Lecture 6

6. Metric spaces

In this section we review the basic facts about metric spaces.

Definitions. A metric on a non-empty set $X$ is a map

$$d : X \times X \to [0, \infty)$$

with the following properties:

(i) If $x, y \in X$ are points with $d(x, y) = 0$, then $x = y$;
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

A metric space is a pair $(X, d)$, where $X$ is a set, and $d$ is a metric on $X$.

Notations. If $(X, d)$ is a metric space, then for any point $x \in X$ and any $r > 0$, we define the open and closed balls:

$$B_r(x) = \{ y \in X : d(x, y) < r \},$$
$$\overline{B}_r(x) = \{ y \in X : d(x, y) \leq r \}.$$

Definition. Suppose $(X, d)$ is a metric space. Then $X$ carries a natural topology constructed as follows. We say that a set $D \subset X$ is open, if it has the property:

- for every $x \in D$, there exists some $r_x > 0$, such that $B_{r_x}(x) \subset D$.

One can prove that the collection

$$T_d = \{ D \subset X : D \text{ open} \}$$

is indeed a topology, i.e. we have

- $\emptyset$ and $X$ are open;
- if $(D_i)_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} D_i$ is again open;
- if $D_1$ and $D_2$ are open, then $D_1 \cap D_2$ is again open.

The topology thus constructed is called the metric topology.

Remark 6.1. Let $(X, d)$ be a metric space. Then for every $p \in X$, and for every $r > 0$, the set $B_r(p)$ is open, and the set $\overline{B}_r(p)$ is closed.

If we start with some $x \in B_r(p)$, an if we define $r_x = r - d(x, p)$, then for every $y \in B_{r_x}(x)$ we will have

$$d(y, p) \leq d(y, x) + d(x, p) < r_x + d(x, p) = r,$$

so $y$ belongs to $B_r(p)$. This means that $B_{r_x}(x) \subset B_r(p)$. Since this is true for all $x \in B_r(p)$, it follows that $B_r(p)$ is indeed open.
To prove that $\mathcal{B}_r(p)$ is closed, we need to show that its complement
\[ X \setminus \mathcal{B}_r(p) = \{ x \in X : d(x, p) > r \} \]
is open. If we start with some $x \in X \setminus \mathcal{B}_r(p)$, an if we define $\rho_x = d(p, x) - r$, then for every $y \in \mathcal{B}_{\rho_x}(x)$ we will have
\[ d(y, p) \geq d(p, x) - d(y, x) > d(p, x) - \rho_x = r, \]
so $y$ belongs to $X \setminus \mathcal{B}_r(p)$. This means that $\mathcal{B}_{\rho_x}(x) \subset X \setminus \mathcal{B}_r(p)$. Since this is true for all $x \in X \setminus \mathcal{B}_r(p)$, it follows that $X \setminus \mathcal{B}_r(p)$ is indeed open.

**Remark 6.2.** The metric topology on a metric space $(X, d)$ is Hausdorff. Indeed, if we start with two points $x, y \in X$, with $x \neq y$, then if we choose $r$ to be a real number, with
\[ 0 < r < \frac{d(x, y)}{2}, \]
then we have $\mathcal{B}_r(x) \cap \mathcal{B}_r(y) = \emptyset$. (Otherwise, if we have a point $z \in \mathcal{B}_r(x) \cap \mathcal{B}_r(y)$, we would have $2r < d(x, y) \leq d(x, z) + d(y, z) < 2r$, which is impossible.)

**Remark 6.3.** Let $(X, d)$ be a metric space, and let $M$ be a subset of $X$. Then $d|_{M \times M}$ is a metric on $M$, and the metric topology on $M$ defined by this metric is precisely the induced topology from $X$. This means that a set $A \subset M$ is open in $M$ if and only if there exists some open set $D \subset X$ with $A = M \cap D$.

The metric space framework is particularly convenient because one can use convergence.

**Definition.** Let $(X, d)$ be a metric space. For a point $x \in X$, we say that a sequence $(x_n)_{n \geq 1} \subset X$ is convergent to $x$, if $\lim_{n \to \infty} d(x_n, x) = 0$.

**Remark 6.4.** Let $(X, d)$ be a metric space, and if the sequence $(x_n)_{n \geq 1} \subset X$ is convergent to some point $x \in X$, then
\[ (1) \quad \lim_{n \to \infty} d(x_n, y) = d(x, y), \ \forall y \in X. \]
This is an immediate consequence of the inequalities
\[ d(x, y) - d(x_n, x) \leq d(x_n, y) \leq d(x, y) + d(x_n, x). \]

Among other things, the equality (1) gives the fact that $(x_n)_{n \geq 1}$ cannot be convergent to any other point $y \neq x$. Therefore, if $(x_n)_{n \geq 1}$ is convergent to some $x$, then $x$ is uniquely determined, and will be denoted by $\lim_{n \to \infty} x_n$.

Convergence is useful for characterizing closure.

**Proposition 6.1.** Let $(X, d)$ be a metric space, and let $A \subset X$ be a non-empty subset. For a point $x \in X$, the following are equivalent:
\begin{enumerate}[(i)]
    \item $x$ belongs to the closure $\overline{A}$ of $A$;
    \item there exists some sequence $(x_n)_{n \geq 1} \subset A$, with $\lim_{n \to \infty} x_n = x$.
\end{enumerate}

**Proof.** $(i) \Rightarrow (ii)$. Assume $x \in \overline{A}$. This means that
\[ (\ast) \quad \text{For every open set } D \subset X \text{ with } D \ni x, \text{ the intersection } D \cap A \text{ is non-empty}. \]
We use this property for the open sets $\mathcal{B}_{1/n}(x), n = 1, 2, \ldots$. So, for every integer $n \geq 1$, we can find a point $x_n \in \mathcal{B}_{1/n}(x) \cap A$. This way we have built a sequence $(x_n)_{n \geq 1} \subset A$, such that

$$d(x_n, x) < \frac{1}{n}, \forall n \geq 1.$$

It is clear that this gives $x = \lim_{n \to \infty} x_n$.

$(ii) \Rightarrow (i)$. Assume $x$ satisfies property $(ii)$. Fix $(x_n)_{n \geq 1} \subset A$ to be a sequence with $\lim_{n \to \infty} x_n = x$. We need to prove property $(\ast)$. Start with some arbitrary open set $D \subset X$, with $x \in D$. Let $\varepsilon > 0$ be chosen such that $\mathcal{B}_\varepsilon(x) \subset D$. Since $\lim_{n \to \infty} d(x_n, x) = 0$, there exists some $n_\varepsilon$ such that $d(x_{n_\varepsilon}, x) < \varepsilon$. It is now clear that

$$x_{n_\varepsilon} \in \mathcal{B}_\varepsilon(x) \cap A \subset D \cap A,$$

so the intersection $D \cap A$ is indeed non-empty. \hfill \Box

Continuity can be characterized using convergence, as follows.

**Proposition 6.2.** Let $X$ and $Y$ be metric spaces, and let $f : X \to Y$ be a function. For a point $p \in X$, the following are equivalent:

(i) $f$ is continuous at $p$;

(ii) for every $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$ such that $d(f(x), f(p)) < \varepsilon$, for all $x \in X$ with $d(x, p) < \delta_\varepsilon$.

