B. Cardinal Arithmetic

In this Appendix we discuss cardinal arithmetic. *We assume the Axiom of Choice is true.*

DEFINITIONS. Two sets A and B are said to have the *same cardinality*, if there exists a bijective map $A \to B$. It is clear that this defines an equivalence relation on the class[1] of all sets.

A *cardinal number* is thought as an equivalence class of sets. In other words, if we write a cardinal number as $\mathfrak{a}$, it is understood that $\mathfrak{a}$ consists of all sets of a given cardinality. So when we write $\text{card}\, A = \mathfrak{a}$ we understand that A belongs to this class, and for another set B we write $\text{card}\, B = \mathfrak{a}$, exactly when B has the same cardinality as A. In this case we write $\text{card}\, B = \text{card}, A$.

NOTATIONS. The cardinality of the empty set $\varnothing$ is zero. More generally the cardinality of a *finite* set is equal to its number of elements. The cardinality of the set $\mathbb{N}$, of all natural numbers, is denoted by $\aleph_0$.

DEFINITION. Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers. We write $\mathfrak{a} \leq \mathfrak{b}$ if there exist sets $A \subset B$ with $\text{card}\, A = \mathfrak{a}$ and $\text{card}\, B = \mathfrak{b}$.

This is equivalent to the fact that, for *any* sets A and B, with $\text{card}\, A = \mathfrak{a}$ and $\text{card}\, B = \mathfrak{b}$, one of the following equivalent conditions holds:

- there exists an *injective* function $f : A \to B$;
- there exists a *surjective* function $g : B \to A$.

For two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$, we use the notation $\mathfrak{a} < \mathfrak{b}$ to indicate that $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \neq \mathfrak{b}$.

THEOREM B.1 (Cantor-Bernstein). *Suppose two cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$ *satisfy* $\mathfrak{a} \leq \mathfrak{b}$ *and* $\mathfrak{b} \leq \mathfrak{a}$. *Then* $\mathfrak{a} = \mathfrak{b}$.

PROOF. Fix two sets A and B with $\text{card}\, A = \mathfrak{a}$ and $\text{card}\, B = \mathfrak{b}$, so there exist injective functions $f : A \to B$ and $g : B \to A$. We shall construct a bijective function $h : A \to B$. Define the sets

$$A_0 = A \smallsetminus g(B) \text{ and } B_0 = A \smallsetminus f(A).$$

Then define recursively the sequences $(A_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$ by

$$A_n = g(B_{n-1}) \text{ and } B_n = f(A_{n-1}), \ \ \forall\, n \geq 1.$$

Claim 1: *One has* $A_m \cap A_n = B_m \cap B_n$, $\forall\, m > n \geq 0$.

Let us first observe that the case when $n = 0$ is trivial, since we have the inclusions $A_m = g(B_{m-1}) \subset g(B) = A \smallsetminus A_0$ and $B_m = f(A_{m-1}) \subset f(A) = B \smallsetminus B_0$. Next we prove the desired property by induction on m. The case $m = 1$ is clear (this forces $n = 0$). Suppose the statement is true for $m = k$, and let us prove it for $m = k + 1$. Start with some $n < k + 1$. If $n = 0$, we are done, by the above discussion. Assume first $n \geq 1$. Since f and g are injective we have

$$A_{k+1} \cap A_n = g(B_k) \cap g(B_{n-1}) = g(B_k \cap B_{n-1}) = \varnothing,$$
$$B_{k+1} \cap B_n = f(A_k) \cap f(A_{n-1}) = f(A_k \cap A_{n-1}) = \varnothing,$$

[1] The term *class* is used, because there is no such thing as the "set of all sets."

and we are done.

Put $C = A \smallsetminus_{n\geq 0} A_n$ and $D = B \smallsetminus \bigcup_{n\geq 0} B_n$.

*Claim 2***:** *One has the equality $f(C) = D$.*

First we prove the inclusion $f(C) \subset D$. Start with some point $c \in C$, but assume $f(c) \notin D$. This means that there exists some $n \geq 0$ such that $f(c) \in B_n$. Since $f(c) \in f(A) = B \smallsetminus B_0$, we must have $n \geq 1$. But then we get $f(c) \in B_n = f(A_{n-1})$, and the injectivity of f will force $c \in A_{n-1}$, which is impossible.

