6. Duals of $L^p$ spaces

This section deals with the problem of identifying the duals of $L^p$ spaces, $p \in [1, \infty)$. There are essentially two cases of this problem: (i) $p = 1$; (ii) $1 < p < \infty$. The major difference between these two cases is the fact that for $1 < p < \infty$, there is a “nice” characterization of the dual of $L^p$ which identifies it with $L^q$ (with $\frac{1}{p} + \frac{1}{q} = 1$), and this identification holds without any restriction on the underlying space. On the other hand, the dual of $L^1$ will be identified with $L^\infty_{loc}$, but this identification will work only if the underlying measure space is decomposable (e.g. $\sigma$-finite). Of course, one can also pose the problem of identifying the duals of the spaces $L^\infty$ and $L^\infty_{loc}$, introduced in Section 5, but we shall not deal with it.

We start off with the study of the finite case.

**Comment.** The finite case is particularly nice, since among other things the $L^p$ spaces contain a “blueprint” of the underlying measure space. More precisely, if $(X, \mathcal{A}, \mu)$ is a finite measure space, then the characteristic functions $\kappa_A$, $A \in \mathcal{A}$, belong to all spaces $L^p_k(X, \mathcal{A}, \mu)$, $1 \leq p \leq \infty$. The following result somehow explains how the union can be “seen in $L^p$.”

**Lemma 6.1.** Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let $p \in [1, \infty)$. If $(A_n)_{n=1}^\infty$ is a pairwise disjoint sequence, and if we put $A = \bigcup_{n=1}^\infty A_n$, then one has the equality $\sum_{n=1}^\infty \kappa_{A_n} = \kappa_A$ in $L^p_k(X, \mathcal{A}, \mu)$. In other words, one has the equality

$$\kappa_A = L^p \lim_{n \to \infty} f_n,$$

where $(f_n)_{n=1}^\infty \subset L^p_k(X, \mathcal{A}, \mu)$ is the sequence of partial sums $f_n = \sum_{k=1}^n \kappa_{A_k}$, $n \geq 1$.

**Proof.** One has $f_n = \kappa_{B_n}$, where $B_n = A_1 \cup \cdots \cup A_n$, $\forall \ n \geq 1$. In particular, we get

$$\int_X |f_n - \kappa_A|^p \, d\mu = \int_X |\kappa_{A \setminus B_n}|^p \, d\mu = \int_X \kappa_{A \setminus B_n} \, d\mu = \mu(A \setminus B_n), \ \forall \ n \geq 1.$$

This computation can be re-written as

$$\|f_n - \kappa_A\|_p = \mu(A \setminus B_n)^{1/p}, \ \forall \ n \geq 1.$$

Notice that using the inclusions $A \setminus B_1 \supset A \setminus B_2 \supset \cdots$, combined with the equality $\bigcap_{n=1}^\infty [A \setminus B_n] = \emptyset$, by the continuity property (it is key here that $\mu$ is finite), we get $\lim_{n \to \infty} \mu(A \setminus B_n) = \mu(\emptyset) = 0$, so the equalities (1) immediately give $\lim_{n \to \infty} \|f_n - \kappa_A\|_p = 0$. □

An important consequence of this fact, we get our first duality result.

**Theorem 6.1.** Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty)$, and let $\phi : L^p_k(X, \mathcal{A}, \mu) \to \mathbb{K}$ be a linear continuous map. Define the number $q = p/(p - 1)$ (with the convention $1/0 = \infty$). Then there exists some function $f \in L^q_k(X, \mathcal{A}, \mu)$, such that

$$\phi(g) = \int_X fg \, d\mu, \ \forall \ g \in L^p_k(X, \mathcal{A}, \mu).$$

Moreover one has the following.
and since both $\psi$ are speaking, we have $\psi = \phi$. We get the equality

We know that $S$ so in fact $L$ subspace is $\text{Ran} L$ which comes from the inclusion $L \subset \text{Ran} L$.

We wish to extend the above equality beyond elementary functions.

Proof. We begin with the existence part.

Start off by considering the correspondence

$$\nu_\phi : A \ni A \longmapsto \phi(\mathcal{X}_A) \in \mathbb{K}$$

By Lemma 6.1, $\nu_\phi$ defines a $\mathbb{K}$-valued measure on $\mathcal{A}$. Note also that $\nu_\phi$ is absolutely continuous with respect to $\mu$. Indeed, if $A \in \mathcal{A}$ has $\mu(A)$, then $\mathcal{X}_A = 0$ in $L^p$, so $\nu_\phi(A) = \phi(\mathcal{X}_A) = 0$.

Use the Radon-Nikodym Theorem (the finite case) to find some function $f \in L^1_k(X, \mathcal{A}, \mu)$, such that

$$\phi(\mathcal{X}_A) = \int_X f \mathcal{X}_A \, d\mu, \quad \forall A \in \mathcal{A}. \tag{3}$$

Since $\phi$ is linear, the equality (3) can be extended to give

$$\phi(g) = \int_X fg \, d\mu, \quad \forall g \in L^p_{K, \text{elem}}(X, \mathcal{A}, \mu). \tag{4}$$

We wish to extend the above equality beyond elementary functions.

Since $(X, \mathcal{A}, \mu)$ is finite, one has an injective linear continuous map

$$J_p : L^\infty_k(X, \mathcal{A}, \mu) \to L^p_k(X, \mathcal{A}, \mu),$$

which comes from the inclusion $L^\infty_k(X, \mathcal{A}, \mu) \subset L^p_k(X, \mathcal{A}, \mu)$. Using the map $J_p$, we identify $L^\infty_k(X, \mathcal{A}, \mu)$ as a linear subspace in $L^p_k(X, \mathcal{A}, \mu)$. (Strictly speaking, this subspace is $\text{Ran} J_p$.) We also know that, one has the obvious inclusions

$$L^p_{\text{elem}, K}(X, \mathcal{A}, \mu) = L^\infty_{\text{elem}, K}(X, \mathcal{A}, \mu) \subset L^\infty_k(X, \mathcal{A}, \mu) \subset L^p_k(X, \mathcal{A}, \mu),$$

so in fact $L^\infty_k(X, \mathcal{A}, \mu)$ is a dense linear subspace of $L^p_k(X, \mathcal{A}, \mu)$.