(iii) if $(x_n)_{n \geq 1} \subset X$ is a sequence with $\lim_{n \to \infty} x_n = p$, then $\lim_{n \to \infty} f(x_n) = f(p)$.

**Proof.** $(i) \Rightarrow (ii)$. The condition that $f$ is continuous at $p$ means

$(\ast)$ for every open set $D \subset Y$, with $D \ni f(p)$, there exists some open set $E \subset X$, with $p \in E \subset f^{-1}(D)$.

Assume $f$ is continuous at $p$. For every $\varepsilon > 0$, we consider the open ball $\mathcal{B}_\varepsilon^Y(f(p))$. Using $(\ast)$, there exists some open set $E \subset X$, with $E \ni p$, and $f(E) \subset \mathcal{B}_\varepsilon^Y(f(p))$. In particular, there exists $\delta > 0$, such that $\mathcal{B}_\delta^X(p) \subset E$, so now we have

$$f(\mathcal{B}_\delta^X(p)) \subset \mathcal{B}_\varepsilon^Y(f(p)),$$

which clearly gives $(ii)$.

$(ii) \Rightarrow (iii)$. Assume $f$ satisfies $(ii)$, and start with some sequence $(x_n)_{n \geq 1} \subset X$, which converges to $p$. For every $\varepsilon > 0$, we choose $\delta_\varepsilon > 0$ as in $(ii)$, and using the fact that $\lim_{n \to \infty} x_n = p$, we can also choose some $N_\varepsilon$ such that

$$d(x_n, p) < \delta_\varepsilon, \forall n \geq N_\varepsilon.$$

Using $(ii)$ this will give

$$d(f(x_n), f(p)) < \varepsilon, \forall n \geq N_\varepsilon.$$

In other words, we get the fact that

$$\lim_{n \to \infty} (f(x_n), f(p)) = 0,$$

which means that we indeed have $\lim_{n \to \infty} f(x_n) = f(p)$.

$(iii) \Rightarrow (i)$. Assume $f$ satisfies $(iii)$, but $f$ is not continuous at $p$. By $(\ast)$ this means that there exists some open set $D_0 \subset Y$ with $D_0 \ni f(p)$, such that

$(\ast')$ for every open set $E \subset X$ with $E \ni p$, we have $f(E) \not\subset D_0$. 

\hfill\Box
It is clear that any other open set $D$, with $f(p) \in D \subset D_0$, will again satisfy property $(\ast')$. Fix then some $r > 0$, such that $B^Y_r(f(p)) \subset D_0$. Using condition $(\ast')$ it follows that for every integer $n \geq 1$, we have

$$f(B^X_{1/n}(p)) \not\subset B^Y_r(f(p)).$$

This means that, for every integer $n \geq 1$, we can find a point $x_n \in X$ such that

$$d(x_n, p) < \frac{1}{n} \text{ and } d(f(x_n), f(p)) \geq r.$$

It is then clear that the sequence $(x_n)_{n \geq 1} \subset X$ is convergent to $p$, but the sequence $(f(x_n))_{n \geq 1} \subset Y$ is not convergent to $f(p)$. This will contradict (iii). □

Convergence can also be used for characterizing compactness.

**Theorem 6.1.** Let $(X, d)$ be a metric space. The following are equivalent:

(i) $X$ is compact in the metric topology;

(ii) every sequence has a convergent subsequence.

**Proof.** (i) $\Rightarrow$ (ii). Assume $X$ is compact. Start with an arbitrary sequence $(x_n)_{n \geq 1} \subset X$. For every $n \geq 1$, we define the closed set

$$T_n = \{x_k : k > n\}.$$

It is obvious that the family of closed sets $(T_n)_{n \geq 1}$ has the finite intersection property, i.e. for every finite set $F$ of indices, we have

$$\bigcap_{n \in F} T_n \neq \emptyset.$$ (This follows from the fact that the $T_n$'s form a decreasing sequence of sets.) By compactness, it follows that

$$\bigcap_{n \geq 1} T_n \neq \emptyset.$$

Take a point $x \in \bigcap_{n \geq 1} T_n$. The key feature of $x$ is the given by the following:

**Claim 1:** For every $\varepsilon > 0$ and every integer $\ell \geq 1$, there exists some integer $N(\varepsilon, \ell) > \ell$ such that $d(x_{N(\varepsilon, \ell)}, x) < \varepsilon$.

This is a consequence of the fact that, for every $\ell \geq 1$, the point $x$ belongs to the closure $\{x_N : N > \ell\}$, so for every $\varepsilon > 0$ we have

$$B_X(x) \cap \{x_N : N > \ell\} \neq \emptyset.$$

Using Claim 1, we define a sequence $(k_n)_{n \geq 0}$ of integers, recursively by

$$k_n = N(\frac{1}{n}, k_{n-1}), \ \forall n \geq 1.$$

(The initial term $k_0$ is chosen arbitrarily.) We have, by construction, $k_0 < k_1 < k_2 < \ldots$, and

$$d(x_{k_n}, x) < \frac{1}{n}, \ \forall n \geq 1,$$

so $(x_{k_n})_{n \geq 1}$ is indeed a subsequence of $(x_k)_{k \geq 1}$, which is convergent (to $x$).

(ii) $\Rightarrow$ (i). Assume (ii). Before we start proving that $X$ is compact, We shall need some preparations.

**Claim 2:** For every $r > 0$ there exists a finite set $F \subset X$, such that

$$X = \bigcup_{x \in F} B_r(x).$$
We prove this by contradiction. Assume there exists some \( r > 0 \), such that 
\[
\bigcup_{x \in F} B_r(x) \subsetneq X,
\]
for every finite set \( F \subset X \). In particular, there exists a sequence \( (x_n)_{n \geq 1} \) such that 
\[
x_{n+1} \in X \setminus \left[ B_r(x_1) \cup \cdots \cup B_r(x_n) \right], \quad \forall n \geq 1.
\]
This will force 
\[d(x_m, x_n) \geq r, \quad \forall m > n \geq 1.\]
Notice that every subsequence \( (x_{k_n})_{n \geq 1} \) will satisfy the same property 
\[d(x_{k_m}, x_{k_n}) \geq r, \quad \forall m > n \geq 1.\]
This proves that no subsequence of \( (x_n)_{n \geq 1} \) is Cauchy, so no subsequence of \( (x_n)_{n \geq 1} \) can be convergent, thus contradicting (ii).

Having proven Claim 2, we choose, for every integer \( n \geq 1 \), finite set \( F_n \) such that 
\[X = \bigcup_{x \in F_n} B_{\frac{1}{n}}(x) .\]

**Claim 3:** The collection \( \mathcal{W} = \{ B_{\frac{1}{n}}(x) : n \in \mathbb{N}, x \in F_n \} \) is a base for the metric topology.

What we need to show is that every open set is a union of sets in \( \mathcal{W} \). Fix an open set \( D \) and a point \( p \in D \). Choose \( r > 0 \), such that \( B_r(p) \subset D \). Choose then some integer \( n \geq 1 \), such that \( \frac{1}{n} < \frac{r}{2} \), and choose some point \( x \in F_n \), such that \( p \in B_{\frac{1}{n}}(x) \). Notice that, for every \( y \in B_{\frac{1}{n}}(x) \), we have 
\[d(y, p) \leq d(y, x) + d(x, p) < \frac{1}{n} + \frac{1}{n} \leq r,
\]
which proves that \( y \in B_r(p) \). Therefore we have 
\[p \in B_{\frac{1}{n}}(x) \subset B_r(p) \subset D.
\]
Since \( p \in D \) is arbitrary, this proves that \( D \) is a union of sets in \( \mathcal{W} \).