Second, we prove that $D \subset f(C)$. Start with some $d \in D$. First of all, since $D \subset B \smallsetminus B_0 = f(A)$, there exists some $c \in A$ with $d = f(c)$. If $c \notin C$, then there exists some $n \geq 0$, such that $c \in A_n$, and then we would get $d = f(c) \in f(A_n) = B_{n+1}$, which is impossible.

We now begin constructing the desired bijection. First we define $\phi : \bigcup_{n\geq 0} B_n \to B$ by

$$\phi(b) = \begin{cases} b & \text{if } b \in B_n \text{ and } n \text{ is } \textit{odd} \\ (f \circ g)(b) & \text{if } b \in B_n \text{ and } n \text{ is } \textit{even} \end{cases}$$

*Claim 3***:** *The map ϕ defines a bijection*

$$\phi : \bigcup_{n\geq 0} B_n \to \bigcup_{n\geq 1} B_n.$$

It is clear that, since $\phi\big|_{B_n}$ is injective, the map ϕ is injective. Notice also that, if $n \geq 0$ is even, then $\phi(B_n) = f\big(g(B_n)\big) = f(A_{n+1}) = B_{n+2}$. When $n \geq 0$ is odd we have $\phi(B_n) = B_n$, so we have indeed the equality

$$\phi\big(\bigcup_{n\geq 0} B_n\big) = \bigcup_{n\geq 1} B_n.$$

Now we define $\psi : \bigcup_{n\geq 0} A_n \to B$ by $\psi = \phi^{-1} \circ f$. Clearly ψ is injective, and

$$\psi\big(\bigcup_{n\geq 0} A_n\big) = \phi^{-1}\big(\bigcup_{n\geq 0} f(A_n)\big) = \phi^{-1}\big(\bigcup_{n\geq 0} B_{n+1}\big) = \phi^{-1}\big(\bigcup_{n\geq 1} B_n\big) = \bigcup_{n\geq 0} B_n,$$

so ψ defines a bijection

$$\psi : \bigcup_{n\geq 0} A_n \to \bigcup_{n\geq 0} B_n.$$

We then combine ψ with the bijection $f : C \to D$, i.e. we define the map $h : A \to B$ by

$$h(x) = \begin{cases} \psi(x) & \text{if } x \in \bigcup_{n\geq 0} A_n \\ f(x) & \text{if } x \in A \smallsetminus \bigcup_{n\geq 0} A_n = C. \end{cases}$$

Clearly h is injective, and

$$h(B) = \psi\big(\bigcup_{n\geq 0} A_n\big) \cup f(C) = \big(\bigcup_{n\geq 0} B_n\big) \cup D = B,$$

so h is indeed bijective. $\square$

THEOREM B.2 (Total ordering for cardinal numbers). *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers. Then one has either $\mathfrak{a} \leq \mathfrak{b}$, or $\mathfrak{b} \leq \mathfrak{a}$.*

PROOF. Choose two sets A and B with $\operatorname{card} A = \mathfrak{a}$ and $\operatorname{card} B = \mathfrak{b}$. In order to prove the theorem, it suffices to construct either an injective function $f : A \to B$, or an injective function $f : B \to A$.

We define the set

$$\mathcal{X} = \{(C, D, g) \,:\, C \subset A,\ D \subset B,\ g : C \to D \text{ bijection}\}.$$

We equip $\mathcal{X}$ with the following order relation:

$$(C, D, g) \prec (C', D', g') \Longleftrightarrow \begin{cases} C \subset C' \\ D \subset D' \\ g = g'\big|_C \end{cases}$$

We now check that $(\mathcal{X}, \prec)$ satisfies the hypothesis of Zorn Lemma. Let $\mathcal{A} \subset \mathcal{X}$ be a totally ordered subset, say $\mathcal{A} = \big\{(C_i, D_i, g_i) \,:\, i \in I\big\}$. Define $C = \bigcup_{i\in I} C_i$, $D = \bigcup_{i\in I} D_i$, and $g : C \to D$ to be the unique function with the property that $g\big|_{C_i} = g_i$, $\forall\, i \in I$. (We use here the fact that for $i, j \in I$ we either have $C_i \subset C_j$ and $g_j\big|_{C_i} = g_i$, or $C_j \subset C_i$ and $g_i\big|_{C_j} = g_j$. In either case, this proves that $g_i\big|_{C_i\cap C_j} = g_j\big|_{C_i\cap C_j}$, $\forall\, i, j \in I$, so such a g exists.) It is then pretty clear that $(C, D, g) \in \mathcal{X}$ and $(C_i, D_i, g_i) \prec (C, D, g)$, $\forall\, i \in I$, i.e. (C, D, g) is an upper bound for $\mathcal{A}$. Use now Zorn Lemma, to find a maximal element (A_0, B_0, f) in $\mathcal{X}$.

