5. Topology preliminaries V: Locally compact spaces

Definition. A locally compact space is a Hausdorff topological space with the property

(lc) Every point has a compact neighborhood.

One key feature of locally compact spaces is contained in the following;

Lemma 5.1. Let $X$ be a locally compact space, let $K$ be a compact set in $X$, and let $D$ be an open subset, with $K \subset D$. Then there exists an open set $E$ with:

(i) $E$ compact;
(ii) $K \subset E \subset E \subset D$.

Proof. Let us start with the following

Particular case: Assume $K$ is a singleton $K = \{x\}$.

Start off by choosing a compact neighborhood $N$ of $x$. Using the results from section 4, when equipped with the induced topology, the set $N$ is normal. In particular, if we consider the closed sets $A = \{x\}$ and $B = N \setminus D$, which are also closed in the induced topology, it follows that there exist sets $U, V \subset N$, such that

- $U \supset \{x\}$, $V \supset B$, $U \cap V = \emptyset$;
- $U$ and $V$ are open in the induced topology on $N$.

The second property means that there exist open sets $U_0, V_0 \subset X$, such that $U = N \cap U_0$ and $V = N \cap V_0$. Let $E = \text{Int}(U)$. By construction $E$ is open, and $E \ni x$. Also, since $E \subset U \subset N$, it follows that

$E \subset \overline{N} = N$.

In particular this gives the compactness of $E$. Finally, since we obviously have

$E \cap V_0 \subset U \cap V_0 = N \cap U_0 \cap V_0 = U \cap V = \emptyset$,

we get $E \subset X \setminus V_0$, so using the fact that $X \setminus V_0$ is closed, we also get the inclusion $\overline{E} \subset X \setminus V_0$. Finally, combining this with (1) and with the inclusion $N \setminus D \subset V \subset V_0$, we will get

$\overline{E} \subset N \cap (X \setminus V_0) \subset N \cap (N \setminus D) \subset D$,

and we are done.

Having proven the particular case, we proceed now with the general case. For every $x \in K$ we use the particular case to find an open set $E(x)$, with $E(x)$ compact, and such that $x \in E(x) \subset \overline{E(x)} \subset D$. Since we clearly have $K \subset \bigcup_{x \in K} E(x)$, by compactness, there exist $x_1, \ldots, x_n \in K$, such that $K \subset E(x_1) \cup \cdots \cup E(x_n)$. Notice that if we take $E = E(x_1) \cup \cdots \cup E(x_n)$, then we clearly have

$K \subset E \subset \overline{E} \subset \overline{E(x_1)} \cup \cdots \cup \overline{E(x_n)} \subset D$,  

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and we are done. \(\square\)

One of the most useful result in the analysis on locally compact spaces is the following.

**Theorem 5.1** (Urysohn’s Lemma for locally compact spaces). Let \(X\) be a locally compact space, and let \(K,F \subset X\) be two disjoint sets, with \(K\) compact, and \(F\) closed. Then there exists a continuous function \(f : X \to [0,1]\) such that \(f|_K = 1\) and \(f|_F = 0\).

**Proof.** Apply Lemma 5.1 for the pair \(K \subset X \setminus F\) and find an open set \(E\), with \(E\) compact, such that \(K \subset E \subset [E \subset X \setminus F\). Apply again Lemma 5.1 for the pair \(K \subset E\) and find another open set \(G\) with \(\overline{G}\) compact, such that \(K \subset G \subset \overline{G} \subset E\).

Let us work for the moment in the space \(E\) (equipped with the induced topology). This is a compact Hausdorff space, hence it is normal. In particular, using Urysohn Lemma (see section 1) there exists a continuous function \(g : E \to [0,1]\) such that \(g|_K = 0\) and \(g|_{E \setminus G} = 0\). Let us now define the function \(f : X \to [0,1]\) by

\[
f(x) = \begin{cases}
g(x) & \text{if } x \in E \\
0 & \text{if } x \in X \setminus E
\end{cases}
\]

Notice that \(f|_E = g|_E\), so \(f|_E\) is continuous. If we take the open set \(A = X \setminus \overline{G}\), then it is also clear that \(f|_A = 0\). So now we have two open sets \(E\) and \(A\), with \(A \cup E = X\), and \(f|_A\) and \(f|_E\) both continuous. Then it is clear that \(f\) is continuous. The other two properties \(f|_K = 1\) and \(f|_F = 0\) are obvious. \(\square\)

We now discuss an important notion which makes the linkage between locally compact spaces and compact spaces.