We now begin proving that \( X \) is compact. Start with a collection \( (D_i)_{i \in I} \) of open sets, with \( \bigcup_{i \in I} D_i = X \). We need to find a finite set of indices \( I_0 \subset I \), such that \( \bigcup_{i \in I_0} D_i = X \). First we show that:

**Claim 4:** There exists a countable set of indices \( I_1 \subset I \), such that 
\[\bigcup_{i \in I_1} D_i = X .\]

The key fact is that the base \( \mathcal{W} \) is countable. Let us enumerate the base \( \mathcal{W} \) as a sequence 
\[\mathcal{W} = \{ W_m : m \in \mathbb{N} \} .\]
For each \( i \in I \), we define the set 
\[M_i = \{ m \geq 1 : W_m \subset D_i \} .\]
By Claim 3, we know that for every \( x \in D_i \) there exists some \( m \in M_i \) such that \( x \in W_m \subset D_i \). In particular this proves the equality 
\[D_i = \bigcup_{m \in M_i} W_m, \quad \forall i \in I .\]
Consider then the union $M = \bigcup_{i \in I} M_i$, which is countable, being a subset of the integers. We clearly have
\[
\bigcup_{m \in M} W_m = \bigcup_{i \in I} \left( \bigcup_{m \in M_i} W_m \right) = \bigcup_{i \in I} D_i = X.
\]
For every $m \in M$ we choose an $i_m \in I$, such that $m \in M_{i_m}$. If we take
\[
I_1 = \{i_m : m \in M\},
\]
then $I_1$ is obviously countable, and since we clearly have $W_m \subset D_{i_m}$, we get
\[
X = \bigcup_{m \in M} W_m \subset \bigcup_{m \in M} D_{i_m} = \bigcup_{i \in I_1} D_i,
\]
so the Claim is proven.

Let us list the countable set $I_1$ as
\[
I_1 = \{i_k : k \geq 1\}.
\]
(Of course, if $I_1$ is already finite, there is nothing to prove. So we will assume that $I_1$ is infinite.) In order to finish the proof, we must find some $k$, such that
\[
D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k} = X,
\]
in other words, if we define for each $k \geq 1$, the close set
\[
A_k = X \setminus (D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k}),
\]
we have
\[
A_k \neq \emptyset, \quad \forall k \geq 1.
\]
For each $k \geq 1$ we choose a point $x_k \in A_k$. This way we have constructed a sequence $(x_k)_{k \geq 1} \subset X$, so using property (i) we can find a convergent subsequence. This means that we have a sequence of integers
\[
1 \leq k_1 < k_2 < \ldots
\]
and a point $x \in X$, such that $\lim_{n \to \infty} x_{k_n} = x$. Notice that, since
\[
k_n \geq n, \quad \forall n \geq 1,
\]
and since the sequence $(A_k)_{k \geq 1}$ is decreasing, we get the fact that, for each $m \geq 1$, we have
\[
x_{k_n} \in A_m, \quad \forall n \geq m.
\]
Since $A_m$ is closed, this forces $x \in A_m$, for all $m \geq 1$. But this is clearly impossible, since
\[
\bigcap_{m \geq 1} A_m = X \setminus \left( \bigcup_{m \geq 1} (D_{i_1} \cup \cdots \cup D_{i_m}) \right) = X \setminus \left( \bigcup_{i \in I_1} D_i \right) = \emptyset.
\]

\textbf{Corollary 6.1 (of the proof).} Every compact metric space is second countable, which means that there exists a sequence $(W_m)_{m \geq 1}$ of open sets, with the property
\[(b) \quad \text{for every open set } D, \text{ there exists a subset } M \subset \mathbb{N} \text{ such that }
D = \bigcup_{m \in M} W_m.
\]

\textbf{Proof.} Use (i) and the steps in the proof of (i) $\Rightarrow$ (ii), up to the proof of Claim 3. \qed
Corollary 6.2. Let \((X, d)\) be a metric space. For a subset \(K \subset X\) the following are equivalent:

(i) every sequence in \(K\) has a subsequence which is convergent to some point in \(K\);
(ii) \(K\) is compact in \(X\).

Proof. (i) \(\Rightarrow\) (ii). By the above Theorem, we know that when we equip \(K\) with the metric \(d|_{K \times K}\), then \(K\) is compact. This means that \(K\) is compact in the induced topology, which means exactly that \(K\) is compact in \(X\).

(ii) \(\Rightarrow\) (i). Argue as above. If \(K\) is compact in \(X\), then \(K\) is compact when equipped with the induced topology, which means that \((K, d|_{K \times K})\) is compact. \(\square\)

Corollary 6.3. Let \(X\) and \(Y\) be metric spaces, and let \(f : X \to Y\) be a continuous map. If \(X\) is compact, then \(f\) is uniformly continuous, that is,

- for every \(\varepsilon > 0\), there exists some \(\delta_\varepsilon > 0\), such that
  \[d(f(x), f(x')) < \varepsilon, \text{ for all } x, x' \in X \text{ with } d(x, x') < \delta_\varepsilon.\]

Proof. Suppose \(f\) is not uniformly continuos, so there exists some \(\varepsilon_0 > 0\), with the property that for any \(\delta > 0\) there exists \(x, x' \in X\), with \(d(x, x') < \delta\), but \(d(f(x), f(x')) \geq \varepsilon_0\). In particular, one can construct two sequences \((x_n)_{n \geq 1}\) and \((x'_n)_{n \geq 1}\) with

\[d(x_n, x'_n) < \frac{1}{n} \text{ and } d(f(x_n), f(x'_n)) \geq \varepsilon_0, \forall n \geq 1. \tag{2}\]

Using compactness, we can find a subsequence \((x_{n_k})_{k \geq 1}\) of \((x_n)_{n \geq 1}\) which converges to some point \(p\). On the one hand, we have

\[d(p, x'_n) \leq d(p, x_{n_k}) + d(x_{n_k}, x'_n) < d(p, x_{n_k}) + \frac{1}{n_k}, \forall k \geq 1, \tag{3}\]

which proves that

\[\lim_{k \to \infty} x'_{n_k} = p.\]

On the other hand, using (2) we also have

\[\varepsilon_0 \leq d(f(x_{n_k}), f(x'_{n_k})) \leq d(f(p), f(x_{n_k})) + d(f(p), f(x'_{n_k})), \]

which leads to a contradiction, because the equalities

\[\lim_{k \to \infty} x_{n_k} = \lim_{k \to \infty} x'_{n_k} = p,\]

together with the continuity of \(f\), will force

\[\lim_{k \to \infty} d(f(p), f(x_{n_k})) = \lim_{k \to \infty} d(f(p), f(x'_{n_k})) = 0. \tag*{\square}\]

Remark 6.5. Let \(X\) be a metric space. Then any compact subset \(K \subset X\) is closed (this is a consequence of the fact that \(X\) is Hausdorff) and bounded, in the sense that for every \(p \in X\) we have

\[\sup_{x \in K} d(x, p) < \infty.\]

This is a consequence of the continuity (see ??) of the map

\[K \ni x \mapsto d(x, p) \in [0, \infty).\]
In general however the converse is not true, i.e. there are metric spaces in which closed bounded sets may fail to be compact.

