A third method of constructing Banach spaces is the completion. If we start with a normed $\mathbb{K}$-vector space $X$, when we regard $X$ as a metric space, its completion $\tilde{X}$ is constructed as follows. One defines

$$\text{cs}(X) = \{ \mathbf{x} = (x_n)_{n \geq 1} : (x_n)_{n \geq 1} \text{ Cauchy sequence in } X \}.$$ 

Two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{x}' = (x'_n)_{n \geq 1}$ are said to be equivalent, if

$$\lim_{n \to \infty} \| x_n - x'_n \| = 0.$$ 

In this case one writes $\mathbf{x} \sim \mathbf{x}'$. The completion $\tilde{X}$ is then defined as the space

$$\tilde{X} = \text{cs}(X) / \sim$$

of equivalence classes. For $\mathbf{x} \in \text{cs}(X)$, one denotes by $[\mathbf{x}]$ its equivalence class in $\tilde{X}$. Finally for an element $x \in X$ one denotes by $[x] \in \tilde{X}$ the equivalence class of the constant sequence $x$.

We know from general theory that $\tilde{X}$ is a complete metric space, with the distance $\tilde{d}$ (correctly) defined by

$$\tilde{d}(\tilde{x}, \tilde{x}') = \lim_{n \to \infty} \| x_n - x'_n \|,$$

for any two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{x}' = (x'_n)_{n \geq 1}$. 

It turns out that, in our situation, the space $\text{cs}(X)$ carries a natural vector space structure, defined by pointwise addition and scalar multiplication. Moreover, the space $\hat{X}$ is identified as a quotient vector space

$$\hat{X} = \text{cs}(X)/\text{ns}(X),$$

where

$$\text{ns}(X) = \{ x = (x_n)_{n \geq 1} : (x_n)_{n \geq 1} \text{ sequence in } X \text{ with } \lim_{n \to \infty} x_n = 0 \}$$

is the linear subspace of null sequences. It then follows that $\hat{X}$ carries a natural vector space structure. More explicitly, if we start with a scalar $\lambda \in K$, and with two elements $p, q \in \hat{X}$, which are represented as $p = \tilde{x}$ and $q = \tilde{y}$, for two Cauchy sequences $x = (x_n)_{n \geq 1}$ and $y = (y_n)_{n \geq 1}$ in $X$, then the sequence

$$w = (\lambda x_n + y_n)_{n \geq 1}$$

is Cauchy in $\mathcal{X}$, and the element $\lambda p + q \in \hat{X}$ is then defined as $\lambda p + q = \tilde{w}$.

Finally, there is a natural norm on $\hat{X}$, (correctly) defined by

$$\|\tilde{x}\| = \hat{d}(\tilde{x}, (0)) = \lim_{n \to \infty} \|x_n\|,$$

for all Cauchy sequences $x = (x_n)_{n \geq 1}$. These considerations then prove that $\hat{X}$ is a Banach space, and the map

$$X \ni x \longmapsto \langle x \rangle \in \hat{X}$$

is linear and isometric, in the sense that

$$\|\langle x \rangle\| = \|x\|, \ \forall \ x \in X.$$

In the context of normed vector spaces, the universality property of the completion is stated as follows:

**Proposition 3.3.** Let $X$ be a normed vector space, let $\hat{X}$ denote its completion, and let $Y$ be a Banach space. For every linear continuous map $T : X \to Y$, there exists a unique linear continuous map $\hat{T} : \hat{X} \to Y$, such that

$$\hat{T}(x) = Tx, \ \forall \ x \in X.$$

Moreover the map

$$\mathcal{L}(X, Y) \ni T \longmapsto \hat{T} \in \mathcal{L}(\hat{X}, Y)$$

is an isometric linear isomorphism.