**Definition.** Let \(X\) be a locally compact space. By a compactification of \(X\) one means a pair \((\theta,T)\) consisting of a compact Hausdorff space \(T\), and of a continuous map \(\theta : X \to T\), with the following properties

(i) \(\theta(X)\) is a dense open subset of \(T\);

(ii) when we equip \(\theta(X)\) with the induced topology, the map \(\theta : X \to \theta(X)\) is a homeomorphism.

Notice that, when \(X\) is already compact, any compactification \((\theta,T)\) of \(X\) is necessarily made up of a compact space \(T\), and a homeomorphism \(\theta : X \to T\).

**Examples 5.1.**

A. Take \([-\infty,\infty] = \mathbb{R} \cup \{-\infty,\infty\}\), with the “usual” topology, in which a set \(D \subset [-\infty,\infty]\) is open if \(D = D_0 \cup D_1 \cup D_2\), where \(D_0\) is open in \(\mathbb{R}\) and \(D_1,D_2 \in \{\emptyset\} \cup \{(a,\infty) : a \in \mathbb{R}\} \cup \{(-\infty,a) : a \in \mathbb{R}\}\). Then \([-\infty,\infty]\) is a compactification of \(\mathbb{R}\).

B. (Alexandrov compactification) Suppose \(X\) is a locally compact space, which is not compact. We form a disjoint union with a singleton \(X^\alpha = X \cup \{\infty\}\), and we equip the space \(X^\alpha\) with the topology in which a subset \(D \subset X^\alpha\) is declared to be open, if either \(D\) is an open subset of \(X\), or there exists some compact subset \(K \subset X\), such that \(D = (X \setminus K) \cup \{\infty\}\). Define the inclusion map \(i : X \hookrightarrow X^\alpha\). Then \((i : X^\alpha\) is a compactification of \(X\), which is called the Alexandrov compactification. The fact that \(i(X)\) is open in \(X^\alpha\), and \(i : X \to i(X)\) is a homeomorphism, is clear. The density of \(i(X)\) in \(X^\alpha\) is also clear, since every open set \(D \subset X^\alpha\), with \(D \ni \infty\),
is of the form \((X \setminus K) \sqcup \{\infty\}\), for some compact set \(K \subset X\), and then we have \(D \cap \iota(X) = \iota(X \setminus K)\), which is non-empty, because \(X\) is not compact.

Remark that, if \(X\) is already compact, we can still define the topological space \(X^\alpha = X \sqcup \{\infty\}\), but this time the singleto set \(\{\infty\}\) will be also be open. Although \(\iota(X)\) will still be open in \(X^\alpha\), it will not be dense in \(X^\alpha\).

Exercise 1 ♦. Let \(K\) be some compact Hausdorff space, and let \(p \in K\) be some point with the property that the set \(X = K \setminus \{p\}\) is not compact. Equip \(X\) with the induced topology.

(i) Show that \(X\) is locally compact (but non-compact).

(ii) If we denote by \(X^\alpha\) the Alexandrov compactification of \(X\), then the map \(\Psi : X^\alpha \to K\), defined by

\[
\Psi(x) = \begin{cases} 
  x & \text{if } x \in X \\
  p & \text{if } x = \infty
\end{cases}
\]

is a homeomorphism.

One should regard the Alexandrov compactification as a minimal one. More precisely, one has the following.

Proposition 5.1. Suppose \(X\) is a locally compact space which is non-compact. Let \((\theta, T)\) be a compactification of \(X\), and let \(X^\alpha = X \sqcup \{\infty\}\) be the Alexandrov compactification. Then there exists a unique continuous map \(\Psi : T \to X^\alpha\), such that \((\Psi \circ \theta)(x) = x, \forall x \in X\). Moreover, the map \(\Psi\) has the property that \(\Psi(y) = \infty\), \(\forall y \in T \setminus \theta(X)\).