**Exercise 1.** Equip $\mathbb{R}$ with the metric
\[ d(x, y) = \frac{|x - y|}{1 + |x - y|}, \quad \forall x, y \in \mathbb{R}. \]
Prove that $d$ is indeed a metric on $\mathbb{R}$, and the metric topology on $\mathbb{R}$ defined by $d$ is the usual topology. Prove that $\mathbb{R}$ is bounded with respect to this metric.

**Exercise 2.** Start with a metric space $X$, and let $(x_n)_{n \geq 1} \subset X$ be a sequence which is convergent to some point $x$. Prove that the set
\[ K = \{x\} \cup \{x_n : n \geq 1\} \]
is compact in $X$.

**Definition.** Let $(X, d)$ be a metric space. For a point $x \in X$ and a non-empty subset $A \subset X$, one defines the distance from $x$ to $A$ as the number
\[ d(x, A) = \inf \{d(x, a) : a \in A\}. \]

**Exercise 3.** Let $(X, d)$ be a metric space, and let $A$ be a non-empty subset of $X$.
(i) For a point $x \in X$, prove that the equality $d(x, A) = 0$ is equivalent to the fact that $x \in \overline{A}$.
(ii) Prove the inequality
\[ |d(x, A) - d(y, A)| \leq d(x, y), \quad \forall x, y \in X. \]
Using (ii) conclude that the map
\[ X \ni x \mapsto d(x, A) \in [0, \infty) \]
is continuous.

**Proposition 6.3.** Let $(X, d)$ be a metric space. When equipped with the metric topology, $X$ is normal.

**Proof.** Let $A$ and $B$ be closed subsets of $X$ with $A \cap B = \emptyset$. We need to find open sets $U, V \subset X$, with $U \supset A$, $V \supset B$, and $U \cap V = \emptyset$. We are going to use a converse of Urysohn Lemma. More explicitly, let us define the function $f : X \to [0, 1]$ by
\[ f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \quad x \in X. \]
Notice that by Exercise 3, both the numerator and denominator are continuous, and the denominator never vanishes. So $f$ is indeed continuous. It is obvious that $f|A = 0$ and $f|B = 1$, so if we take the open sets $U = f^{-1}((-\infty, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, \infty))$, we clearly get the desired result. \qed

We continue now with a discussion on completeness.

**Definitions.** Let $(X, d)$ be a metric space. A sequence $(x_n)_{n \geq 1} \subset X$ is said to be a Cauchy sequence, if it has the following property.

(C) For every $\varepsilon > 0$, there exists some integer $N_\varepsilon \geq 1$ such that
\[ d(x_m, x_n) < \varepsilon, \quad \forall m, n \geq N_\varepsilon. \]
The metric space \((X, d)\) is said to be *complete*, if every Cauchy sequence is convergent.

The following result summarizes some equivalent characterizations of completeness.

**Proposition 6.4.** Let \((X, d)\) be a metric space. The following are equivalent.

(i) \((X, d)\) is complete.

(ii) Every sequence \((x_n)_{n \geq 1} \subset X\), with

\[
\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty,
\]

is convergent.

(iii) Every Cauchy sequence has a convergent subsequence.

**Proof.** (i) \(\Rightarrow\) (ii). Assume \(X\) is complete. Let \((x_n)_{n \geq 1} \subset X\) be a sequence with property (4). To prove (ii) it suffices to show that \((x_n)_{n \geq 1}\) is Cauchy. For every \(N \geq 1\) we define

\[
R_N = \sum_{n=N}^{\infty} d(x_{n+1}, x_n).
\]

Using (4) we get \(\lim_{N \to \infty} R_N = 0\), so for every \(\varepsilon > 0\) there exists some \(N(\varepsilon)\) with \(R_N(\varepsilon) < \varepsilon\). Notice also that the sequence \((R_N)_{N \geq 1}\) is decreasing. If \(m > n \geq N(\varepsilon)\), then

\[
d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{\infty} d(x_{k+1}, x_k) = R_n \leq R_{N(\varepsilon)} < \varepsilon,
\]

so \((x_n)_{n \geq 1}\) is indeed Cauchy.

(ii) \(\Rightarrow\) (iii). Start with some Cauchy sequence \((y_k)_{k \geq 1}\). For every \(n \geq 1\) choose an integer \(N(n) \geq 1\) such that

\[
d(x_k, x_\ell) < \frac{1}{2^n}, \quad \forall k, \ell \geq N(n).
\]

Start with some arbitrary \(k_1 \geq N(1)\) and define recursively an entire sequence \((k_n)_{n \geq 1}\) of integers, by

\[
k_{n+1} = \max\{k_n + 1, N(n + 1)\}, \quad n \geq 1.
\]

Clearly we have \(k_1 < k_2 < \ldots\), and since we have

\[
k_{n+1} > k_n \geq N(n), \quad \forall n \geq 1,
\]

using (5), we get

\[
d(y_{k_{n+1}}, y_{k_n}) < \frac{1}{2^n}, \quad \forall n \geq 1.
\]

So if we define the subsequence \(x_n = y_{k_n}, \ n \geq 1\), we will have

\[
\sum_{n=1}^{\infty} d(x_{n+1}, x_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,
\]

so the subsequence \((x_n)_{n \geq 1}\) satisfies condition (4). By (ii) the subsequence \((x_n)_{n \geq 1}\) is convergent.
(iii) \(\Rightarrow\) (i). Assume condition (iii) holds. Start with some Cauchy sequence \((x_n)_{n\geq 1}\). For every integer \(n \geq 1\) we put
\[
S_n = \sup_{\ell, m \geq n} d(x_\ell, x_m).
\]
Since \((x_n)_{n\geq 1}\) is Cauchy, we have
\[
\lim_{n \to \infty} S_n = 0.
\]
Using the assumption, we can find a subsequence \((x_{k_n})_{n\geq 1}\) (defined by an increasing sequence of integers \(1 \leq k_1 < k_2 < \ldots\)) which is convergent to some point \(x\). We are going to prove that the entire sequence \((x_n)_{n\geq 1}\) is convergent to \(x\). Fix for the moment \(n \geq 1\). For every \(m \geq n\), we have
\[
S_n \geq d(x_n, x_{k_m}), \quad \forall m \geq n.
\]
By Remark 3.4, we also know that
\[
\lim_{m \to \infty} d(x_n, x_{k_m}) = d(x_n, x),
\]
so if we take \(\lim_{m \to \infty}\) in (7) we will get
\[
d(x_n, x) \leq S_n.
\]
Since this estimate holds for arbitrary \(n \geq 1\), using (6) we immediately get the fact that \((x_n)_{n\geq 1}\) is indeed convergent to \(x\). \(\square\)

**Proposition 6.5.** Suppose \((X, d)\) is a complete metric space, and \(Y\) is a subset of \(X\). The following are equivalent:

(i) \(Y\) is complete, when equipped with the metric from \(X\);
(ii) \(Y\) is closed in \(X\), in the metric topology.