**Proof.** If $T : X \to Y$ is linear and continuous, then $T$ is a Lipschitz map with Lipschitz constant $\|T\|$, because

$$\|Tx - Tx'\| \leq \|T\| \cdot \|x - x'\|, \ \forall x, x' \in X.$$ 

We know, from the theory of metric spaces, that there exists a unique continuous map $\hat{T} : \hat{X} \to Y$, such that

$$\hat{T}(x) = Tx, \ \forall \ x \in X.$$

We also know that $\hat{T}$ is Lipschitz, with Lipschitz constant $\|T\|$. The only thing we need to prove is the fact that $\hat{T}$ is linear. Start with two points $p, q \in \hat{X}$, represented as $p = \tilde{x}$ and $q = \tilde{z}$, for some Cauchy sequences $x = (x_n)_{n \geq 1}$ and $z = (z_n)_{n \geq 1}$ in $X$. If $\lambda \in K$, then $\lambda p + q = \tilde{w}$, where $w = (\lambda x_n + z_n)_{n \geq 1}$. We then have

$$\hat{T}(\lambda p + q) = \lim_{n \to \infty} T(\lambda x_n + z_n) = [\lambda \cdot \lim_{n \to \infty} Tx_n] + [\lim_{n \to \infty} Tz_n] = \lambda Tp + Tq.$$
Let us prove now that \( \|\tilde{T}\| = \|T\| \). Since \( \tilde{T} \) is Lipschitz, with Lipschitz constant \( \|T\| \), we will have \( \|\tilde{T}\| \leq \|T\| \). To prove the other inequality, let us consider the sets

\[ \mathcal{B}_0 = \{ p \in \tilde{X} : \|p\| \leq 1 \}, \mathcal{B}_1 = \{ \langle x \rangle : x \in X, \|x\| \leq 1 \}. \]

By definition, we have

\[ \|\tilde{T}\| = \sup_{p \in \mathcal{B}_0} \|\tilde{T}p\|. \]

Since we clearly have \( \mathcal{B}_0 \supset \mathcal{B}_1 \), we get

\[ \|\tilde{T}\| = \sup_{p \in \mathcal{B}_1} \|\tilde{T}p\| \geq \sup \{ \|Tx\| : x \in X, \|x\| \leq 1 \} = \|T\|. \]

The fact that the map \( \mathcal{L}(X, Y) \ni T \mapsto \tilde{T} \in \mathcal{L}(\tilde{X}, Y) \) is linear is obvious.

To prove the surjectivity, start with some \( S \in \mathcal{L}(\tilde{X}, Y) \). Consider the map \( \iota : X \ni x \mapsto \langle x \rangle \in \tilde{X} \).

Since \( \iota \) is linear and isometric, in particular it is continuous, so the composition \( T = S \circ \iota \) is linear and continuous. Notice that

\[ S(\langle x \rangle) = S(\iota(x)) = (S \circ \iota)x = Tx, \forall x \in X, \]

so by uniqueness we have \( S = \tilde{T} \).

**Corollary 3.4.** Let \( X \) be a normed space, let \( Y \) be a Banach space, and let \( T : X \to Y \) be an isometric linear map.

(i) Let \( \tilde{T} : \tilde{X} \to Y \) be the linear continuous map defined in the previous result. Then \( \tilde{T} \) is linear, isometric, and \( \tilde{T}(\tilde{X}) = \overline{T(X)} \).

(ii) \( X \) is complete, if and only of \( T(X) \) is closed in \( Y \).

**Proof.** (i). The fact that \( \tilde{T} \) is isometric, and has the range equal to \( \overline{T(X)} \) is true in general (i.e. for \( X \) metric space, and \( Y \) complete metric space). The linearity follows from the previous result.

(ii). This is obvious. \( \square \)

**Example 3.2.** Let \( X \) be a normed vector space. For every \( x \in X \) define the map \( \epsilon_x : X^* \to \mathbb{K} \) by

\[ \epsilon_x(\phi) = \phi(x), \forall \phi \in X^*. \]

Then \( \epsilon_x \) is a linear and continuous. This is an immediate consequence of the inequality

\[ |\epsilon_x(\phi)| = |\phi(x)| \leq \|x\| \cdot \|\phi\|, \forall \phi \in X^*. \]

Notice that this also proves

\[ \|\epsilon_x\| \leq \|x\|, \forall x \in X. \]

Interestingly enough, we actually have

\[ \|\epsilon_x\| = \|x\|, \forall x \in X. \]