Claim: *Either $A_0 = A$ or $B_0 = B$.*

We prove this by contradiction. If we have strict inclusions $A_0 \subsetneq A$ and $B_0 \subsetneq B$, then if we choose $a \in A \smallsetminus A_0$ and $b \in B \smallsetminus B_0$, we can define a bijection $g : A_0 \cup \{a\} \to B_0 \cup \{b_0\}$ by $g(a) = b$ and $g\big|_{A_0} = f$. This would then produce a new element $(A_0 \cup \{a\}, B_0 \cup \{b\}, g) \in \mathcal{X}$, which would contradict the maximality of (A_0, B_0, f).

The theorem now follows immediately from the Claim. If $A_0 = A$, then $f : A \to B$ is injective, and if $B_0 = B$, then $f : B \to A$ is injective. $\square$

We now define the operations with cardinal numbers.

Definitions. Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers.

- We define $\mathfrak{a}+\mathfrak{b} = \operatorname{card} S$, where S is any set which is of the form $S = A\cup B$ with $\operatorname{card} A = \mathfrak{a}$, $\operatorname{card} B = \mathfrak{b}$, and $A \cap B = \varnothing$.
- We define $\mathfrak{a}\cdot\mathfrak{b} = \operatorname{card} P$, where P is any set which is of the form $P = A\times B$ with $\operatorname{card} A = \mathfrak{a}$ and $\operatorname{card} B = \mathfrak{b}$.
- We define $\mathfrak{a}^{\mathfrak{b}} = \operatorname{card} X$, where X is any set of the form X which is of the form
$$X = \prod_{i\in I} A_i,$$
with $\operatorname{card} I = \mathfrak{b}$ and $\operatorname{card} A_i = \mathfrak{a}$, $\forall\, i \in I$. Equivalently, if we take two sets A and B with $\operatorname{card} A = \mathfrak{a}$, and $\operatorname{card} B = \mathfrak{b}$, and if we define
$$A^B = \prod_B A = \{f \,:\, f \text{ function from } B \text{ to } A\},$$
then $\mathfrak{a}^{\mathfrak{b}} = \operatorname{card}(A^B)$.

It is pretty easy to show that these definitions are correct, in the sense that they do not depend on the particular choices of the sets involved. Moreover, these operations are consistent with the usual operations with natural numbers.

Remark B.1. The operations with cardinal numbers, defined above, satisfy:

- $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$,
- $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{d} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{d})$,

- $\mathfrak{a} + 0 = \mathfrak{a}$,
- $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}$,
- $(\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{d} = \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{d})$,
- $\mathfrak{a} \cdot 1 = \mathfrak{a}$,
- $\mathfrak{a} \cdot (\mathfrak{b} + \mathfrak{d}) = (\mathfrak{a} \cdot \mathfrak{b}) + (\mathfrak{a} \cdot \mathfrak{d})$,
- $(\mathfrak{a} \cdot \mathfrak{b})^{\mathfrak{d}} = (\mathfrak{a}^{\mathfrak{d}}) \cdot (\mathfrak{b}^{\mathfrak{d}})$,
- $\mathfrak{a}^{\mathfrak{b}+\mathfrak{d}} = (\mathfrak{a}^{\mathfrak{b}}) \cdot (\mathfrak{a}^{\mathfrak{d}})$,
- $(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{d}} = (\mathfrak{a}^{\mathfrak{b} \cdot \mathfrak{d}}$,

for all cardinal numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{d} \geq 1$.

Remark B.2. The order relation $\leq$ is compatible with all the operations, in the sense that, if $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{b}_1$, and $\mathfrak{b}_2$ are cardinal numbers with $\mathfrak{a}_1 \leq \mathfrak{a}_2$ and $\mathfrak{b}_1 \leq \mathfrak{b}_2$, then

- $\mathfrak{a}_1 + \mathfrak{b}_1 \leq \mathfrak{a}_2 + \mathfrak{b}_2$,
- $\mathfrak{a}_1 \cdot \mathfrak{b}_1 \leq \mathfrak{a}_2 \cdot \mathfrak{b}_2$,
- $\mathfrak{a}_1^{\mathfrak{b}_1} \leq \mathfrak{a}_2^{\mathfrak{b}_2}$.