**Proof.** (i) \(\Rightarrow\) (ii). Assume \(Y\) is complete, and let us prove that \(Y\) is closed. Start with a point \(x \in \overline{Y}\). Then there exists a sequence \((y_n)_{n\geq 1} \subset Y\) with \(\lim_{n \to \infty} y_n = x\). Notice that \((y_n)_{n\geq 1}\) is Cauchy in \(Y\), so by assumption, \((y_n)_{n\geq 1}\) is convergent to some point in \(Y\). This will then clearly force \(x \in Y\).

(ii) \(\Rightarrow\) (i). Assume \(Y\) is closed, and let us prove that \(Y\) is complete. Start with a Cauchy sequence \((y_n)_{n\geq 1} \subset Y\). Since \(X\) is complete, the sequence \((y_n)_{n\geq 1}\) is convergent to some point \(x \in X\). Since \(Y\) is closed, this forces \(x \in Y\). \(\square\)

**Remark 6.6.** Using Theorem 6.1, we immediately see that a metric space, which is compact in the metric topology, is automatically complete.

The next result identifies those complete metric spaces that are compact. In order to formulate it, we need the following:

**Definition.** Let \((X, d)\) be a metric space, and let \(\varepsilon > 0\). A subset \(A \subset X\) is said to be \(\varepsilon\)-rare, if
\[
d(a, b) \geq \varepsilon, \text{ for all } a, b \in A \text{ with } a \neq b.
\]

**Proposition 6.6.** Let \((X, d)\) be a complete metric space. The following are equivalent:

(i) \(X\) is compact in the metric topology;
(ii) for each \(\varepsilon > 0\), all \(\varepsilon\)-rare subsets of \(X\) are finite;
(iii) for any \(\varepsilon > 0\), there exist finitely many points \(p_1, p_2, \ldots, p_n \in X\), such that
\[
X = B_{\varepsilon}(p_1) \cup B_{\varepsilon}(p_2) \cup \cdots \cup B_{\varepsilon}(p_n).
\]
Proof. (i) ⇒ (ii). Assume $X$ is compact. We prove (ii) by contradiction. Assume there exists some $\varepsilon > 0$ and an infinite $\varepsilon$-rare set $A \subset X$. It then follows that there exists a sequence $(a_n)_{n \geq 1} \subset A$, such that
\[d(a_m, a_n) \geq \varepsilon, \quad \forall m > n \geq 1.\]
It is clear that no subsequence of $(a_n)_{n \geq 1}$ is Cauchy, which means that $(a_n)_{n \geq 1}$ does not have any convergent subsequence, thus contradicting the fact that $X$ is compact.

(ii) ⇒ (iii). Assume property (ii) and let us prove (iii) by contradiction. Assume there exists some $\varepsilon > 0$, such that, for every finite set $F \subset X$, one has a strict inclusion
\[\bigcup_{x \in F} \mathcal{B}_\varepsilon(x) \subsetneq X.\]
Start with some arbitrary point $a_1 \in X$, and construct recursively a sequence $(a_n)_{n \geq 1} \subset X$, by choosing
\[a_{n+1} \in X \setminus \left[ \mathcal{B}_\varepsilon(a_1) \cup \cdots \cup \mathcal{B}_\varepsilon(a_n) \right], \quad \forall n \geq 1.\]
This will then force
\[d(a_m, a_n) \geq \varepsilon, \quad \forall m > n \geq 1,\]
so $A = \{a_n : n \in \mathbb{N}\}$ will be an infinite $\varepsilon$-rare set, thus contradicting (ii).

(iii) ⇒ (i). Assume property (iii), and let us prove that $X$ is compact. We are going to use Theorem 6.1. Start with an arbitrary sequence $(x_n)_{n \geq 1} \subset X$, and let us construct a convergent subsequence.

Claim: There exists a sequence $(p_n)_{n \geq 1} \subset X$, such that for every integer $k \geq 1$, the set
\[M_k = \{n \in \mathbb{N} : x_n \in \bigcap_{\ell=1}^{k} \mathcal{B}_{\varepsilon}(p_\ell)\}\]
is infinite.
The sequence $(p_n)_{n \geq 1}$ is constructed recursively. To start, we use (ii) to find a finite set $F_1 \subset X$, such that
\[X = \bigcup_{p \in F_1} \mathcal{B}_1(p).\]
If we define, for each $p \in F_1$, the set
\[S_1(p) = \{n \in \mathbb{N} : x_n \in \mathcal{B}_1(p)\},\]
then we clearly have
\[\bigcup_{p \in F_1} S_1(p) = \mathbb{N},\]
so in particular one of the sets $S_1(p)$, $p \in F_1$, is infinite.

Suppose now we have constructed points $p_1, p_2, \ldots, p_{m-1}$, such that, for every $k \in \{1, \ldots, m-1\}$, the set
\[M_k = \{n \in \mathbb{N} : x_n \in \bigcap_{\ell=1}^{k} \mathcal{B}_{\varepsilon}(p_\ell)\}\]
is infinite, and let us indicate how the next term \( p_m \) is to be constructed. Start with a finite set \( F_m \subset X \), such that
\[
X = \bigcup_{p \in F_m} B_{\frac{1}{m}}(p),
\]
and define, for each \( p \in F_m \), the set
\[
S_m(p) = \{n \in M_{m-1} : x_n \in B_{\frac{1}{m}}(p)\}.
\]
It is clear that
\[
M_{m-1} = \bigcup_{p \in F_m} S_m(p),
\]
and since \( M_{m-1} \) is infinite, it follows that one of the sets \( S_m(p), p \in F_m \) is infinite. We then choose \( p_m \in F_m \) to be one point for which \( S_m(p_m) \) is infinite.

Having proven the Claim, let us us construct a sequence of integers \( 1 \leq n_1 < n_2 < \cdots \) as follows. Start with some arbitrary \( n \in M_1 \). Once \( n_1 < n_2 < \cdots < n_k \) have been constructed, we choose the integer \( n_{k+1} \in M_{k+1} \), such that \( n_{k+1} > n_k \). (It is here that we use the fact that \( M_{k+1} \) is infinite.) By construction, we have \( n_k \in M_k, \forall k \geq 1 \).

Suppose \( k \geq \ell \geq 1 \). Then by construction we have \( n_k \in M_k \subset M_\ell \) and \( n_\ell \in M_\ell \). In particular we get
\[
d(x_{n_k}, x_{n_\ell}) \leq d(x_{n_k}, p_\ell) + d(x_{n_\ell}, p_\ell) < \frac{2}{\ell}.
\]
The above estimate clearly proves that the subsequence \( (x_{n_k})_{k \geq 1} \) is Cauchy. Since \( X \) is complete, it follows that \( (x_{n_k})_{k \geq 1} \) is convergent. 

\[ \tag*{□} \]

**Corollary 6.4.** Let \((X, d)\) be a complete metric space, and let \( A \) be a subset of \( X \). The following are equivalent:

(i) the closure \( \overline{A} \) is compact in \( X \);

(ii) for each \( \varepsilon > 0 \), all \( \varepsilon \)-rare subsets of \( A \) are finite.