To prove this fact, we start with an arbitrary \( x \in X \), and we consider the linear subspace

\[ Y = \mathbb{K}x = \{ \lambda x : \lambda \in \mathbb{K} \}. \]
If we define \( \phi_0 : \mathcal{Y} \to \mathcal{K} \), by
\[
\phi_0(\lambda x) = \lambda \|x\|, \quad \forall \lambda \in \mathcal{K},
\]
then it is clear that \( \phi_0(x) = \|x\| \), and
\[
|\phi_0(y)| \leq \|y\|, \quad \forall y \in \mathcal{Y}.
\]
Use then the Hahn-Banach Theorem to find \( \phi : \mathcal{X} \to \mathcal{K} \) such that \( \phi|_\mathcal{Y} = \phi_0 \), and
\[
|\phi(z)| \leq \|z\|, \quad \forall z \in \mathcal{X}.
\]
This will clearly imply \( \|\phi\| \leq 1 \), while the first condition will give \( \phi(x) = \phi_0(x) = \|x\| \). In particular, we will have
\[
\|x\| = |\phi(x)| = |\epsilon_x(\phi)| \leq \|\epsilon_x\| \cdot \|\phi\| \leq \|\phi\|.
\]

Having proven (4), we now have a linear isometric map
\[
E : \mathcal{X} \ni x \mapsto \epsilon_x \in \mathcal{X}^{**}.
\]
Since \( \mathcal{X}^{**} \) is a Banach space, we now see that \( \tilde{E} : \overline{\mathcal{X}} \to \overline{\mathcal{E}(\mathcal{X})} \) is an isometric linear isomorphism. In particular, \( \mathcal{X} \) is Banach, if and only if \( \mathcal{E}(\mathcal{X}) \) is closed in \( \mathcal{X}^{**} \).

We conclude with a series of results, which are often regarded as the “principles of Banach space theory.” These results are consequences of Baire Theorem.

**Theorem 3.1 (Uniform Boundedness Principle).** Let \( \mathcal{X} \) be a Banach space, let \( \mathcal{Y} \) be a normed vector space, and let \( \mathcal{M} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y}) \). The following are equivalent

(i) \( \sup \{\|T\| : T \in \mathcal{M}\} < \infty \);
(ii) \( \sup \{\|Tx\| : T \in \mathcal{M}\} < \infty, \forall x \in \mathcal{X} \).

**Proof.** The implication (i) \( \Rightarrow \) (ii) is trivial, because if we define
\[
\mathcal{M} = \sup \{\|T\| : T \in \mathcal{M}\},
\]
then by the definition of the norm, we clearly have
\[
\sup \{\|Tx\| : T \in \mathcal{M}\} \leq M\|x\|, \forall x \in \mathcal{X}.
\]

(ii) \( \Rightarrow \) (i). Assume \( \mathcal{M} \) satisfies condition (ii). For each integer \( n \geq 1 \), let us define the set
\[
\mathcal{F}_n = \{x \in \mathcal{X} : \|Tx\| \leq n, \forall T \in \mathcal{M}\}.
\]
It is obvious that \( \mathcal{F}_n \) is a closed subset of \( \mathcal{X} \), for each \( n \geq 1 \). Moreover, by (ii) we clearly have \( \bigcup_{n=1}^\infty \mathcal{F}_n = \mathcal{X} \). Using Baire’s Theorem, there exists some \( n \geq 1 \), such that \( \text{Int}(\mathcal{F}_n) \neq \emptyset \). This means that there exists some \( x_0 \in \mathcal{X} \) and some \( r > 0 \), such that
\[
\mathcal{F}_n \supset \overline{B}_r(x_0) = \{y \in \mathcal{X} : \|x - x_0\| \leq r\}.
\]
Put \( M_0 = \sup \{\|Tx_0\| : T \in \mathcal{M}\} \). Fix for the moment some arbitrary \( x \in \mathcal{X} \), with \( \|x\| \leq 1 \), and some arbitrary element \( T \in \mathcal{M} \). The vector \( y = x_0 + rx \) clearly belongs to \( \overline{B}_r(x_0) \), so we have \( \|Ty\| \leq n \). We then get
\[
\|Tx\| = \|T(\frac{1}{r}(y - x_0))\| = \frac{1}{r}\|Ty - Tx_0\| \leq \frac{1}{r}(\|Ty\| + \|Tx_0\|) \leq \frac{1}{r}(n + M_0).
\]
Keep \( T \) fixed, and use the above estimate, which gives
\[
\sup \{\|Tx\| : x \in \mathcal{X}, \|x\| \leq 1\} \leq \frac{n + M_0}{r},
\]
to conclude that \( \|T\| \leq \frac{n + M_0}{r} \). Since \( T \in \mathcal{M} \) is arbitrary, we finally get
\[
\sup \{\|T\| : T \in \mathcal{M}\} \leq \frac{n + M_0}{r} < \infty. \quad \Box
\]
Consider the sequence of closed sets 

\[ \varepsilon > T \]