Proposition B.1. *Let $\mathfrak{a} \geq 1$ be a cardinal number.*

(i) *If A is a set with* $\operatorname{card} A = \mathfrak{a}$*, and if we define*

$$\mathcal{P}(A) = \{B : B \text{ subset of } A\},$$

then $2^{\mathfrak{a}} = \operatorname{card} \mathcal{P}(A)$.

(ii) $\mathfrak{a} < 2^{\mathfrak{a}}$.

Proof. (i). Put

$$P = \{0,1\}^A = \big\{f : f \text{ function from } A \text{ to } \{0,1\}\big\},$$

so that $2^{\mathfrak{a}} = \operatorname{card} P$. We need to define a bijection $\phi : P \to \mathcal{P}(A)$. We take

$$\phi(f) = \{a \in A : f(a) = 1\}, \ \ \forall f \in P.$$

It is clear that, since a function $f : A \to \{0,1\}$ is completely determined by the set $\{a \in A : f(a) = 1\}$, the map ϕ is indeed bijective.

(ii). The map $A \ni a \longmapsto \{a\} \in \mathcal{P}(A)$ is clearly injective. This prove the inequality $\mathfrak{a} \leq 2^{\mathfrak{a}}$. We now prove that $\mathfrak{a} \neq 2^{\mathfrak{a}}$, by contradiction. Assume there is a bijection $\theta : A \to \mathcal{P}(A)$. Define the set

$$B = \{a \in A : a \notin \theta(a)\},$$

and choose $b \in A$ such that $B = \theta(b)$. If $b \in B$, then by construction we get $b \notin \theta(b) = B$, which is impossible. If $b \notin B$, we have $b \notin \theta(b)$, which forces $b \in B$, again an impossibility. $\square$

We now discuss the properties of these operations, when infinite cardinal numbers are used.

Lemma B.1 (Properties of $\aleph_0$).

(i) *For any infinite cardinal number $\mathfrak{a}$, one has the inequality $\aleph_0 \leq \mathfrak{a}$.*

(ii) $\aleph_0 + \aleph_0 = \aleph_0$*;*

(iii) $\aleph_0 \cdot \aleph_0 = \aleph_0$*;*

Proof. (i). Let $\mathfrak{a}$ be an infinite cardinal number, and let A be an infinite set A, with $\operatorname{card} A = \mathfrak{a}$. Since for every finite subset $F \subset A$, there exists some $x \in A \smallsetminus F$, one to construct a sequence $(x_n)_{n\in\mathbb{N}} \subset A$, with $x_m \neq x_n$, $\forall\, m > n \geq 1$. Then the subset $B = \{x_n : n \in \mathbb{N}\}$ has $\operatorname{card} B = \aleph_0$, so the inclusion $B \subset A$ gives the desired inequality.

(ii). Consider the sets

$$A_0 = \{n \in \mathbb{N} : n, \text{ even}\} \text{ and } A_1 = \{n \in \mathbb{N} : n, \text{ odd}\}.$$

Then clearly $\operatorname{card} A_0 = \operatorname{card} A_1 = \aleph_0$, and the equality $A_0 \cup A_1 = \mathbb{N}$ gives

$$\aleph_0 + \aleph_0 = \operatorname{card} A_0 + \operatorname{card} A_1 = \operatorname{card}(A_0 \cup A_1) = \operatorname{card} \mathbb{N} = \aleph_0.$$

(iii). Take the set $P = \mathbb{N} \times \mathbb{N}$, so that $\aleph_0 \cdot \aleph_0 = \operatorname{card} P$. It is obvious that $\operatorname{card} P \geq \aleph_0$. To prove the other inequality, we define a surjection $\phi : \mathbb{N} \to P$ as follows. For each $n \geq 1$ we take $s_n = n(n-1)/2$, we set

$$B_n = \{m \in \mathbb{N} : s_n < m \leq s_{n+1}\},$$

and we define $\phi_n : B_n \to P$ by

$$\phi(m) = (n + s_n - m, m - s_n + 1),\ \ \forall\, m \in B_n.$$

Notice that

$$\phi_n(B_n) = \{(p, q) \in \mathbb{N} \times \mathbb{N} : p + q = n + 1\}. \tag{1}$$

Notice also that $\bigcup_{n\geq 1} B_n = \mathbb{N}$, and $B_j \cap B_k = \varnothing$, $\forall\, j > k \geq 1$, so there exists a (unique) function $\phi : \mathbb{N} \to P$, such that $\phi\big|_{B_n} = \phi_n$, for all $n \geq 1$. By (1) it is clear that ϕ is surjective.