Claim 1: One has the equality

$$\phi(g) = \int_X fg \, d\mu, \quad \forall g \in L^\infty_k(X, \mathcal{A}, \mu). \tag{5}$$

Consider the linear map $S_f : L^\infty_k(X, \mathcal{A}, \mu) \to \mathbb{K}$, defined by

$$S_f(g) = \int_X fg \, d\mu, \quad \forall g \in L^\infty_k(X, \mathcal{A}, \mu).$$

We know that $S_f$ is continuous (Proposition 5.5). Notice that the restriction $\psi = \phi|_{L^\infty_k(X, \mathcal{A}, \mu)}$ gives rise to a linear continuous map $\psi : L^\infty_k(X, \mathcal{A}, \mu) \to \mathbb{K}$. (Strictly speaking, we have $\psi = \phi \circ J_p$.)

Then the equality (4) gives

$$\psi(g) = \int_X fg \, d\mu = S_f(g), \quad \forall g \in L^\infty_{\text{elem}, K}(X, \mathcal{A}, \mu),$$

and since both $\psi$ and $S_f$ are continuous, and $L^\infty_{\text{elem}, K}(X, \mathcal{A}, \mu)$ is dense in $L^\infty_k(X, \mathcal{A}, \mu)$, we get the equality $\psi = S_f$, which is precisely (5).

Claim 2: The function $f$ belongs to $L^p_k(X, \mathcal{A}, \mu)$. 

\section{Duals of $L^p$ spaces}
We start with the easy case $p = 1$. In this case we observe that, since $\|\kappa_A\|_1 = \mu(A), \forall A \in \mathcal{A}$, by the continuity of $\phi$, we get

$$|\nu_{\phi}(A)| = |\phi(\kappa_A)| \leq \|\phi\| \cdot \|\kappa_A\|_1 = \|\phi\| \cdot \mu(A), \forall A \in \mathcal{A},$$

so we can apply the “easy” Radon-Nikodym Theorem 4.1, to get the fact that

$$|f| \leq \|\phi\|, \mu\text{-a.e.}$$

We continue with the case $p > 1$. We are going to give an estimate (from above) for $\int_X |f|^q \, d\mu$.

By definition, we know that

$$\int_X |f|^q \, d\mu = \sup \left\{ \int_X h \, d\mu : h \in \mathcal{A}\text{-Elem}_\mathbb{R}(X), \ 0 \leq h \leq |f|^q \right\}.$$

Fix for the moment an $\mathcal{A}$-elementary function $h : X \to \mathbb{R}$, such that $0 \leq h \leq |f|^q$. Consider the measurable function $u : X \to \mathbb{K}$ by

$$u(x) = \begin{cases} |f(x)|/|f(x)| & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

so that one has the equality $uf = |f|$. Consider now the function $g = uh^{1/p}$. Since $h^{1/p}$ is elementary, and, $|u| \leq 1$, it follows that $g$ is measurable, and bounded, hence in $L^\infty$. By Claim 1, we then have

$$\phi(g) = \int_X fg \, d\mu = \int_X fuh^{\frac{1}{p}} \, d\mu = \int_X |f|h^{\frac{1}{p}} \, d\mu.$$

Among other things, this proves that $\phi(g) = |\phi(g)| \geq 0$, as well as (use $|f| \geq h^{\frac{1}{p}}$) the inequality

$$|\phi(g)| \geq \int_X h^{\frac{1}{2}}h^{\frac{1}{p}} \, d\mu = \int_X h \, d\mu.$$ 

By the continuity of $\phi$, this inequality forces

$$\int_X h \, d\mu \leq \|\phi\| \cdot \|g\|_p. \quad (6)$$

Let us estimate the norm $\|g\|_p$. Since $|g|^p = |u|^p h \leq h$, we clearly have

$$\|g\|_p \leq \left[ \int_X h \, d\mu \right]^{\frac{1}{p}},$$

so if we denote $\int_X h \, d\mu$ simply by $\alpha$, then (6) reads

$$\alpha \leq \|\phi\| \cdot \alpha^{\frac{1}{p}},$$

so we immediately get $\alpha \leq \|\phi\|^q$. Having proven the inequality

$$\int_X h \, d\mu \leq \|\phi\|^q,$$

for all elementary functions $h$, with $0 \leq h \leq |f|^q$, it now follows that $|f|^q$ is integrable, and

$$\int_X |f|^q \, d\mu \leq \|\phi\|^q,$$

so $f$ indeed belongs to $L^q_{\mathcal{A}}(X, \mathcal{A}, \mu)$ (and moreover, $\|f\|_q \leq \|\phi\|$).

Having proven the existence part, the other properties (i) and (ii) are clear from the results in Sections 3 and 5. \qed
Our next goal is to extend the above result beyond the finite case. In preparation for the proof in the general case, we need to develop a technique for “breaking” a linear continuous map \( \phi : L^p_k(X, \mathcal{A}, \mu) \to \mathbb{K} \) into (small) pieces. For this purpose we introduce the following.

**Notations.** Let \((X, \mathcal{A}, \mu)\) be a measure space, let \(\mathbb{K}\) be one of the fields \(\mathbb{R}\) or \(\mathbb{C}\), and let \(p \in [1, \infty)\). For any set \(A \in \mathcal{A}\), we define the map
\[
P_A : L^p_k(X, \mathcal{A}, \mu) \ni f \mapsto f|_A \in L^p_k(A, \mathcal{A}, \mu).
\]

It turns out that \(P_A\) is surjective, linear, continuous, and it has a right inverse defined as follows. For \(f \in L^p_k(A, \mathcal{A}, \mu)\), we define the function \(\hat{f} : X \to \mathbb{K}\) by
\[
\hat{f} = \begin{cases} 
    f(x) & \text{if } x \in A \\
    0 & \text{if } x \in X \setminus A
\end{cases}
\]
It turns out that \(\hat{f}\) belongs to \(L^p_k(X, \mathcal{A}, \mu)\), and \(\|\hat{f}\|_p = \|f\|_p\), so the correspondence
\[
L^p_k(A, \mathcal{A}, \mu) \ni f \mapsto \hat{f} L^p_k(X, \mathcal{A}, \mu)
\]
gives rise to a linear continuous map
\[
R_A : L^p_k(A, \mathcal{A}, \mu) \to L^p_k(X, \mathcal{A}, \mu),
\]
which satisfies \(P_A \circ R_A = \text{Id}\). Remark also that the composition the other way can be described as the multiplication map by \(\kappa_A\) (see Section 5):
\[
R_A \circ P_A = M^p_{\kappa_A}.
\]