Start off by choosing \( T \). The Claim will follow, once we prove the inclusion 

\[ v \in \text{Int}(T) \Rightarrow \bigcup_{k \geq 1} T(kA) = T(kA), \quad \forall k \geq 1. \]

In particular, we have

\[ \bigcup_{k=1}^{\infty} kT(A) = \bigcup_{k=1}^{\infty} T(kA) \supset \bigcup_{k=1}^{\infty} T(kA) = T(\bigcup_{k=1}^{\infty} kA). \]

Since we obviously have \( \bigcup_{k=1}^{\infty} kA = X \), and \( T \) is surjective, the above equality shows that \( \bigcup_{k=1}^{\infty} kT(A) = Y \). Using Baire’s Theorem, there exists some \( k \geq 1 \), such that \( \text{Int}(kT(A)) \neq \emptyset \). Again using the fact that \( v \mapsto kv \) is a homeomorphism, this gives \( \text{Int}(T(A)) \neq \emptyset \). Fix now some point \( y \in \text{Int}(T(A)) \), and some \( r > 0 \), such that \( T(A) \) contains the open ball

\[ B_r(y) = \{ z \in Y : \| z - y \| < r \}. \]

The proof of the Claim is then finished, once we prove the inclusion 

\[ T(A) \supset B_r(0). \]

To prove this inclusion, start with some arbitrary \( v \in B_r(0) \), i.e. \( v \in Y \) and \( \| v \| < \frac{r}{2} \). Since \( \| (2v + y) - y \| = 2\| v \| < r \), using (5) it follows that \( 2v + y \in T(A) \). i.e. there exists a sequence \( (x_n)_{n=1}^{\infty} \subset X \) with \( \| x_n \| < 1 \), \( \forall n \geq 1 \), and \( 2v + y = \lim_{n \to \infty} T x_n \). Since \( y \) itself belongs to \( T(A) \), there also exists some sequence \( (z_n)_{n=1}^{\infty} \subset X \), with \( \| z_n \| < 1 \), \( \forall n \geq 1 \), and \( y = \lim_{n \to \infty} T z_n \). On the one hand, if we consider the sequence \( (u_n)_{n=1}^{\infty} \subset X \) given by \( u_n = \frac{1}{2}(x_n - z_n) \), then it is clear that

\[ \| u_n \| \leq \frac{1}{2}(\| x_n \| + \| z_n \|) < 1, \quad \forall n \geq 1, \]

i.e. \( (u_n)_{n=1}^{\infty} \subset A \). On the other hand, we have

\[ \lim_{n \to \infty} T u_n = \lim_{n \to \infty} \frac{1}{2}(T x_n - T z_n) = \frac{1}{2}(2v + y - y) = v, \]

so \( v \) indeed belongs to \( T(A) \).

The next step is a slight (but crucial) improvement of Claim 1.

**Claim 2:** \( T(A) \) is a neighborhood of 0.

Start off by choosing \( \varepsilon > 0 \), such that

\[ T(A) \supset B_{\varepsilon}(0). \]

The Claim will follow, once we prove the inclusion

\[ T(A) \supset B_{\frac{\varepsilon}{2}}(0). \]
To prove this inclusion, we start with some arbitrary \( y \in B_r(0) \). We want to construct a sequence of vectors \((x_n)_{n=1}^\infty \subset A\), such that, for every \( n \geq 1 \), we have the inequality

\[
\left\| y - \frac{1}{2^n} \sum_{k=1}^n T(x_k) \right\| \leq \frac{\varepsilon}{2^{n+1}}.
\]