$\square$

Theorem B.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinal numbers, with $1 \leq \mathfrak{b} \leq \mathfrak{a}$, and $\mathfrak{a}$ infinite. Then:*

(i) $\mathfrak{a} + \mathfrak{b} = \mathfrak{a}$;
(ii) $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{a}$.

Proof. It is clear that

$$\mathfrak{a} \leq \mathfrak{a} + \mathfrak{b} \leq \mathfrak{a} + \mathfrak{a},$$
$$\mathfrak{a} \leq \mathfrak{a} \cdot \mathfrak{b} \leq \mathfrak{a} \cdot \mathfrak{a},$$

so in order to prove the theorem, we can assume that $\mathfrak{a} = \mathfrak{b}$.

(i). Fix some set A with $\operatorname{card} A = \mathfrak{a}$. Use Zorn Lemma, to find a maximal non-empty family $\{A_i : i \in I\}$ of subsets of A with

(a) $\operatorname{card} A_i = \aleph_0$, for all $i, j \in I$;
(b) $A_i \cap A_j = \varnothing$, for all $i, j \in I$ with $i \neq j$.

If we put $B = A \smallsetminus \big(\bigcup_{i\in I} A_i\big)$, then by maximality it follows that B is finite. In particular, if we take $i_0 \in I$ then obviously $\operatorname{card}(A_{i_0} \cup B) = \aleph_0$, so if we replace A_{i_0} with $A_{i_0} \cup B$, we will still have the above properties (a) and (b), but also $A = \bigcup_{i\in I} A_i$. This proves that $\mathfrak{a} = \operatorname{card} A = \aleph_0 \cdot \mathfrak{d}$, where $\mathfrak{d} = \operatorname{card} I$. In other words, we have $\mathfrak{a} = \operatorname{card}(\mathbb{N} \times I)$. Consider then the sets

$$C_0 = \{n \in \mathbb{N} : n \text{ even}\} \text{ and } C_1 = \{n \in \mathbb{N} : n \text{ odd}\},$$

so that $(C_0 \times I) \cup (C_1 \times I) = I \times \mathbb{N}$, and $(C_0 \times I) \cap (C_1 \times I) = \varnothing$. In particular, we get

$$\begin{aligned} \mathfrak{a} &= \operatorname{card}(C_0 \times I) + \operatorname{card}(C_1 \times I) = \\ &= (\operatorname{card} C_0) \cdot (\operatorname{card} I) + (\operatorname{card} C_1) \cdot (\operatorname{card} I) = \\ &= \aleph_0 \cdot \mathfrak{d} + \aleph_0 \cdot \mathfrak{d} = \mathfrak{a} + \mathfrak{a}. \end{aligned}$$

(ii). Fix A a set with $\operatorname{card} A = \mathfrak{a}$. We are going to employ Zorn Lemma to find a bijection $A \to A \times A$. Define

$$\mathcal{X} = \{(D, f) : D \subset A,\ f : D \to D \times D \text{ bijective}\}.$$

Equip $\mathcal{X}$ with the following order

$$(D, f) \prec (D', f') \Longleftrightarrow \begin{cases} D \subset D' \\ f = f'\big|_D \end{cases}$$

Notice that $\mathcal{X}$ is non-empty, since we can find at leas one set $D \subset A$ with $\operatorname{card} D = \aleph_0$. We now check that $\mathcal{X}$ satisfies the hypothesis of Zorn Lemma. Let $\mathcal{T} = \{(D_i, f_i) : i \in I\}$ be a totally ordered subset of $\mathcal{X}$. It is fairly clear that if one takes $D = \bigcup_{i \in I}$ and one defines $f : D \to D \times D$ as the unique function with $f\big|_{D_i} = f_i$, $\forall\, i \in I$, then f is injective, and

$$f(D) = \bigcup_{i \in I} f(D_i) = \bigcup_{i \in I} f_i(D_i) = \bigcup_{i \in I} (D_i \times D_i) = D \times D,$$

so the pair (D, f) indeed belongs to $\mathcal{X}$, and is an upper bound for $\mathcal{T}$.