**Definitions.** Let \((X, \mathcal{A}, \mu)\) be a measure space, let \(\mathbb{K}\) be one of the fields \(\mathbb{R}\) or \(\mathbb{C}\), and let \(p \in [1, \infty)\). Given any linear continuous map \(\phi : L^p_k(X, \mathcal{A}, \mu) \to \mathbb{K}\), we define, for every \(A \in \mathcal{A}\) the linear continuous maps
\[
\phi|_A = \phi \circ R_A : L^p_k(A, \mathcal{A}, \mu) \to \mathbb{K},
\]
\[
\phi_A = \phi \circ M^p_{\kappa_A} : L^p_k(X, \mathcal{A}, \mu) \to \mathbb{K}.
\]

One has the obvious equalities
\[
\phi_A = (\phi|_A) \circ P_A \text{ and } \phi|_A = \phi_A \circ R_A.
\]
Notice also that one has the equality
\[
\phi_A(f) = \phi_A(f \kappa_A), \quad \forall f \in L^p_k(X, \mathcal{A}, \mu).
\]
Remark that, since \(\|P_A\| = \|R_A\| \leq 1\), one also has
\[
\|\phi|_A\| = \|\phi_A\| \leq \|\phi\|.
\]
If we take \(A = X\), we clearly have \(\phi_X = \phi|_X = \phi\). Note also, that given sets \(A, B \in \mathcal{A}\) with \(A \supset B\), then one has
\[
(\phi_A)_B = \phi_B \text{ and } (\phi|_A)_B = \phi|_B.
\]

In particular one has \(\|\phi_A\| = \|\phi|_A\| \geq \|\phi_B\| = \|\phi|_B\|\).

**Remarks 6.1.** Use the notations above. Suppose \(A_1, \ldots, A_n \in \mathcal{A}\) are pairwise disjoint, and let \(A = A_1 \cup \cdots \cup A_n\).

A. For any linear continuous map \(\phi : L^p_k(X, \mathcal{A}, \mu)\), one has the equality
\[
\phi_A = \phi_{A_1} + \cdots + \phi_{A_n}.
\]
This is a consequence of the fact that the correspondence
\[ L^\infty_{\text{loc}}(X,\mathcal{A},\mu) \ni f \mapsto M^p_f \in \mathcal{L}(L^p_{\mathbb{K}}(X,\mathcal{A},\mu)) \]
is linear, so in particular one has
\[ M^p_{\mathcal{A}_g} = M^p_{\mathcal{A}_1} + \cdots + M^p_{\mathcal{A}_n}. \]

B. For any \( g \in L^p_{\mathbb{K}}(X,\mathcal{A},\mu) \), one has the equality
\[ \|g \mathcal{A}_g\|_p = \left(\left\|g \mathcal{A}_1\right\|_p^p + \cdots + \left\|g \mathcal{A}_n\right\|_p^p\right)^{\frac{1}{p}}. \]

This is quite clear, since
\[ \left(\left\|g \mathcal{A}_g\right\|_p^p\right) = \int_X \left\|g \mathcal{A}_g\right\|_p d\mu = \sum_{k=1}^n \int_X \left\|g \mathcal{A}_k\right\|_p d\mu = \sum_{k=1}^n \left\|g \mathcal{A}_k\right\|_p^p. \]

The following result gives an important property of these operations.

**Lemma 6.2.** Let \((X,\mathcal{A},\mu)\) be a measure space, let \(\mathbb{K}\) be either \(\mathbb{R}\) or \(\mathbb{C}\), and let \(A, B \in \mathcal{A}\) be disjoint sets.

A. For any linear continuous map \(\phi : L^p_{\mathbb{K}}(X,\mathcal{A},\mu) \to \mathbb{K}\), one has
\[ \|\phi_{A\cup B}\| = \max\left\{\|\phi_A\|,\|\phi_B\|\right\}. \]

B. If \(p, q \in (1, \infty)\) are such that \(\frac{1}{p} + \frac{1}{q} = 1\), then for any linear continuous map \(\phi : L^p_{\mathbb{K}}(X,\mathcal{A},\mu) \to \mathbb{K}\), one has
\[ \|\phi_{A\cup B}\| = \left[\|\phi_A\|^p + \cdots + \|\phi_A\|^q\right]^{\frac{1}{q}}. \]

**Proof.** One useful observation is the fact that, for every \(p \in [1, \infty)\) and every linear continuous map \(\phi : L^p_{\mathbb{K}}(X,\mathcal{A},\mu) \to \mathbb{K}\), one has
\[ (1) \quad \phi_{A\cup B}(g) = \phi_A(g) + \phi_B(g) = \phi_A(g \mathcal{A}_A) + \phi_B(g \mathcal{A}_B), \quad \forall g \in L^p_{\mathbb{K}}(X,\mathcal{A},\mu). \]

A. Let us remark first that, since \((\phi_{A\cup B})_A = \phi_A\) and \((\phi_{A\cup B})_B = \phi_B\), we clearly have \(\|\phi_{A\cup B}\| \geq \max\left\{\|\phi_A\|,\|\phi_B\|\right\}\). To prove the other inequality \(\|\phi_{A\cup B}\| \leq \max\left\{\|\phi_A\|,\|\phi_B\|\right\}\), we must show that
\[ (2) \quad \|\phi_{A\cup B}(g)\| \leq \max\left\{\|\phi_A\|,\|\phi_B\|\right\} \cdot \|g\|_1, \quad \forall g \in L^1_{\mathbb{K}}(X,\mathcal{A},\mu). \]

Start with some \(g \in L^1_{\mathbb{K}}(X,\mathcal{A},\mu)\). Using (1), we clearly have
\[ \|\phi_{A\cup B}(g)\| \leq \|\phi_A(g \mathcal{A}_A)\| + \|\phi_B(g \mathcal{A}_B)\| \leq \|\phi_A\| \cdot \|g \mathcal{A}_A\|_1 + \|\phi_B\| \cdot \|g \mathcal{A}_B\|_1 \leq \max\left\{\|\phi_A\|,\|\phi_B\|\right\} \cdot (\|g \mathcal{A}_A\|_1 + \|g \mathcal{A}_B\|_1) \]
\[ \leq \max\left\{\|\phi_A\|,\|\phi_B\|\right\} \cdot (\|g \mathcal{A}_A\|_1 + \|g \mathcal{A}_B\|_1) \]
Using Remark 6.1.B we also have
\[ \|g \mathcal{A}_A\|_1 + \|g \mathcal{A}_B\|_1 = \|g \mathcal{A}_{A\cup B}\|_1 \leq \|g\|_1, \]
and then using (9) we immediately get (8)