Proof. The uniqueness part is pretty obvious, since \(\theta(X)\) is dense in \(T\). For the existence, we use the map \(\theta : X \to T\) to identify \(X\) with an open dense subset of \(T\), and we define \(\Psi : T \to X^\alpha\) by

\[
\Psi(x) = \begin{cases} 
  x & \text{if } x \in X \\
  \infty & \text{if } x \in T \setminus X
\end{cases}
\]

so that all we have to prove is the fact that \(\Psi\) is continuous. Start with some open set \(D\) in \(X^\alpha\), and let us prove that \(\Psi^{-1}\) is open in \(T\). There are two cases.

Case I: \(D \subset X\).

This case is trivial, \(\Phi^{-1}(D) = D\), and \(D\) is open in \(X\), hence also open in \(T\).

Case II: \(D \not\subset X\).

In this case \(D \ni \infty\), so there exists some compact set \(K\) in \(X\), such that \(D = X^\alpha \setminus K = (X \setminus K) \cup \{\infty\}\). We then have

\[
\Psi^{-1}(D) = \Psi^{-1}(X \setminus K) \cup \Psi^{-1}(\{\infty\}) = (X \setminus K) \cup (T \setminus X) = T \setminus K.
\]

Since \(K\) is compact in \(X\), it will be compact in \(T\) as well. In particular, \(K\) is closed in \(T\), hence the set \(\Psi^{-1}(D) = T \setminus K\) is indeed open in \(T\).

It turns out that there exists another compactification which is described below, which can be regarded as the largest.

Theorem 5.2 (Stone-Čech). Let \(X\) be a locally compact space. Consider the set

\[
F = \{ f : X \to [0, 1] : f \text{ continuous} \},
\]
and consider the product space
\[ T = \prod_{f \in F} [0,1], \]
equipped with the product topology, and define the map \( \theta : X \to T \) by
\[ \theta(x) = (f(x))_{f \in F}, \forall x \in X. \]
 Equip the closure \( \bar{\theta}(X) \) with the topology induced from \( T \). Then the pair \( (\theta, \bar{\theta}(X)) \) is a compactification of \( X \).

**Proof.** For every \( f \in F \), let us denote by \( \pi_f : T \to [0,1] \) the coordinate map.

Remark that \( \theta : X \to T \) is continuous. This is immediate from the definition of the product topology, since the continuity of \( \theta \) is equivalent to the continuity of all compositions \( \pi_f \circ \theta \), \( f \in F \). The fact that these compositions are continuous is however trivial, since we have \( \pi_f \circ \theta = f, \forall f \in F \).

Denote for simplicity \( \bar{\theta}(X) \) by \( B \). By Tihonov’s Theorem, the space \( T \) is compact (and obviously Hausdorff), so the set \( B \) is compact as well, being a closed subset of \( T \). By construction, \( \bar{\theta}(X) \) is dense in \( B \), and \( \theta \) is continuous.

At this point, it is interesting to point out the following property

**Claim 1:** For every \( f \in F \), there exists a unique continuous map \( \tilde{f} : B \to [0,1] \), such that \( \tilde{f} \circ \theta = f \).

The uniqueness is trivial, since \( \bar{\theta}(X) \) is dense in \( B \). The existence is also trivial, because we can take \( \tilde{f} = \pi_f \mid_B \).

We can show now that \( \theta \) is injective. If \( x, y \in X \) are such that \( x \neq y \), then using Urysohn Lemma we can find \( f \in F \), such that \( f(x) \neq f(y) \). The function \( \tilde{f} \) given by Claim 1, clearly satisfies
\[ \tilde{f}(\theta(x)) = f(x) \neq f(y) = \tilde{f}(\theta(y)), \]
which forces \( \theta(x) \neq \theta(y) \).

In order to show that \( \bar{\theta}(X) \) is open in \( B \), we need some preparations. For every compact subset \( K \subset X \), we define
\[ F_K = \{ f : X \to [0,1] : f \text{ continuous, } f\mid_{X \setminus K} = 0 \}. \]

On key observation is the following.

**Claim 2:** If \( K \subset X \) is compact, and if \( f \in F_K \), then the continuous function \( \tilde{f} : B \to [0,1] \), given by Claim 1, has the property \( \tilde{f} \mid_{B \setminus \theta(K)} = 0 \).