Use Zorn Lemma to produce a maximal element $(D, f) \in \mathcal{X}$. Notice that, if we take $\mathfrak{d} = \operatorname{card} D$, then by construction we have

$$\mathfrak{d} \cdot \mathfrak{d} = \mathfrak{d}. \tag{2}$$

We would like to prove that $D = A$. In general this is not the case (for example, when $A = \mathbb{N}$, every $(D, f) \in \mathcal{X}$, with $\mathbb{N} \smallsetminus D$ finite, is automatically maximal). We notice however that all we need to show is the equality

$$\mathfrak{d} = \mathfrak{a}. \tag{3}$$

We prove this equality by contradiction. We know that we already have $\mathfrak{d} \leq \mathfrak{a}$. Suppose $\mathfrak{d} < \mathfrak{a}$. Put $G = A \smallsetminus D$ notice that $\mathfrak{d} + \operatorname{card} G = \mathfrak{a}$. Since $\mathfrak{d} < \mathfrak{a}$, by (i) we see that we must have the equality $\operatorname{card} G = \mathfrak{a}$. Then there exists a subset $E \subset G$ with $\operatorname{card} E = \mathfrak{d}$. Consider the set

$$P = (E \times E) \cup (E \times D) \cup (D \times E).$$

Since $E \cap D = \varnothing$, the three sets above are pairwise disjoint, so using (2) combined again with part (i), we get

$$\begin{aligned} \operatorname{card} P &= \operatorname{card}(E \times E) + \operatorname{card}(E \times D) + \operatorname{card}(D \times E) = \\ &= \mathfrak{d} \cdot \mathfrak{d} + \mathfrak{d} \cdot \mathfrak{d} + \mathfrak{d} \cdot \mathfrak{d} = \mathfrak{d} + \mathfrak{d} + \mathfrak{d} = \mathfrak{d} = \operatorname{card} E. \end{aligned}$$

This means that there exists a bijection $g : E \times P$, which combined with the fact that $E \cap D = P \cap (D \times D) = \varnothing$, will produce a bijection $h : D \cup E \to P \cup (D \times D)$, such that $h\big|_D = f$ and $h\big|_E = g$. Since we have $P \cup (D \times D) = (D \cup E) \times (D \cup E)$, the pair $(D \cup E, h) \in \mathcal{X}$ will contradict the maximality of (D, f). $\square$

Corollary B.1. *If $\mathfrak{a}$ is an infinite cardinal number, and if $\mathfrak{b}$ is a cardinal number with $2 \le \mathfrak{b} \le 2^{\mathfrak{a}}$, then*

$$\mathfrak{b}^{\mathfrak{a}} = 2^{\mathfrak{a}}.$$

Proof. We have

$$2^{\mathfrak{a}} \le \mathfrak{b}^{\mathfrak{a}} \le (2^{\mathfrak{a}})^{\mathfrak{a}} = 2^{\mathfrak{a}\cdot\mathfrak{a}} = 2^{\mathfrak{a}},$$

and the desired equality follows from the Cantor-Bernstein Theorem. □

Corollary B.2. *Let $\mathfrak{a}$ be an infinite cardinal number, let A be a set with* $\operatorname{card} A = \mathfrak{a}$*, and define*

$$\mathcal{P}_{fin}(A) = \{F \in \mathcal{P}(A) : F \text{ finite}\}.$$

Then $\operatorname{card} \mathcal{P}_{fin}(A) = \mathfrak{a}$.

Proof. First of all, the map $A \ni a \longmapsto \{a\} \in \mathcal{P}_{fin}(A)$ is injective, so $\mathfrak{a} \le \operatorname{card} \mathcal{P}_{fin}(A)$.

We now prove the other inequality. For every integer $n \ge 1$, let A^n denote the n-fold cartesian product. We treat the sequence $A^1, A^2, \dots$ as pairwise disjoint. For every $n \ge 1$ we define the map

$$\phi_n : A^n \to \mathcal{P}_{fin}(A),$$

by

$$\phi(a_1, \dots, a_n) = \{a_1, \dots, a_n\},$$

and we define the map $\phi : \bigcup_{n=1}^{\infty} A^n \to \mathcal{P}_{fin}(A)$ as the unique map such that $\phi\big|_{A^n} = \phi_n$, $\forall\, n \ge 1$. Notice now that, since

$$\operatorname{card} A^n = \mathfrak{a}^n = \mathfrak{a}, \ \ \forall\, n \ge 1,$$