This sequence is constructed inductively as follows. We start by using (6), and we pick \( x_1 \in A \) such that \( \|2y - T x_1\| < \frac{\varepsilon}{2} \). Once \( x_1, \ldots, x_p \) are constructed, such that (8) holds with \( n = p \), we consider the vector

\[
z = 2^{p+1} \left[ y - \sum_{k=1}^p T \left( \frac{1}{2^p} T x_k \right) \right] \in B_r(0),
\]

and we use again (6) to find \( x_{p+1} \in A \), such that \( \|z - T x_{p+1}\| \leq \frac{\varepsilon}{2} \). We then clearly have

\[
\left\| y - \frac{1}{2^p+1} \sum_{k=1}^{p+1} T \left( \frac{1}{2^p} T x_k \right) \right\| = \left\| z - \frac{T x_{p+1}}{2^{p+1}} \right\| \leq \frac{\varepsilon}{2^{p+2}}.
\]

Consider now the series \( \sum_{k=1}^\infty \frac{1}{2^k} x_k \). Since \( \|x_k\| < 1 \), \( \forall k \geq 1 \), and \( X \) is a Banach space, by Remark 3.1, the sequence of \( (w_n)_{n=1}^\infty \subset X \) of partial sums

\[
w_n = \sum_{k=1}^n \frac{1}{2^k} x_k, \quad n \geq 1,
\]

is convergent to some point \( x \in X \). Moreover, since we have

\[
\|w_n\| \leq \sum_{k=1}^n \frac{\|x_k\|}{2^k} \leq \sum_{k=1}^\infty \frac{\|x_k\|}{2^k}, \quad \forall n \geq 1,
\]

we get the inequality

\[
\|x\| \leq \sum_{k=1}^\infty \frac{\|x_k\|}{2^k} < 1,
\]

which means that \( x \in A \). Note also that using these partial sums, the inequality (8) reads

\[
\|y - T w_n\| \leq \frac{\varepsilon}{2^{n+2}}, \quad \forall n \geq 1,
\]

so by the continuity of \( T \), we have \( y = T x \in T(A) \).

Let us show now that \( T^{-1} \) is continuous. Use Claim 2, to find some \( r > 0 \) such that

\[
T(A) \supset B_r(0),
\]

and let \( y \in Y \) be an arbitrary vector with \( \|y\| \leq 1 \). Consider the vector \( v = \frac{r}{2} y \), which has \( \|v\| \leq \frac{r}{2} \). By (9), there exists \( x \in A \), such that \( T x = v \), which means that \( T^{-1} y = \frac{r}{2} x \). This forces \( \|T^{-1} y\| \leq \frac{r}{2} \). This argument shows that

\[
\sup \{ \|T^{-1} y\| : y \in Y, \|y\| \leq 1 \} \leq \frac{2}{r} < \infty,
\]

and the continuity of \( T^{-1} \) follows from Proposition 2.4.

The following two exercises deal with two more “principles of Banach space theory.”
Exercise 5\(\blacklozenge\). (Closed Graph Theorem). Let \(X\) and \(Y\) be Banach spaces, and let \(T : X \to Y\) be a linear map. Prove that the following are equivalent:

(i) \(T\) is continuous.

(ii) The graph of \(T\)

\[ G_T = \{(x, Tx) : x \in X\} \]

is a closed subset of \(X \times Y\), in the product topology.

Hint: For the implication \((ii) \Rightarrow (i)\), use Exercise 3, to get the fact that \(G_T\) is a Banach space. Then \(T\) is exactly the inverse of \(\pi_X|_{G_T}\), where \(\pi_X : X \times Y \to X\) is the projection onto the first coordinate. Use Theorem 3.2.

Exercise 6\(\blacklozenge\). (Open Mapping Theorem). Let \(X\) and \(Y\) be Banach spaces, and let \(T : X \to Y\) be a surjective linear continuous map. Prove that \(T\) is an open map, in the sense that

- whenever \(D \subset X\) is open, it follows that \(T(D)\) is open in \(Y\).

Hint: Consider the linear map

\[ S : X \times Y \ni (x, y) \mapsto (x, Tx + y) \in X \times Y. \]

Prove that \(S\) is linear, continuous, bijective, hence by Theorem 3.2, it is a homeomorphism. Use this fact to prove that for every open set \(D \subset X\), there exists some open set \(E \subset X \times Y\), such that \(T(D) = \pi_Y(E)\), where \(\pi_Y : X \times Y \to Y\) is the projection onto the second coordinate. This reduces the problem to proving the fact that \(\pi_Y\) is an open map.