B. We first prove the inequality
\[ (3) \quad \|\phi_{A\cup B}\| \geq \left[\|\phi_A\|^p + \|\phi_B\|^q\right]^{\frac{1}{q}}. \]
We assume \(\|\phi_A\|,\|\phi_B\| > 0\) (otherwise there is nothing to prove). Start with some \(\varepsilon\) with \(0 < \varepsilon < \min\{\|\phi_A\|,\|\phi_B\|\}\), and choose two functions \(g, h \in L^p_{\mathbb{K}}(X,\mathcal{A},\mu)\) with \(\|g\|_p,\|h\|_p \leq 1\) such that
\[ |\phi_A(g)| \geq \|\phi_A\| - \varepsilon \quad \text{and} \quad |\phi_B(h)| \geq \|\phi_B\| - \varepsilon. \]
Replacing \( g \) with \( \frac{\phi_A(g)}{\phi_A(g)} \), we can assume that \( \phi_A(g) = |\phi_A(g)| \). Similarly, we can assume \( \phi_B(h) = |\phi_B(h)| \).

Fix for the moment an arbitrary pair \((\alpha, \beta) \in [0, \infty)^2\), with \( \alpha^p + \beta^p = 1 \), and define the function

\[
f = \alpha \varkappa_A + \beta \varkappa_B.
\]

By construction we have \( f = f \varkappa_{A \cup B} \), as well as \( f \varkappa_A = \alpha \varkappa_A \) and \( f \varkappa_B = \beta \varkappa_B \), so by Remark 6.1.B, we have

\[
\|f\|_p = \left[ (\|\alpha \varkappa_A\|_p)^p + (\|\beta \varkappa_B\|_p)^p \right]^{\frac{1}{p}} \leq \left[ (\|\alpha\|_p)^p + (\|\beta\|_p)^p \right]^{\frac{1}{p}} \leq [\alpha^p + \beta^p]^{\frac{1}{p}} = 1,
\]

so we get

\[
\|\phi_{A \cup B}\| \geq |\phi_{A \cup B}(f)|.
\]

Notice that, since \( \varkappa_A \varkappa_B = 0 \), clearly have have

\[
\phi_A(f) = \phi_A(\alpha \varkappa_A) = \alpha |\phi_A(g)| \quad \text{and} \quad \phi_B(f) = \phi_B(\beta \varkappa_B) = \beta |\phi_B(h)|,
\]

which yields

\[
\phi_{A \cup B}(f) = \alpha |\phi_A(g)| + \beta |\phi_B(g)| \geq \alpha (\|\phi_A\| - \varepsilon) + \beta (\|\phi_B\| - \varepsilon).
\]

Since \( \|f\|_p \leq 1 \), by the definition of the norm, the above estimate forces

\[
\|\phi_{A \cup B}\| \geq \alpha (\|\phi_A\| - \varepsilon) + \beta (\|\phi_B\| - \varepsilon).
\]

In the above inequality \( \alpha, \beta \in [0, \infty) \) are arbitrary, subject to \( \alpha^p + \beta^p = 1 \). In particular, if we consider numbers

\[
\alpha = \left[ \frac{(\|\phi_A\| - \varepsilon)^q}{(\|\phi_A\| - \varepsilon)^q + (\|\phi_B\| - \varepsilon)^q} \right]^{\frac{1}{q}} \quad \text{and} \quad \beta = \left[ \frac{(\|\phi_B\| - \varepsilon)^q}{(\|\phi_A\| - \varepsilon)^q + (\|\phi_B\| - \varepsilon)^q} \right]^{\frac{1}{q}},
\]

we get

\[
\|\phi_{A \cup B}\| \geq \left( \|\phi_A\| - \varepsilon \right)^{\frac{1}{q} + 1} + \left( \|\phi_B\| - \varepsilon \right)^{\frac{1}{q} + 1} \left[ \left( \|\phi_A\| - \varepsilon \right)^q + \left( \|\phi_B\| - \varepsilon \right)^q \right]^{\frac{1}{q}} = \left( \|\phi_A\| - \varepsilon \right)^{\frac{1}{q} + 1} \left[ \left( \|\phi_A\| - \varepsilon \right)^q + \left( \|\phi_B\| - \varepsilon \right)^q \right]^{\frac{1}{q}}.
\]

Since the inequality

\[
\|\phi_{A \cup B}\| \geq \left[ \left( \|\phi_A\| - \varepsilon \right)^q + \left( \|\phi_B\| - \varepsilon \right)^q \right]^{\frac{1}{q}}
\]

holds for every \( \varepsilon \in (0, \min\{\|\phi_A\|, \|\phi_B\|\}) \), it will clearly force (10).

To prove the other inequality \( \|\phi_{A \cup B}\| \leq \left[ \|\phi_A\|^q + \|\phi_B\|^q \right]^{\frac{1}{q}} \), we must show that

\[
|\phi_{A \cup B}(g)| \leq \left[ \|\phi_A\|^q + \|\phi_B\|^q \right]^\frac{1}{q} \cdot \|g\|_p, \quad \forall \ g \in L^p_X(X, A, \mu).
\]

Start with some \( g \in L^p_X(X, A, \mu) \). Using (7), combined with Hölder’s inequality, we have

\[
|\phi_{A \cup B}(g)| \leq |\phi_A(\varkappa_A)| + |\phi_B(\varkappa_B)| \leq \|\phi_A\| \cdot \|g\|_p + \|\phi_B\| \cdot \|g\|_p \leq \left[ \|\phi_A\|^q + \|\phi_B\|^q \right]^\frac{1}{q} \cdot \left( \|g\|_p \right)^p \left( \|g\|_p \right)^p \leq \left[ \|\phi_A\|^q + \|\phi_B\|^q \right]^\frac{1}{q} \cdot \left( \|g\|_p \right)^p \leq \left( \|g\|_p \right)^p \left[ \|\phi_{A \cup B}\|_p \right]^\frac{1}{q} = \left( \|g\|_p \right)^p \left[ \|g\|_p \right]^{\frac{1}{q}}.
\]
and then using (12) we immediately get (11).