We start with some \( \alpha \in B \setminus \theta(K) \), and we use Urysohn Lemma to find some continuous function \( \phi : B \to [0,1] \) such that \( \phi(\alpha) = 1 \) and \( \phi\mid_{\theta(K)} = 0 \). Consider the function \( \psi = \phi \cdot \tilde{f} \). Notice that \( (\phi \circ \theta)\mid_K = 0 \), which combined with the fact that \( f\mid_{X \setminus K} = 0 \), gives
\[ \psi \circ \theta = (\phi \circ \theta) \cdot (\tilde{f} \circ \theta) = (\phi \circ \theta) \cdot f = 0, \]
so using Claim 1 (the uniqueness part), we have \( \psi = 0 \). In particular, since \( \phi(\alpha) = 1 \), this forces \( \tilde{f}(\alpha) = 0 \), thus proving the Claim.

We define now the collection
\[ F_c = \bigcup_{K \subset X}^{K \text{ compact}} F_K. \]
Define the set

\[ S = \bigcap_{f \in F_c} \pi_f^{-1}(\{0\}). \]

By the definition of the product topology, it follows that \( S \) is closed in \( T \). The fact that \( \theta(X) \) is open in \( B \), is then a consequence of the following fact.

**Claim 3:** One has the equality \( \theta(X) = B \setminus S \).

Start first with some point \( x \in X \), and let us show that \( \theta(x) \not\in S \). Choose some open set \( D \subset X \), with \( D \) compact, such that \( D \ni x \), and apply Urysohn Lemma to find some continuous map \( f : X \to [0,1] \) such that \( f(x) = 1 \) and \( f|_{X \setminus D} = 0 \).

It is clear that \( f \in F_D \subset F_c \), but \( \pi_f(x) = f(x) = 1 \neq 0 \), which means that \( \theta(x) \not\in \pi_f^{-1}(\{0\}) \), hence \( \theta(x) \not\in S \). Conversely, let us start with some point \( \alpha = (\alpha_f)_{f \in F} \in B \setminus S \), and let us prove that \( \alpha \in \theta(X) \). Since \( \alpha \not\in S \), there exists some \( f \in F_c \), such that \( \pi_f(\alpha) > 0 \). Since \( f \in F_c \), there exists some compact subset \( K \subset X \), such that \( f|_{X \setminus K} = 0 \). Using Claim 2, we know that \( f|_{B \setminus \theta(K)} = 0 \). Since \( \tilde{f}(\alpha) = \pi_f(\alpha) \neq 0 \), this forces \( \alpha \in \theta(K) \subset \theta(X) \).

To finish the proof of the Theorem, all we need to prove now is the fact that \( \theta : X \to \theta(X) \) is a homeomorphism, which amounts to proving that, whenever \( D \subset X \) is open, it follows that \( \theta(D) \) is open in \( B \). Fix an open subset \( D \subset X \). In order to show that \( \theta(D) \) is open in \( B \), we need to show that \( \theta(D) \) is a neighborhood for each of its points. Fix some point \( \alpha \in \theta(D) \), i.e. \( \alpha = \theta(x) \), for some \( x \in D \). Choose some compact subset \( K \subset D \), such that \( x \in \text{Int}(K) \), and apply Urysohn Lemma to find a function \( f \in F_K \), with \( f(x) = 1 \). Consider the continuous function \( \tilde{f} : B \to [0,1] \) given by Claim 1, and apply Claim 2 to conclude that \( \tilde{f}|_{B \setminus \theta(K)} = 0 \).

In particular the open set

\[ N = \tilde{f}^{-1}\left((1/2, \infty)\right) \subset B \]

is contained in \( \theta(K) \subset \theta(D) \). Since \( \tilde{f}(\alpha) = f(x) = 1 \), we clearly have \( x \in N \). \( \Box \)

**Definition.** The compactification \((\theta, \bar{\theta}(X))\), constructed in the above Theorem, is called the *Stone-Cech compactification of X*. The space \( \bar{\theta}(X) \) will be denoted by \( X^\beta \). Using the map \( \theta \), we shall identify from now on \( X \) with a dense open subset of \( X^\beta \). Remark that if \( X \) is compact, then \( X^\beta = X \).