**Proof.** (i) \(\Rightarrow\) (ii). This is trivial from the above result.

(ii) \(\Rightarrow\) (i). Assume (ii), and let us prove that \( \overline{A} \) is compact. Since \( \overline{A} \) is complete, it suffices to prove that, for each \( \varepsilon > 0 \), all \( \varepsilon \)-rare subsets of \( \overline{A} \) are finite. Fix \( \varepsilon > 0 \), and let \( B \) be an \( \varepsilon \)-rare subset of \( \overline{A} \). For each \( x \in B \), let us choose a point \( a_x \in A \), such that \( x \in B_{\varepsilon/3}(a_x) \). Suppose \( x, y \in B \) are such that \( x \neq y \). Then
\[
d(a_x, a_y) \geq d(x, y) - d(a_x, x) - d(a_y, y) > \varepsilon - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.
\]
In particular, this shows that the map
\[
f : B \ni x \mapsto a_x \in A
\]
is injective, and the set \( f(B) \) is an \((\varepsilon/3)\)-rare subset of \( A \). By condition (ii) this forces \( B \) to be finite. 
\[ \tag*{□} \]

We continue with an important construction.

**Definitions.** Let \((X, d)\) be a metric space. We define
\[
\text{cs}(X, d) = \{x = (x_n)_{n \geq 1} : x \text{ Cauchy sequence in } X\}.
\]
We say that two Cauchy sequences \( x = (x_n)_{n \geq 1} \) and \( y = (y_n)_{n \geq 1} \) in \( X \) are equivalent, if
\[
\lim_{n \to \infty} d(x_n, y_n) = 0.
\]
In this case we write \( x \sim y \). (It is fairly obvious that \( \sim \) is indeed an equivalence relation.) We define the quotient space

\[
\widetilde{X} = \text{cs}(X, d)/\sim.
\]

For an element \( x \in \text{cs}(X, d) \), we denote its equivalence class by \( \widetilde{x} \).

Finally, for a point \( x \in X \), we define \( \langle x \rangle \in \widetilde{X} \), to be the equivalence class of the constant sequence \( x \) (which is obviously Cauchy).

**Remark 6.7.** Let \((X, d)\) be a metric space. If \( x = (x_n)_{n \geq 1} \) and \( y = (y_n)_{n \geq 1} \) are Cauchy sequences in \( X \), then the sequence of real numbers \((d(x_n, y_n))_{n \geq 1}\) is convergent. Indeed, for any \( m, n \) we have

\[
|d(x_m, y_m) - d(x_n, y_n)| \leq |d(x_m, y_m) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n).
\]

We can then define

\[
\delta(x, y) = \lim_{n \to \infty} d(x_n, y_n).
\]

**Proposition 6.7.** Let \((X, d)\) be a metric space.

A. The map \( \delta : \text{cs}(X, d) \times \text{cs}(X, d) \to [0, \infty) \) has the following properties:
   (i) \( \delta(x, y) = \delta(y, x) \), \( \forall x, y \in \text{cs}(X, d) \);
   (ii) \( \delta(x, y) \leq \delta(x, z) + \delta(z, y) \), \( \forall x, y, z \in \text{cs}(X, d) \);
   (iii) \( \delta(x, y) = 0 \Rightarrow x \sim y \);
   (iv) If \( x, x', y, y' \in \text{cs}(X, d) \) are such that \( x \sim x' \) and \( y \sim y' \), then \( \delta(y, x) = \delta(x', y') \).

B. The map \( \widetilde{d} : \widetilde{X} \times \widetilde{X} \to [0, \infty) \), correctly defined by

\[
\widetilde{d}(\widetilde{x}, \widetilde{y}) = \delta(x, y), \ \forall x, y \in \text{cs}(X, d),
\]

is a metric on \( \widetilde{X} \).

C. The map \( X \ni x \mapsto \langle x \rangle \in \widetilde{X} \) is isometric, in the sense that

\[
\widetilde{d}(\langle x \rangle, \langle y \rangle) = d(x, y), \ \forall x, y \in X.
\]

**Proof.** A. Properties (i), (ii) and (iii) are obvious. To prove property (iv) let \( x = (x_n)_{n \geq 1}, x' = (x'_n)_{n \geq 1}, y = (y_n)_{n \geq 1}, \) and \( y' = (y'_n)_{n \geq 1} \). The inequality

\[
d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n),
\]

combined with \( \lim_{n \to \infty} d(x'_n, x_n) = \lim_{n \to \infty} d(y_n, y'_n) = 0 \) immediately gives

\[
\delta(x', y') = \lim_{n \to \infty} d(x'_n, y'_n) \leq \lim_{n \to \infty} d(x_n, y_n) = \delta(x, y).
\]

By symmetry we also have \( \delta(x, y) \leq \delta(x', y') \), and we are done.

B. This is immediate from A.

C. Obvious, from the definition. \( \square \)

**Proposition 6.8.** Let \((X, d)\) be a metric space.

(i) For any Cauchy sequence \( x = (x_n)_{n \geq 1} \) in \( X \), one has

\[
\lim_{n \to \infty} (x_n) = \bar{x}, \text{ in } \widetilde{X}.
\]

(ii) The metric space \((\widetilde{X}, \widetilde{d})\) is complete.
Proof. (i). For every \( n \geq 1 \), we have
\[
\tilde{d}(\langle x_n \rangle, \bar{x}) = \lim_{m \to \infty} d(x_n, x_m).
\]
Now if we start with some \( \varepsilon > 0 \), and we choose \( N_\varepsilon \) such that
\[
d(x_n, x_m) < \varepsilon, \quad \forall m, n \geq N_\varepsilon,
\]
then (8) shows that
\[
\tilde{d}(\langle x_n \rangle, \bar{x}) \leq \varepsilon, \quad \forall n \geq N_\varepsilon,
\]
so we indeed have
\[
\lim_{n \to \infty} \tilde{d}(\langle x_n \rangle, \bar{x}) = 0.
\]
(ii). Let \((p_k)_{k \geq 1}\) be a Cauchy sequence in \( \bar{X} \). Using (i), we can choose, for each \( k \geq 1 \), an element \( x_k \in X \), such that
\[
\tilde{d}(\langle x_k \rangle, p_k) \leq \frac{1}{2^k}.
\]
Claim 1: The sequence \( x = (x_k)_{k \geq 1} \) is Cauchy in \( X \).
Indeed, for \( k \geq \ell \geq 1 \) we have
\[
d(x_k, x_\ell) = \tilde{d}(\langle x_k \rangle, \langle x_\ell \rangle) \leq \tilde{d}(\langle x_k \rangle, p_k) + \tilde{d}(p_k, \langle x_\ell \rangle) \leq \tilde{d}(p_k, p_\ell) + \frac{1}{2^\ell}.
\]
This clearly gives
\[
\lim_{n \to \infty} \left[ \sup_{k, \ell \geq N} d(x_k, x_\ell) \right] \leq \lim_{n \to \infty} \left[ \sup_{k, \ell \geq N} \tilde{d}(p_k, p_\ell) \right] = 0,
\]
so \( x = (x_k)_{k \geq 1} \) is indeed Cauchy.

The proof of (ii) will then be finished, once we prove:

Claim 2: We have \( \lim_{n \to \infty} p_k = \bar{x} \) in \( \bar{X} \).