**Theorem 6.2 (Finite approximation).** Let \((X, \mathcal{A}, \mu)\) be a measure space, let \(\mathbb{K}\) be either \(\mathbb{R}\) or \(\mathbb{C}\), and let \(p \in [1, \infty)\). For every linear continuous map \(\phi : L^p_{\text{elem}}(X, \mathcal{A}, \mu) \to \mathbb{K}\), one has

\[
\|\phi\| = \sup \{\|\phi_A\| : A \in \mathcal{A}, \mu(A) < \infty\}.
\]

Moreover, if \(p > 1\), one also has

\[
\inf \{\|\phi - \phi_A\| : A \in \mathcal{A}, \mu(A) < \infty\} = 0.
\]

**Proof.** It is obvious that

\[
\|\phi\| \geq \sup \{\|\phi_A\| : A \in \mathcal{A}, \mu(A) < \infty\}.
\]

To prove the other inequality, we start with some \(\varepsilon > 0\). Since \(L^p_{\text{elem}}(X, \mathcal{A}, \mu)\) is dense in \(L^p(X, \mathcal{A}, \mu)\), there exists \(f \in L^p_{\text{elem}}(X, \mathcal{A}, \mu)\), with \(\|f\|_p \leq 1\), and \(|\phi(f)| \geq \|\phi\| - \varepsilon\). Notice that the fact that \(f \in L^p_{\text{elem}}(X, \mathcal{A}, \mu)\) means that \(f = \lambda_1 \cdot \chi_{A_1} + \cdots + \lambda_n \cdot \chi_{A_n}\), with \(A_k \in \mathcal{A}\) and \(\mu(A_k) < \infty\), \(\forall k = 1, \ldots, n\). In particular, if one considers the integration map \(\int_{X} f \, d\mu \in \mathbb{K}\), with \(\mu(A_k) < \infty\), \(\forall k = 1, \ldots, n\), then \(\phi(f) = \phi(A)\), which gives \(\phi_A(f) = \phi(f)\). Therefore, we get

\[
\|\phi\| - \varepsilon \leq |\phi_A(f)| \leq \|\phi_A\| \cdot \|f\|_p = \|\phi_A\|
\]

thus proving that we have in fact equality in (13).

To prove the equality (??), we use (13) to find, for every integer \(n \geq 1\), some set \(A_n \in \mathcal{A}\), with \(\mu(A_n) < \infty\), such that

\[
\|\phi_{\mathcal{A} \setminus A_n}\| > \|\phi\| - \frac{1}{n}.
\]

Since we clearly have \(\phi_{\mathcal{A} \setminus A_n} = \phi - \phi_{A_n}\), by Lemma 6.2 we get

\[
\|\phi\| = \left[ \|\phi_{A_n}\|^q + \|\phi - \phi_{A_n}\|^q \right]^\frac{1}{q},
\]

where \(q = p/(p - 1)\). This yields

\[
\|\phi - \phi_{A_n}\| = \left[ \|\phi\|^q - \|\phi_{A_n}\|^q \right]^\frac{1}{q},
\]

and using (15) we get

\[
\|\phi - \phi_{A_n}\| \leq \left[ \|\phi\|^q - \left( \|\phi\| - \frac{1}{n} \right)^q \right]^\frac{1}{q},
\]

which clearly forces \(\lim_{n \to \infty} \|\phi - \phi_{A_n}\| = 0\).

**Remark 6.2.** The equality (??) does not hold for \(p = 1\), the reason being the fact that in this case one only has

\[
\|\phi - \phi_A\| = \|\phi_{\mathcal{A} \setminus A}\|.
\]

For example, suppose one works with a \(\sigma\)-finite measure space \((X, \mathcal{A}, \mu)\), with \(\mu(X) = \infty\), and if one considers the integration map

\[
\phi : L^1_{\text{elem}}(X, \mathcal{A}, \mu) \ni f \mapsto \int_X f \, d\mu \in \mathbb{K},
\]

then it is pretty clear that \(\|\phi_B\| = 1\), \(\forall B \in \mathcal{A}\), with \(\mu(B) > 0\). In particular, we see that for this map one has

\[
\inf \{\|\phi - \phi_A\| : A \in \mathcal{A}, \mu(A) < \infty\} = \inf \{\|\phi_{\mathcal{A} \setminus A}\| : A \in \mathcal{A}, \mu(A) < \infty\} = 1.
\]
Exercise 1. Suppose \((X, \mathcal{A}, \mu)\) is a measure space, and \(p \in [1, \infty)\). Prove that, for every linear continuous map \(\phi : L^p(X, \mathcal{A}, \mu) \to K\), there exists a \(\mu\)-\(\sigma\)-finite set \(A \in \mathcal{A}\), with \(\|\phi_A\| = \|\phi\|\). Moreover, if \(p > 1\), then any set with this property forces the equality \(\phi_A = \phi\).

We are now ready to prove our first duality result. Before we do so, let us recall some notations and results from Section 3.

Notations. Let \((X, \mathcal{A}, \mu)\) be a measure space, let \(K\) be one of the fields \(\mathbb{R}\) or \(\mathbb{C}\), and let \(p, q \in (1, \infty)\) be Hölder conjugate numbers, i.e. \(\frac{1}{p} + \frac{1}{q} = 1\). For any \(f \in L^p(X, \mathcal{A}, \mu)\) we consider the map

\[
\Lambda_f : L^q(X, \mathcal{A}, \mu) \ni g \mapsto \int_X fg \, d\mu \in K.
\]

We know from the results in Section 3 that

- for every \(f \in L^p(X, \mathcal{A}, \mu)\), the map \(\Lambda_f : L^p(X, \mathcal{A}, \mu) \to K\) is linear continuous, and has \(\|\Lambda_f\| = \|f\|_q\).
- the correspondence

\[
\Lambda : L^q(X, \mathcal{A}, \mu) \ni f \mapsto \Lambda_f \in L^p(X, \mathcal{A}, \mu)^*
\]

is linear continuous and isometric.