**Comment.** The Stone-Cech compactification is inherently “Zorn Lemma type” construction. For example, if \( X \) is a locally compact space, then every ultrafilter on \( X \) gives rise to a point in \( X^\beta \), constructed as follows. If \( \theta : X \to X^\beta \) denotes the inclusion map, then for every ultrafilter \( \mathcal{U} \) on \( X \), we consider the ultrafilter \( \theta_* \mathcal{U} \) on \( X^\beta \), and by compactness this ultrafilter converges to some (unique) point in \( X^\beta \). This way one gets a correspondence

\[ \lim_{X} : \{ \mathcal{U} \subset \mathcal{F}(X) : \mathcal{U} \text{ ultrafilter on } X \} \to X^\beta. \]

The next two exercises discuss the features of this map.
With these notations, we have $X$ as well as the maps $\Phi$ of the product spaces such that $\Phi$ is a continuous map, then there exists a unique continuous map $\Psi$ such that $\Phi(\alpha) = \Psi(\alpha)\Phi(\alpha)$.

Exercise 2. Let $X$ be a locally compact space.

A. Prove that, for an ultrafilter $U$ on $X$, the condition $\lim_{U} X \in X$ is equivalent to the condition that $U$ contains a compact subset of $X$.

B. Prove that, for two ultrafilters $U_1$, $U_2$, the condition $\lim_{U_1} X \neq \lim_{U_2} X$ is equivalent to the existence of two sets $A_1 \in U_1$ and $A_2 \in U_2$, that are “separated by a continuous function,” that is, for which there exists a continuous function $f : X \rightarrow \mathbb{R}$, and numbers $\alpha_1 < \alpha_2$, such that $f(A_1) \subset (-\infty, \alpha_1]$ and $f(A_2) \subset [\alpha_2, \infty)$.

C. Prove that the correspondence $\lim_{U} X$ is surjective.

Exercise 3. Suppose a set $X$ is equipped with the discrete topology. Prove that the correspondence $\lim_{U} X$ is bijective.

The Stone-Cech compactification is functorial, in the following sense.

Proposition 5.2. If $X$ and $Y$ are locally compact spaces, and if $\Phi : X \rightarrow Y$ is a continuous map, then there exists a unique continuous map $\Phi^\beta : X^\beta \rightarrow Y^\beta$, such that $\Phi^\beta|_X = \Phi$.

Proof. We use the notations from Theorem 5.2. Define $F = \{f : X \rightarrow [0, 1] : f \text{ continuous}\}$ and $G = \{g : Y \rightarrow [0, 1] : g \text{ continuous}\}$, the product spaces $T_X = \prod_{f \in F} [0, 1]$ and $T_Y = \prod_{g \in G} [0, 1]$, as well as the maps $\theta_X : X \rightarrow T_X$ and $\theta_Y : Y \rightarrow T_Y$, defined by

$$\theta_X(x) = (f(x))_{f \in F}, \quad \forall x \in X;$$

$$\theta_Y(y) = (g(y))_{g \in G}, \quad \forall y \in Y.$$ 

With these notations, we have $X^\beta = \overline{\theta_X(X)} \subset T_X$ and $Y^\beta = \overline{\theta_Y(Y)} \subset T_Y$. Using the fact that we have a correspondence $G \ni g \rightarrow g \circ \Phi \in F$, we define the map

$$\Psi : T_X \ni (\alpha_f)_{f \in F} \mapsto (\alpha_{g \circ \Phi})_{g \in G} \in T_Y.$$ 

Remark that $\Psi$ is continuous. This fact is pretty obvious, because when we compose with coordinate projections $\pi_g : T_Y \rightarrow [0, 1], \ g \in G$, we have $\pi_g \circ \Psi = \pi_{g \circ \Phi}$ where $\pi_{g \circ \Phi} : T_X \rightarrow [0, 1]$ is the coordinate projection, which is automatically continuous. Remark that if we start with some point $x \in X$, then

$$(2) \quad \Psi(\theta_X(x)) = ((g \circ \Phi)(x))_{g \in G} = \theta_Y(\Phi(x)),$$

which means that we have the equality $\Psi \circ \theta_X = \theta_Y \circ \Phi$. Remark first that, since $Y^\beta$ is closed, it follows that $\Psi^{-1}(Y^\beta)$ is closed in $T_X$. Second, using (2), we clearly have the inclusion $\theta_X(X) \subset \Psi^{-1}(\theta_Y(Y)) \subset \Psi^{-1}(Y^\beta)$, so using the fact that $\Psi^{-1}(Y^\beta)$ is closed, we get the inclusion