it follows that

$$\operatorname{card}\Big(\bigcup_{n=1}^{\infty} A^n\Big) = \aleph_0 \cdot \mathfrak{a} = \mathfrak{a},$$

which gives

$$\operatorname{card}(\operatorname{Range} \phi) \le \mathfrak{a}.$$

But it is clear that

$$\{\varnothing\} \cup \operatorname{Range} \phi = \mathcal{P}_{fin}(A),$$

and the fact that $\mathcal{P}_{fin}(A)$ is infinite, proves that

$$\operatorname{card} \mathcal{P}_{fin}(A) = \operatorname{card}(\operatorname{Range} \phi) \le \mathfrak{a}.$$

□

We conclude with a result on the cardinal number $\mathfrak{c} = \operatorname{card} \mathbb{R}$.

Proposition B.2.

(i) *For two real numbers $a < b$, one has*

$$\operatorname{card}(a,b) = \operatorname{card}[a,b) = \operatorname{card}(a,b] = \operatorname{card}[a,b] = \mathfrak{c}.$$

(ii) $\mathfrak{c} = 2^{\aleph_0}$.

PROOF. (i). It is clear that, since (a, b) is infinite, we have

$$\text{card}[a, b] = 2 + \text{card}(a, b) = \text{card}(a, b).$$

The inclusions $(a, b) \subset [a, b) \subset [a, b]$ and $(a, b) \subset (a, b] \subset [a, b]$, combined with the Cantor-Bernstein Theorem, immediately give

$$\text{card}[a, b) = \text{card}(a, b] = \text{card}(a, b).$$

Finally, the bijection

$$(a, b) \ni t \longmapsto \tan\left(\frac{\pi(2t - a - b)}{2(b - a)}\right) \in \mathbb{R}$$

shows that $\text{card}(a, b) = \mathfrak{c}$.

(ii). The proof of this result uses a certain construction, which is useful for many other purposes. Therefore we choose to work in full generality. Consider the set

$$T = \{0, 1\}^{\aleph_0} = \{a = (\alpha_n)_{n\in\mathbb{N}} : \alpha_n \in \{0, 1\}, \ \forall\, n \in \mathbb{N}\},$$

so $2^{\aleph_0} = \text{card}\, P$. For any real number $r \geq 2$, we define the map $\phi_r : T \to [0, 1]$ by

$$\phi(a) = (r - 1) \sum_{n=1}^{\infty} \frac{\alpha_n}{r^n}, \ \ \forall\, a = (\alpha_n)_{n\in\mathbb{N}} \in T.$$

The maps ϕ_r, $r \geq 2$ are "almost" injective. To clarify this, we define the set

$$T_0 = \{a = (\alpha_n)_{n\in\mathbb{N}} \in T : \ \text{the set } \{n \in \mathbb{N} : \alpha_n = 0\} \text{ is infinite}\}.$$

Note that

$$T \smallsetminus T_0 = \{(\alpha_n)_{n\in\mathbb{N}} \in T : \ \text{there exists } N \in \mathbb{N}, \text{ such that } \alpha_n = 1, \ \forall\, n \geq N\}.$$

Clearly ϕ is surjective. In fact ϕ is "almost" bijective.

Claim 1: *Fix $r \geq 2$. For elements $a = (\alpha_n)_{n\in\mathbb{N}}$, $b = (\beta_n)_{n\in\mathbb{N}} \in T_0$, the following are equivalent*

($*$) $\phi_r(a) > \phi_r(b)$;

($**$) *there exists $k \in \mathbb{N}$, such that $alpha_k > \beta_k$, and $\alpha_j = \beta_j$, for all $j \in \mathbb{N}$ with $j < k$.*

We first prove the implication $(**) \Rightarrow (*)$. If $a, b \in T_0$ satisfiy $(**)$, then

$$\phi_r(a) - \phi_r(b) = \frac{r - 1}{r^k} + (r - 1) \sum_{n=k+1}^{\infty} \frac{\alpha_n - \beta_n}{r^n} \geq \frac{r - 1}{r^k} - (r - 1) \sum_{n=k+1}^{\infty} \frac{\beta_n}{2^n}. \tag{4}$$

Notice now that there are infinitely many indices $n \geq k + 1$ such that $\beta_n = 0$. This gives the fact that

$$\sum_{n=k+1}^{\infty} \frac{\beta_n}{r^n} < \sum_{n=k+1}^{\infty} \frac{1}{r^n} = \frac{1}{(r - 1)r^k},$$

so if we go back to (4) we get

$$\phi_r(a) - \phi_r(b) \geq \frac{r - 1}{r^k} - (r - 1) \sum_{n=k+1}^{\infty} \frac{\beta_n}{r^n} > \frac{r - 1}{r^k} - \frac{1}{r^k} = \frac{r - 2}{r^k} \geq 0,$$

so in particular we get $\phi_r(a) > \phi_r(b$.