Theorem 6.3 (Duality Theorem for \(L^p\), \(p \in (1, \infty)\)). With the notations above, the correspondence \((16)\) is an isometric linear isomorphism.

Proof. All we need to prove is the surjectivity of \((16)\). Start with some linear continuous map \(\phi : L^p(X, \mathcal{A}, \mu) \to K\), and let us construct a function \(f \in L^p(X, \mathcal{A}, \mu)\), such that \(\Lambda_f = \phi\). The key step is contained in the following.

Claim 1: For every \(A \in \mathcal{A}\) with \(\mu(A) < \infty\), there exists some \(f \in L^p(X, \mathcal{A}, \mu)\), such that \(\Lambda_f = \phi |_A\).

Indeed, if one considers the linear continuous map \(\phi |_A : L^p(X, \mathcal{A}, \mu) \to K\), then by Theorem 6.1, there exists a function \(f_0 \in L^p(A, \mathcal{A}|_A, \mu)\) with

\[
(\phi |_A)(g) = \int_A f_0 g \, d\mu, \quad \forall g \in L^p(A, \mathcal{A}|_A, \mu).
\]

If we define the function \(f = R_A f_0 \in L^p(X, \mathcal{A}, \mu)\), then we have \(f |_A = f_0\) and \(f = f \sigma_A\), so for every \(g \in L^p(X, \mathcal{A}, \mu)\) we have

\[
\int_X fg \, d\mu = \int_A f_0(g|_A) \, d\mu = (\phi |_A)(g|_A) = \phi_A(g).
\]

Having proven Claim 1, we now use Theorem 6.2 to find a sequence \((A_n)_{n=1}^{\infty} \subset \mathcal{A}\), with \(\mu(A_n) < \infty, \forall n \geq 1\), such that \(\lim_{n \to \infty} \|\phi - \phi_{A_n}\| = 0\). For each \(n \geq 1\) we use Claim 1 to find some \(f_n \in L^p(X, \mathcal{A}, \mu)\) such that \(\Lambda_{f_n} = \phi_{A_n}\), so now we have

\[
\lim_{n \to \infty} \|\phi - \Lambda_{f_n}\| = 0.
\]

Claim 2: The sequence \((f_n)_{n=1}^{\infty} \subset L^p(X, \mathcal{A}, \mu)\) is Cauchy.

Indeed, from \((17)\) it follows that the sequence \((\Lambda_{f_n})_{n=1}^{\infty}\) is Cauchy in the Banach space \(L^p(X, \mathcal{A}, \mu)^*\), i.e.

\[
\lim_{N \to -\infty} \left[ \sup \{ \|\Lambda_{f_m} - \Lambda_{f_n}\| : m, n \geq N \} \right] = 0.
\]
Since $\Lambda$ is linear and isometric, we have $\|\Lambda_{f_m} - \Lambda_{f_n}\| = \|f_m - f_n\|_q$, $\forall m, n \geq 1$, so the above estimate gives
\[
\lim_{N \to \infty} \left[ \sup \left\{ \|f_m - f_n\|_q : m, n \geq N \right\} \right] = 0.
\]

Use now the fact that $L^q_{\infty}(X,A,\mu)$ is a Banach space, to conclude that
\[
L^q_{\infty} \lim_{n \to \infty} f_n = f,
\]
for some $f \in L^q_{\infty}(X,A,\mu)$. Since $\Lambda$ is isometric, we have
\[
\lim_{n \to \infty} \|\Lambda_f - \Lambda_{f_n}\| = 0,
\]
and then (17) forces $\phi = \Lambda_f$. \hfill \box

Before we deal with the duality problem for $L^1$, we recall some notations and results from Section 5.

**Notations.** Let $(X,A,\mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For any $f \in L^\infty_{\mathbb{K}}(X,A,\mu)$ we consider the map
\[
\Lambda_f : L^1_{\mathbb{K}}(X,A,\mu) \ni g \longmapsto \int_X fg \, d\mu \in \mathbb{K}.
\]
We know from the results in Section 5 that

- for every $f \in L^\infty_{\mathbb{K}}(X,A,\mu)$, the map $\Lambda_f : L^1_{\mathbb{K}}(X,A,\mu) \to \mathbb{K}$ is linear continuous, and has $\|\Lambda_f\| = \|f\|_{\infty}^\mathbb{K}.
- the correspondence (18)
\[
\Lambda : L^\infty_{\mathbb{K}}(X,A,\mu) \ni f \longmapsto \Lambda_f \in L^1_{\mathbb{K}}(X,A,\mu)^*
\]
is linear continuous and isometric.

**Theorem 6.4 (Duality Theorem for $L^1$).** Use the notations above, and assume there exists $X_0 \in A$ such that

(i) the measure space $(X_0,A\mid_{X_0},\mu)$ is decomposable;

(ii) the measure space $(X \setminus X_0,A\mid_{X \setminus X_0},\mu)$ is degenerate, i.e. $\mu(A) \in \{0, \infty\}$, for all $A \in A$ with $A \subset X \setminus X_0$.

Then the correspondence (18) is an isometric linear isomorphism.

**Proof.** As before, all we have to show is the surjectivity of the map (20). First of all, since the measure space $(X \setminus X_0,A\mid_{X \setminus X_0},\mu)$ is degenerate, it is clear that the restriction maps

- $L^1_{\mathbb{K}}(X,A,\mu) \ni f \longmapsto f\mid_{X_0} \in L^1_{\mathbb{K}}(X_0,A\mid_{X_0},\mu)$
- $L^\infty_{\mathbb{K}}(X,A,\mu) \ni f \longmapsto f\mid_{X_0} \in L^\infty_{\mathbb{K}}(X_0,A\mid_{X_0},\mu)$

are isometric linear isomorphisms, so we can assume that in fact we have $X_0 = X$. Recall that the decomposability condition means that there exists a collection $\mathcal{F} \subset A$ with

- $\mu(F) < \infty$, $\forall F \in \mathcal{F}$;
- $\bigcup_{F \in \mathcal{F}} F = X$;
- if a set $A \subset X$ has $A \cap F \in A$, $\forall F \in \mathcal{F}$, then $A \in A$;
- for every $A \in A$ with $\mu(A) < \infty$, one has $\mu(A) = \sum_{F \in \mathcal{F}} \mu(A \cap F)$.  