$$X^\beta = \overline{\theta_X(X)} \subset \Psi^{-1}(Y^\beta).$$ 

In other words, we get now a continuous map $\Phi^\beta = \Psi|_{X^\beta} : X^\beta \rightarrow Y^\beta$, which clearly satisfies $\Phi^\beta \circ \theta_X = \theta_Y \circ \Phi$, which using our conventions means that $\Phi^\beta|_X = \Phi$. The uniqueness is obvious, by the density of $X$ in $X^\beta$. $\square$
Remark 5.1. Suppose $X$ is a locally compact space which is not compact, and $Y$ is a compact Hausdorff space. By the above result, combined with the identification $Y^\beta \simeq Y$, we see that any continuous map $\Phi : X \to Y$ has a unique extension to a continuous map $\Phi^\beta : X^\beta \to Y$. In particular, if one takes $(\theta, T)$ to be a compactification of $X$, then $\theta : X \to T$ extends to a unique continuous map $\theta^\beta : X^\beta \to T$. This explains why the Stone-Cech compactification is sometimes referred to as the “largest” compactification. In particular, if we take $(\iota, X^\alpha)$ to be the Alexandrov compactification, we have a continuous map $\iota^\beta : X^\beta \to X^\alpha$, which is given by $\iota^\beta(x) = \infty$, $\forall x \in X^\beta \setminus X$.

Exercise 4. Let $X$ be a locally compact space, let $X^\beta$ denote its Stone-Cech compactification, and let $(\theta, T)$ be an arbitrary compactification of $X$. Denote by $\theta^\beta : X^\beta \to T$ the map described in the above remark. Prove that for a topological space $Y$ and a map $f : T \to Y$, the following are equivalent

(i) $f$ is continuous;
(ii) the composition $f \circ \theta^\beta : X^\beta \to Y$ is continuous.

This explains how the topology of $T$ can be reconstructed using the map $\theta^\beta$. More precisely, the topology of $T$ is the strong topology defined by $\theta^\beta$ (see Lemma 3.2).

Exercise 5. The Alexandrov compactification is not functorial. In other words, given locally compact spaces $X$ and $Y$, and a continuous map $f : X \to Y$, in general there does not exist a continuous map $f^\alpha : X^\alpha \to Y^\alpha$, with $f^\alpha|_X = f$. Give an example of such a situation.

Hint: Consider $X = Y = \mathbb{N}$, equipped with the discrete topology, and define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

It turns out that one can define a certain type of continuous maps, with respect to which the Alexandrov compactification is functorial.

Definition. Let $X, Y$ be locally compact spaces, and let $\Phi : X \to Y$ be a continuous map. We say that $\Phi$ is proper, if it satisfies the condition

$$K \subset Y, \text{ compact } \Rightarrow \Phi^{-1}(K) \text{ compact in } X.$$ 

Exercise 6. (Functoriality of Alexandrov compactification). Let $X$ and $Y$ be a locally compact spaces, which are non-compact, and let $X^\alpha$ and $Y^\alpha$ denote their respective Alexandrov compactifications. For a continuous map $\Phi : X \to Y$, prove that the following are equivalent:

(i) $\Phi$ is proper;
(ii) the map $\Phi^\alpha : X^\alpha \to Y^\alpha$ defined by $\Phi^\alpha|_X = \Phi$ and $\Phi^\alpha(\infty) = \infty$ is continuous.

The following is an interesting property of proper maps, which will be exploited later, is the following.

Proposition 5.3. Let $X, Y$ be locally compact spaces, let $\Phi : X \to Y$ be a proper continuous map, and let $T \subset X$ be a closed subset. Then the set $\Phi(T)$ is closed in $X$.

Proof. Start with some point $y \in \Phi(T)$. This means that

$$D \cap \Phi(T) \neq \emptyset,$$

for every open set $D \subset Y$, with $D \ni y$.