Conversely, if $\phi_r(a) > \phi_r(b)$, we choose

$$k = \min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\}.$$

Using the implication $(**) \Rightarrow (*)$ we see that we cannot have $\beta_k > \alpha_k$, because this would force $\phi(b) > \phi(a)$. Therefore we must have $\alpha_k > \beta_k$, and we are done.

Using Claim 1, we now see that $\phi_r\big|_{T_0} : T_0 \to [0,1]$ is *injective*

Claim 2: $\operatorname{card}(T \smallsetminus T_0) = \aleph_0$.

This is pretty clear, since we can write

$$T \smallsetminus T_0 = \bigcup_{k=1}^{\infty} R_k,$$

where

$$R_n = \big\{a = (\alpha_n)_{n\in\mathbb{N}} \in T \,:\, \alpha_n = 1, \ \ \forall\, n \geq 1\big\}.$$

Since each R_n is finite, the desired result follows.

Using Claim 2, we have

$$2^{\aleph_0} = \operatorname{card} T = \operatorname{card}(T \smallsetminus T_0) + \operatorname{card} T_0 = \aleph_0 + \operatorname{card} T_0.$$

Since $\aleph_0 < 2^{\aleph_0}$, the above equality forces

$$2^{\aleph_0} = \operatorname{card} T_0.$$

For every $r \geq 2$, we also have $\operatorname{card} \phi_r(T \smallsetminus T_0) \leq \aleph_0$, which then gives $\operatorname{card}\big[\phi_r(T) \smallsetminus \phi_r(T_0)\big] \leq \aleph_0$, hence using the injectivity of $\phi_r\big|_{T_0}$, we have $\operatorname{card} \phi_r(T_0) = \operatorname{card} T_0 = 2^{\aleph_0}$, so we get

$$2^{\aleph_0} = \operatorname{card} \phi_r(T_0) \leq \operatorname{card} \phi_r(T) = \operatorname{card} \phi_r(T_0) + \operatorname{card}\big[\phi_r(T) \smallsetminus \phi_r(T_0)\big] \leq \operatorname{card} phi_r(T_0) + \aleph_0 = 2^{\aleph_0} + \aleph_0 = 2^{\aleph_0}.$$

By the Cantor-Bernstein Theorem this forces $\operatorname{card} \phi_r(T) = 2^{\aleph_0}$.

Now we are done, since for $r = 2$ we clearly have $\phi_2(T) = [0,1]$. □

COROLLARY B.3.

(i) $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$.

(ii) *If we define the set*

$$\mathcal{P}_{count} = \{C \subset \mathbb{R} \,:\, \operatorname{card} F \leq \aleph_0\},$$

then $\operatorname{card} \mathcal{P}_{count}(\mathbb{R}) = \mathfrak{c}$.

PROOF. (i). This is immediate from the equality $2^{\aleph_0} = \mathfrak{c}$ and from Corollary B.1.

(ii). Using the inclusion $\mathcal{P}_{fin}(\mathbb{R}) \subset \mathcal{P}_{count}(\mathbb{R})$, combined with Corollary B.2, we see that we have the inequality

$$\mathfrak{c} \leq \operatorname{card}\mathcal{P}_{count}(\mathbb{R}).$$

To prove the other inequality, we define a map $\phi : \mathbb{R}^{\mathbb{N}} \to \mathcal{P}_{count}(\mathbb{R})$, as follows. If $a \in \mathbb{R}^{\mathbb{N}}$ is a sequence, say $a = (\alpha_n)_{n\in\mathbb{N}}$, we put

$$\phi(a) = \{\alpha_n \,:\, n \in \mathbb{N}\}.$$

Since ϕ is clearly surjective, using part (i) we get

$$\operatorname{card} \mathcal{P}_{count}(\mathbb{R}) \leq \operatorname{card} \mathbb{R}^{\mathbb{N}} = \mathfrak{c}^{\aleph_0} = \mathfrak{c}.$$

□