To prove the surjectivity of (20) we start with some linear continuous map \( \phi : L^p_K(X, A, \mu) \to \mathbb{K} \), and we wish to find some function \( f \in L^\infty_{\text{loc}}(X, A, \mu) \), with \( \phi = \Lambda_f \). The following first step is proven the exact same way as the first step in the proof of Theorem 6.2.

**Claim 1:** For every \( A \in A \) with \( \mu(A) < \infty \), there exists \( g \in L^\infty_{\text{loc}}(X, A, \mu) \), such that \( \Lambda_f = \phi \).

To construct the desired function \( f \), we use the above Claim to find, for each \( F \in \mathcal{F} \) a function \( f_F \in L^\infty_{\text{loc}}(X, A, \mu) \), such that \( \Lambda_{f_F} = \phi_F \). Using Lemma 5.1, we can in fact assume that \( f_F \) is in fact bounded, with \( \|f_F\|_{\text{sup}} = \|f_F\|_{\text{loc}}^{\infty} \). Remark that, using the fact that (18) is isometric, it follows that, for every \( F \in \mathcal{F} \), one has the inequality

\[
\|f_F\|_{\text{sup}} = \|\Lambda_{f_F}\| = \|\phi_F\| \leq \|\phi\|.
\]

Use now the patching property (see ???) to produce a measurable function \( f : X \to \mathbb{K} \) such that \( f|_F = f_F \), \( \forall F \in \mathcal{F} \). Obviously \( f \) is bounded, so in particular it belongs to \( L^\infty_{\text{loc}}(X, A, \mu) \), and one has the equalities

\[
(\Lambda f)_F = \Lambda_{f_F} = \phi_F, \quad \forall F \in \mathcal{F}.
\]

**Claim 2:** For any set \( A \in A \) with \( \mu(A) < \infty \), one has the equality

\[\Lambda_f(\mathcal{A}_A) = \phi(\mathcal{A}_A).\]

To prove this we first observe that, using decomposability, we have

\[
\sum_{F\in\mathcal{F}} \mu(A \cap F) = \mu(A) < \infty,
\]

so in particular the collection \( \mathcal{F}_A = \{F \in A : \mu(A \cap F) > 0\} \) is at most countable. If we list it as a sequence \( \mathcal{F}_A = \{F_1, F_2, \ldots\} \) (finite or infinite) then \( \mu(A) = \sum_n \mu(A \cap F_n) \), and it is pretty clear (use Remark 6.1.B for example) that one has

\[\mathcal{A}_A = L^1 - \sum_n \mathcal{A}_A \cap F_n.\]

(The sum in the right hand side is either an ordinary sum, if \( \mathcal{F}_A \) is finite, or a series, convergent in \( L^1 \), if \( \mathcal{F}_A \) is infinite.) In any event, using the equalities \( \mathcal{A}_A \cap F = 0 \), \( \forall F \in \mathcal{F} \setminus \mathcal{F}_A \), the continuity of both \( \Lambda_f \) and \( \phi \) will give the fact that we have the equalities

\[
\Lambda_f(\mathcal{A}_A) = \sum_{F \in \mathcal{F}} \Lambda_f(\mathcal{A}_A \cap F) \quad \text{and} \quad \phi(\mathcal{A}_A) = \sum_{F \in \mathcal{F}} \phi(\mathcal{A}_A \cap F).
\]

This means that, in order to prove the Claim, all we have to show are the equalities

\[
\Lambda_f(\mathcal{A}_A \cap F) = \phi(\mathcal{A}_A \cap F), \quad \forall F \in \mathcal{F}.
\]

But these equalities immediately follow from (20):

\[
\Lambda_f(\mathcal{A}_A \cap F) = \Lambda_f(\mathcal{A}_A \mathcal{A}_F) = (\Lambda f)_F(\mathcal{A}_A) = \phi_F(\mathcal{A}_A) = \phi(\mathcal{A}_A \cap F).
\]

We now prove the equality \( \Lambda_f = \phi \). Using Claim 2, and linearity, it follows that

\[
(\Lambda f)_A = 0, \quad \forall A \in \mathcal{F}.
\]

Since \( \mathcal{F}_2 = \{A \in A : \mu(A) < \infty\} = L^\infty_{\text{elem},K}(X, A, \mu) \) is dense in \( L^1_{\text{elem},K}(X, A, \mu) \), the continuity of both \( \Lambda_f \) and \( \phi \), combined with (21) will force \( \Lambda_f = \phi \).
Consider the measure space \((X, \mathcal{M}, \mu)\) (on which Theorem 6.4 is true!), and we pick a “small” \(\sigma\)-algebra \(A \subset \mathcal{M}\). If we consider the measure space \((X, \mathcal{A}, \mu|_A)\), then by Exercise 6 in Section 4 we have an inclusion \(L^1_K(X, \mathcal{A}, \mu|_A) \subset L^1_K(X, \mathcal{M}, \mu)\), so if one starts with some \(f_0 \in L^\infty_{K,\text{loc}}(X, \mathcal{M}, \mu)\), we get a linear continuous map \(\phi = \Lambda^M_\mu|_{L^1_K(X, \mathcal{A}, \mu|_A)}\). (We use superscripts to indicate the corresponding measure space.) It may happen that we have no way of representing \(\phi\) as \(\Lambda^A_\mu\) with \(f \in L^\infty_{K,\text{loc}}(X, \mathcal{A}, \mu|_A)\), simply because the only candidate might be \(f = f_0\).