Denote by $\mathcal{V}$ the collection of all compact neighborhoods of $y$. In other words, $V \in \mathcal{V}$, if and only if $V \subset Y$ is compact, and $y \in \text{Int}(V)$. For each $V \in \mathcal{V}$ we define
the set $\tilde{V} = \Phi^{-1}(V) \cap T$. Since $\Phi$ is proper, all sets $\tilde{V}$, $V \in \mathcal{V}$, are compact. Notice also that, for every finite number of sets $V_1, \ldots, V_n \in \mathcal{V}$, if we form the intersection $V = V_1 \cap \cdots \cap V_n$, then $V \in \mathcal{V}$, and $\tilde{V} \subset V$, $\forall j = 1, \ldots, n$. Remark now that, by (3), we have $\tilde{V} \neq \emptyset$, $\forall V \in \mathcal{V}$. Indeed, if we start with some $V \in \mathcal{V}$ and we choose some point $x \in T$, such that $\Phi(x) \in V$, then $x \in \tilde{V}$. Use now the finite intersection property, to get the fact that $\bigcap_{V \in \mathcal{V}} \tilde{V} \neq \emptyset$. Pick now a point $x \in \bigcap_{V \in \mathcal{V}} \tilde{V}$. This means that $x \in T$, and

\begin{equation}
\Phi(x) \in V, \quad \forall V \in \mathcal{V}.
\end{equation}

But now we are done, because this forces $\Phi(x) = y$. Indeed, if $\Phi(x) \neq y$, using the Hausdorff property, one could find some $V \in \mathcal{V}$ with $\Phi(x) \not\in V$, thus contradicting (4).

\textbf{Comment.} When one deals with various compactifications of a non-compact locally compact space, the following extension problem is often discussed.

\textbf{Question:} Let $(\theta, T)$ be a compactification of a locally compact space $X$, let $Y$ be some topological Hausdorff space, and let $\Phi : X \to Y$ be a continuous map. \textit{When does there exist a continuous map $\Psi : T \to Y$, such that $\Psi \circ \theta = \Phi$?}

Of course, such a map (if it exists) is unique. Obviously, by density the existence of $\Psi$ will force $\Psi(T) = \Phi(X)$, so we see that a necessary condition is the fact that $\Phi(X)$ is \textit{compact}. In the case of the Stone-Cech compactification, this condition is also sufficient, by Remark 5.1.

For the Alexandrov compactification, the answer is given by the following.

\textbf{Proposition 5.4.} Let $X$ be a non-compact locally compact space, let $Y$ be a topological Hausdorff space, and let $\Phi : X \to Y$ be a continuous map. \textit{The following are equivalent.}

(i) There exists a continuous map $\Psi : X^\alpha \to Y$ with $\Psi|_X = \Phi$.

(ii) There exists some point $p \in Y$ such that

(*) for every neighborhood $V$ of $p$, there exists some compact subset $K_V \subset X$ with $\Phi(X \setminus K_V) \subset V$.

Moreover, the map $\Psi$ in (i) is unique, the point $p$ mentioned in (ii) is also unique, and $p = \Psi(\infty)$.

\textbf{Proof.} (ii) $\Rightarrow$ (i). Assume $\Psi$ is as in (ii), and let us prove (i). Take $p = \Psi(\infty)$. Start with some neighborhood $V$ of $p$. Since $\Psi$ is continuous at $\infty$, the set $\Psi^{-1}(V)$ is a neighborhood of $\infty$ in $X^\alpha$. In particular there exists some compact set $K \subset X$, such that $\Psi^{-1}(V) \supset (X \setminus K) \cup \{\infty\}$. We then obviously have $\Phi(x) = \Psi(x) \in V$, $\forall x \in X \setminus K$.

(i) $\Rightarrow$ (ii). Assume $p \in Y$ satisfies condition (*). Define the map $\Psi : X^\alpha \to Y$ by

$$\Psi(x) = \begin{cases} 
\Phi(x) & \text{if } x \in X \\
p & \text{if } x = \infty
\end{cases}$$

and let us show that $\Psi$ is continuous. Since $\Psi|_X = \Phi$, and $\Phi$ is continuous, all we need to show is the fact that $\Psi$ is continuous at $\infty$. Let $V$ be some neighborhood of $p = \Psi(\infty)$, and let us show that $\Psi^{-1}(V)$ is a neighborhood of $\infty$ in $X^\alpha$. Take $D$ an open set in $Y$ with $p \in D \subset V$, and use condition (*) to choose some compact set $K$ in $X$, such that $\Phi(X \setminus K) \subset D$, i.e. $\Phi^{-1}(D) \supset X \setminus K$. We then have
Moreover, for every continuous function \( f : X \rightarrow [0,1] \), one has the equality \( \lim_{\lambda \in \Lambda} f(x_\lambda) = f^\beta(p) \), where \( f^\beta : X^\beta \rightarrow [0,1] \) is the continuous extension of \( f \).