To see this, we observe that, for \( \ell \geq k \geq 1 \) we have the inequality
\[
\tilde{d}(p_k, \langle x_\ell \rangle) \leq \tilde{d}(p_k, \langle x_k \rangle) + \tilde{d}(\langle x_k \rangle, \langle x_\ell \rangle) \leq \frac{1}{2^k} + d(x_k, x_\ell).
\]
If we now start with some \( \varepsilon > 0 \), and we choose \( N_\varepsilon \) such that
\[
d(x_k, x_\ell) < \varepsilon, \quad \forall k, \ell \geq N_\varepsilon,
\]
then (9) gives
\[
\tilde{d}(p_k, \langle x_\ell \rangle) \leq \frac{1}{2^k} + \varepsilon, \quad \forall \ell \geq k \geq N_\varepsilon.
\]
If we keep \( k \geq N_\varepsilon \) fixed and take \( \lim_{\ell \to \infty} \), using (i) we get
\[
\tilde{d}(p_k, \bar{x}) = \lim_{\ell \to \infty} \tilde{d}(p_k, \langle x_\ell \rangle) \leq \frac{1}{2^k} + \varepsilon, \quad \forall k \geq N_\varepsilon.
\]
The above estimate clearly proves that
\[
\lim_{k \to \infty} \tilde{d}(p_k, \bar{x}) = 0,
\]
so the sequence \((p_k)_{k \geq 1}\) is convergent (to \( \bar{x} \)).

Definition. The metric space \((\bar{X}, \tilde{d})\) is called the completion of \((X, d)\).

The completion has a certain universality property. In order to formulate this property we need the following
**Definition.** Let \( (X, d) \) and \( (Y, \rho) \) be metric spaces. A map \( f : X \to Y \) is said to be a **Lipschitz function**, if there exists some constant \( C \geq 0 \), such that

\[
\rho(f(x), f(x')) \leq C \cdot d(x, x'), \quad \forall x, x' \in X.
\]

Such a constant \( C \) is then called a **Lipschitz constant** for \( f \).

**Proposition 6.9.** Let \( (X, d) \) be a metric space, and let \( (\tilde{X}, \tilde{d}) \) be its completion. If \( (Y, \rho) \) is a complete metric space, and \( f : X \to Y \) is a Lipschitz function with Lipschitz constant \( C \geq 0 \), then there exists a unique continuous function \( \tilde{f} : \tilde{X} \to Y \), such that

\[
\tilde{f}(\langle x \rangle) = f(x), \quad \forall x \in X.
\]

Moreover, \( \tilde{f} \) is Lipschitz, with Lipschitz constant \( C \).

**Proof.** Start with some Cauchy sequence \( x = (x_n)_{n \geq 1} \) in \( X \). Using the inequality

\[
\rho(f(x_m), f(x_n)) \leq C \cdot d(x_m, x_n), \quad \forall m, n \geq 1,
\]

it is obvious that \( (f(x_n))_{n \geq 1} \) is a Cauchy sequence in \( Y \). Since \( Y \) is complete, this sequence is convergent. Define,

\[
\phi(x) = \lim_{n \to \infty} f(x_n).
\]

This way we have constructed a map \( \phi : \text{cs}(X, d) \to Y \).

**Claim:** If \( x \sim x' \), then \( \phi(x) = \phi(x') \).

Indeed, if \( x = (x_n)_{n \geq 1} \) and \( x' = (x'_n)_{n \geq 1} \), then the Lipschitz property will give

\[
\rho(f(x_n), f(x'_n)) \leq C \cdot d(x_n, x'_n), \quad \forall n \geq 1,
\]

and using the fact that \( \lim_{n \to \infty} d(x_n, x'_n) = 0 \), we get \( \lim_{n \to \infty} \rho(f(x_n), f(x'_n)) = 0 \). This clearly forces

\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(x'_n).
\]

Having proven the claim, we now see that we have a correctly define map \( \tilde{f} : \tilde{X} \to Y \), with the property that

\[
\tilde{f}(\langle x \rangle) = \phi(x), \quad \forall x \in \text{cs}(X, d).
\]

The equality

\[
\tilde{f}(\langle x \rangle) = f(x), \quad \forall x \in X
\]

is trivially satisfied.

Let us check now that \( \tilde{f} \) is Lipschitz, with Lipschitz constant \( C \). Start with two points \( p, p' \in \tilde{X} \), represented as \( p = \tilde{x} \) and \( p' = \tilde{x}' \), for two Cauchy sequences \( x = (x_n)_{n \geq 1} \) and \( x' = (x'_n)_{n \geq 1} \) in \( X \). Using the definition, we have

\[
\tilde{f}(p) = \lim_{n \to \infty} f(x_n) \quad \text{and} \quad \tilde{f}(p') = \lim_{n \to \infty} f(x'_n).
\]

This will give

\[
\rho(f(p), f(p')) = \lim_{n \to \infty} \rho(f(x_n), f(x'_n)).
\]

Notice however that

\[
\rho(f(x_n), f(x'_n)) \leq C \cdot d(x_n, x'_n), \quad \forall n \geq 1,
\]

so taking the limit yields

\[
\rho(f(p), f(p')) = \lim_{n \to \infty} \rho(f(x_n), f(x'_n)) \leq C \cdot \lim_{n \to \infty} d(x_n, x'_n) = C \cdot \tilde{d}(p, p').
\]
Finally, let us show that \( \tilde{f} \) is unique. Let \( F : \tilde{X} \to Y \) be another continuous function with \( F(x) = f(x) \), for all \( x \in X \). Start with an arbitrary point \( p \in \tilde{X} \), represented as \( p = x \), for some Cauchy sequence \( x = (x_n)_{n \geq 1} \) in \( X \). Since \( \lim_{n \to \infty} (x_n) = p \) in \( \tilde{X} \), by continuity we have
\[
F(p) = \lim_{n \to \infty} F((x_n)) = \lim_{n \to \infty} f(x_n) = \phi(x) = \bar{f}(p)
\]
\( \square \)

**Corollary 6.5.** Let \( (X, d) \) be a metric space, let \( (Y, \rho) \) be a complete metric space, and let \( f : X \to Y \) be an isometric map, that is
\[
\rho(f(x), f(x')) = d(x, x'), \quad \forall x, x' \in X.
\]
Then the map \( \tilde{f} : \tilde{X} \to Y \), given by the above result, is isometric and \( \tilde{f}(\tilde{X}) = \bar{f}(X) \) - the closure of \( f(X) \) in \( Y \).

**Proof.** To show that \( \tilde{f}(\tilde{X}) = \bar{f}(X) \), start with some arbitrary point \( y \in \bar{f}(X) \). Then there exists a sequence \( (x_n)_{n \geq 1} \subset X \), with \( \lim_{n \to \infty} f(x_n) = y \). Since \( (f(x_n))_{n \geq 1} \) is Cauchy in \( Y \), and
\[
d(x_m, x_n) = \rho(f(x_m), f(x_n)), \quad \forall m, n \geq 1,
\]
it follows that the sequence \( x = (x_n)_{n \geq 1} \) is Cauchy in \( X \). We then have
\[
y = \lim_{n \to \infty} f(x_n) = \tilde{f}(\tilde{x}).
\]
Finally, we show that \( \tilde{f} \) is isometric. Start with two points \( p, q \in \tilde{X} \), represented as \( p = \tilde{x} \) and \( q = \tilde{z} \), for some Cauchy sequences \( x = (x_n)_{n \geq 1} \) and \( z = (z_n)_{n \geq 1} \) in \( X \). Then by construction we have
\[
\rho(\tilde{f}(p), \tilde{f}(q)) = \lim_{n \to \infty} \rho(\tilde{f}((x_n)), \tilde{f}((z_n))) = \lim_{n \to \infty} \rho(f(x_n), f(z_n)) = \\
= \lim_{n \to \infty} d(x_n, z_n) = \tilde{d}(\tilde{x}, \tilde{z}) = \tilde{d}(p, q).
\]
\( \square \)

**Corollary 6.6.** If \( (X, d) \) is a complete metric space, and \( \tilde{X} \) is its completion, then the map \( \iota : X \ni x \mapsto \langle x \rangle \in \tilde{X} \) is bijective.