**Example 6.1.** Let \(X\) be an uncountable set, and let \(\mu : \mathcal{P}(X) \to [0, \infty]\) be the counting measure. For the measure space \((X, \mathcal{P}(X), \mu)\) we have the obvious identification \(L^1_K(X, \mathcal{P}(X), \mu) \simeq \ell^1_K(X)\), with the integration corresponding to the map
\[
\ell^1_K(X) \ni g \mapsto \sum_{x \in X} g(x) \in \mathbb{K}.
\]
Consider the \(\sigma\)-algebra
\[
A = \{ A \subset X : \text{either } A \text{ or } X \setminus A \text{ is countable} \}.
\]
Remark that, although \(A \subsetneq \mathcal{P}(X)\), one has the equality \(\ell^1_K(X, \mathcal{A}, \mu|_A) = \ell^1_K(X)\).
Remark also that both measure spaces \((X, \mathcal{P}(X), \mu)\) and \((X, \mathcal{A}, \mu|_A)\) have the finite subset property, and in fact one has the equivalence
\[
(\mu\text{-l.a.e}) \leftrightarrow (\mu\text{-a.e}) \leftrightarrow (\text{everywhere}).
\]
Let \(B \subset X\) be a set which does not belong to \(A\), and consider the map
\[
\phi : L^1_K(X, \mathcal{A}, \mu) \ni g \mapsto \sum_{x \in B} g(x) \in \mathbb{K}.
\]
We claim that there is no function \(f \in L^\infty_X(X, \mathcal{A}, \mu|_A)\) such that
\[
\phi(g) = \int_X f \, g \, d(\mu|_A), \quad \forall g \in L^1_K(X, \mathcal{A}, \mu|_A).
\]
Indeed, if such a function exists, then if one considers the functions like \(\kappa_\{x\}\), \(x \in X\), which clearly belong to \(L^1_K(X, \mathcal{A}, \mu|_A)\), then we immediately get
\[
f(x) = \phi(\kappa_\{x\}) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}
\]
i.e. \(f = \kappa_B\), which is impossible, since \(B \notin A\).

**Exercise 2.** Let \((X, \mathcal{A}, \mu)\) be a measure space, and let \(\mathcal{B} \subset \mathcal{A}\) be a \(\sigma\)-algebra. Consider the measure space \((X, \mathcal{B}, \mu|_B)\). Let \(\mathbb{K}\) be either \(\mathbb{R}\) or \(\mathbb{C}\). As seen in Exercise 6 from Section 3, for every \(q \in [1, \infty)\), one has the inclusion
\[
L^p_K(X, \mathcal{B}, \mu|_B) \subset L^p_K(X, \mathcal{A}, \mu).
\]
A. Assume \(1 < q < \infty\), and let \(p = q/(q - 1)\). Prove that for every \(f \in L^p_K(X, \mathcal{A}, \mu)\), there exists a (unique \(\mu\text{-a.e.})\) function \(\tilde{f} \in L^p_K(X, \mathcal{B}, \mu|_B)\) such that
\[
\int_X f \, g \, d\mu = \int_X \tilde{f} \, g \, d\mu, \quad \forall g \in L^p_K(X, \mathcal{B}, \mu|_B).
\]
B. Prove that, if the measure space \((X, \mathcal{B}, \mu|_B)\) is decomposable, then for every \(f \in L^1(X, \mathcal{A}, \mu)\), there exists a (unique \(\mu\)-a.e.) function \(\tilde{f} \in L^1(X, \mathcal{B}, \mu|_B)\) such that
\[
\int_X fg\,d\mu = \int_X \tilde{f}g\,d\mu, \quad \forall g \in L^\infty,loc(X, \mathcal{B}, \mu|_B).
\]
Hints: A. For each \(f \in L^2(X, \mathcal{A}, \mu)\) consider the map
\[
\Phi_f : L^2(X, \mathcal{B}, \mu|_B) \ni g \mapsto \int_X fg\,d\mu \in \mathbb{K},
\]
and use Theorem 6.3. applied to the measure space \((X, \mathcal{B}, \mu|_B)\).
B. Use the Radon-Nikodym Theorem, on the measure space \((X, \mathcal{B}, \mu|_B)\) for the \(\mathbb{K}\)-valued measure \(\nu_f(B) = \int_X f\chi_B\,d\mu, \ B \in \mathcal{B} \).

Comment: Using the notations from the above Exercise, (if \(q = 1\) we also require the extra hypothesis as in part B) the map
\[
E^q_{A|B} : L^q(X, \mathcal{A}, \mu) \ni f \mapsto \tilde{f} \in L^q(X, \mathcal{B}, \mu|_B)
\]
is called the conditional expectation map. This construction is often employed in Probability Theory. In the case \(q = 2\) the map \(E^2_{A|B}\) can also be described as the orthogonal projection of \(L^2(X, \mathcal{A}, \mu)\) onto the closed linear subspace \(L^2(X, \mathcal{B}, \mu|_B)\).

Exercise 3\textsuperscript{c}. Use the notations from Exercise 2. (If \(q = 1\) assume also the same hypothesis as in part B.) Prove that the conditional expectation map
\[
E^q_{A|B} : L^q(X, \mathcal{A}, \mu) \to L^q(X, \mathcal{B}, \mu|_B)
\]
is linear, continuous, and has \(\|E^q_{A|B}\| \leq 1\). Also prove that
(i) \(E^q_{A|B}(f) = f, \forall f \in L^q(X, \mathcal{B}, \mu|_B)\);
(ii) \(E^q_{A|B}(h \cdot f) = h \cdot E^q_{A|B}(f), \forall h \in L^\infty,loc(X, \mathcal{B}, \mu|_B), f \in L^q(X, \mathcal{A}, \mu)\).

Conditional expectations can also be defined on \(L^\infty,loc\).

Exercise 4\textsuperscript{c}. Use the notations from Exercise 2. Prove that, if the measure space \((X, \mathcal{B}, \mu|_B)\) is decomposable, then for every \(f \in L^\infty,loc(X, \mathcal{A}, \mu)\), there exists a (unique \(\mu\)-a.e.) function \(\tilde{f} \in L^\infty,loc(X, \mathcal{B}, \mu|_B)\) such that
\[
\int_X fg\,d\mu = \int_X \tilde{f}g\,d\mu, \quad \forall g \in L^\infty,loc(X, \mathcal{B}, \mu|_B).
\]
Prove that the map
\[
E_{A|B} : L^\infty,loc(X, \mathcal{A}, \mu) \ni f \mapsto \tilde{f} \in L^\infty,loc(X, \mathcal{B}, \mu|_B)
\]
is again linear continuous and contractive. Moreover, it also satifies the properties:
(i) \(E_{A|B}(f) = f, \forall f \in L^\infty,loc(X, \mathcal{B}, \mu|_B)\);
(ii) \(E_{A|B}(h \cdot f) = h \cdot E_{A|B}(f), \forall h \in L^\infty,loc(X, \mathcal{B}, \mu|_B), f \in L^\infty,loc(X, \mathcal{A}, \mu)\).

Hint: For the existence of \(\tilde{f}\) argue as in the hint to Exercise 2, but use Theorem 6.4. instead.