**Proof.** Apply the previous result to the map \( \text{Id} : X \to X \), to get a bijective (isometric) map \( \text{Id} : X \to X \). Since the map \( \text{Id} \) is obviously a left inverse for \( \iota \), it follows that \( \iota \) itself is bijective. \( \square \)

In the remainder of this section we will address the following question: Given a topological Hausdorff space \( X \), when does there exists a metric \( d \) on \( X \), such that the given topology coincides with the metric topology defined by \( d \)? A topological Hausdorff space with the above property is said to be metrizable. It is difficult to give non-trivial necessary and sufficient conditions for metrizability. One instance in which this is possible is the compact case (see the Urysohn Metrizability Theorem later in these notes). Here is a useful result, which is an example of a sufficient condition for metrizability.

**Proposition 6.10 (Metrizability of Countable Products).** Let \( (X_i, d_i)_{i \in I} \) be a countable family of metric spaces. Then the product space \( X = \prod_{i \in I} X_i \), equipped with the product topology, is metrizable.
PROOF. Denote by $\mathcal{T}$ the product topology on $X$. What we need is a metric $d$ on $X$, such that the maps

$$\text{Id} : (X, d) \to (X, \mathcal{T}) \quad \text{and} \quad \text{Id} : (X, \mathcal{T}) \to (X, d)$$

are continuous. (Here the notation $(X, d)$ signifies that $X$ is equipped with the metric topology defined by $d$.) For each $i \in I$, let $\pi_i : X \to X_i$ denote the projection onto the $i^{\text{th}}$ coordinate.

Case I: Assume $I$ is finite. In this case we define the metric $d$ on $X$ as follows. If $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ are elements in $X$, we put

$$d(x, y) = \max_{i \in I} d_i(x_i, y_i).$$

The continuity of the map $\text{Id} : (X, d) \to (X, \mathcal{T})$ is equivalent to the fact that all maps

$$\pi_i : (X, d) \to (X_i, d_i), \quad i \in I$$

are continuous. This is obvious, because by construction we have

$$d_i(\pi_i(x), \pi_i(y)) \leq d(x, y), \quad \forall x, y \in X.$$

Conversely, to prove the continuity of $\text{Id} : (X, \mathcal{T}) \to (X, d)$, we are going to prove that every $d$-open set is open in the product topology. It suffices to prove this only for open balls. Fix then $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$ and $r > 0$, and consider the open ball $B_r(x)$. If we define, for each $i \in I$, the open ball $B_r^{X_i}(x_i)$, then it is obvious that

$$B_r(x) = \bigcap_{i \in I} \pi_i^{-1}(B_r^{X_i}(x_i)),$$

and since $\pi_i$ are all continuous, this proves that $B_r(x)$ is indeed open in the product topology.

Case II: Assume $I$ is infinite. In this case we identify $I = \mathbb{N}$. For every $n \in \mathbb{N}$ we define a new metric $\delta_n$ on $X_n$, as follows. If

$$\sup_{p, q \in X_n} d_n(p, q) \leq 1,$$

we put $\delta_n = d_n$. Otherwise, we define

$$\delta_n(p, p) = \frac{d_n(p, q)}{1 + d_n(p, q)}, \quad \forall p, q \in X_n.$$  

It is not hard to see that the metric topology defined by $\delta_n$ coincides with the one defined by $d_n$. The advantage is that $\delta_n$ takes values in $[0, 1]$. We define the metric $d : X \times X \to [0, \infty)$, as follows. If $x = (x_n)_{n \in \mathbb{N}}$ and $y = (y_n)_{n \in \mathbb{N}}$ are elements in $\prod_{n \in \mathbb{N}} X_n$, we define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} = \sum_{n=1}^{\infty} \frac{\delta_n(x_n, y_n)}{2^n}.$$

Due to the fact that $\delta_n$ takes values in $[0, 1]$, the above series is convergent, and it obviously defines a metric on $X$.

As above, the continuity of the map $\text{Id} : (X, d) \to (X, \mathcal{T})$ is equivalent to the continuity of all the maps $\pi_n : (X, d) \to (X_n, d_n)$, or equivalently for $\pi_n : (X, d) \to (X_n, \delta_n)$, $n \in \mathbb{N}$. But this is an immediate consequence of the (obvious) inequalities

$$\delta_n(\pi_n(x), \pi_n(y)) \leq 2^n \cdot d(x, y), \quad \forall x, y \in X.$$
As before, in order to prove the continuity of the other map \( \text{Id} : (X, \mathcal{T}) \to (X, d) \), we start with some \( d \)-open set \( D \), and we show that \( D \) is open in the product topology. Since \( D \) is a union of \( d \)-open balls, we need to prove that for any \( x \in X \) and any \( r > 0 \), the open ball \( \mathcal{B}_r(x) \), in \((X, d)\), is a neighborhood of \( x \) in the product topology. Fix \( x = (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n \), as well as \( r > 0 \). Choose some integer \( N \geq 1 \), such that
\[
\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{r}{2},
\]
and define, for each \( k \in \{1, 2, \ldots, N\} \) the set
\[
D_k = \{ y = (y_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n : \delta_n(x_k, y_k) < \frac{r}{2} \},
\]
It is clear that \( D_k \) is open in the product topology, for each \( k = 1, 2, \ldots, N \). (This is a consequence of the fact that \( D_k = \pi_k^{-1}(\mathcal{B}_{r/2}(x_k)) \), where \( \mathcal{B}_{r/2}(x_k) \) is the \( \delta_k \)-open ball in \( X_k \) of radius \( r/2 \), centered at \( x_k \).) Then the set \( D = D_1 \cap D_2 \cap \cdots \cap D_N \) is also open in the product topology. Obviously we have \( x \in D \). We now prove that \( D \subset \mathcal{B}_r(x) \). Start with some arbitrary \( y \in D \), say \( y = (y_n)_{n \in \mathbb{N}} \). On the one hand, we have
\[
\delta_k(x_k, y_k) < \frac{r}{2} \quad \forall \ k \in \{1, 2, \ldots, N\},
\]
so we get
\[
\sum_{n=1}^{N} \frac{1}{2^n} \delta_n(x_n, y_n) < \frac{r}{2} \sum_{n=1}^{N} \frac{1}{2^n} < \frac{r}{2}.\]
On the other hand, since \( \delta_n \) takes values in \([0, 1)\), we also have
\[
\sum_{n=N+1}^{\infty} \frac{1}{2^n} \delta_n(x_n, y_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} < \frac{r}{2},\]
so we get
\[
d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_n(x_n, y_n) < r,
\]
thus proving that \( y \) indeed belongs to \( \mathcal{B}_r(x) \). \( \square \)