3. The $L^p$ spaces ($1 \leq p < \infty$)

In this section we discuss an important construction, which is extremely useful in virtually all branches of Analysis. In Section 1, we have already introduced the space $\mathcal{L}^1$. The first construction deals with a generalization of this space.

**Definitions.** Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$.

A. For a number $p \in (1, \infty)$, we define the space

$$L^p(X, \mathcal{A}, \mu) = \{ f : X \to \mathbb{K} : f \text{ measurable, and } \int_X |f|^p \, d\mu < \infty \}.$$ 

Here we use the convention introduced in Section 1, which defines $\int_X h \, d\mu = \infty$, for those measurable functions $h : X \to [0, \infty]$, that are not integrable.

Of course, in this definition we can allow also the value $p = 1$, and in this case we get the familiar definition of $L^1(X, \mathcal{A}, \mu)$.

B. For $p \in [1, \infty)$, we define the map $Q_p : L^p(X, \mathcal{A}, \mu) \to [0, \infty)$ by

$$Q_p(f) = \int_X |f|^p \, d\mu, \quad \forall f \in L^p(X, \mathcal{A}, \mu).$$

**Remark 3.1.** The space $L^1(X, \mathcal{A}, \mu)$ was studied earlier (see Section 1). It has the following features:

(i) $L^1(X, \mathcal{A}, \mu)$ is a $\mathbb{K}$-vector space.

(ii) The map $Q_1 : L^1(X, \mathcal{A}, \mu) \to [0, \infty)$ is a seminorm, i.e.

(a) $Q_1(f + g) \leq Q_1(f) + Q_1(g), \quad \forall f, g \in L^1(X, \mathcal{A}, \mu)$;

(b) $Q_1(\alpha f) = |\alpha| \cdot Q_1(f), \quad \forall f \in L^1(X, \mathcal{A}, \mu), \quad \alpha \in \mathbb{K}$.

(iii) $\int_X |f| \, d\mu \leq Q_1(f), \quad \forall f \in L^1(X, \mathcal{A}, \mu)$.

Property (b) is clear. Property (a) immediately follows from the inequality $|f + g| \leq |f| + |g|$, which after integration gives

$$\int_X |f + g| \, d\mu \leq \int_X (|f| + |g|) \, d\mu = \int_X |f| \, d\mu + \int_X |g| \, d\mu.$$

In what follows, we aim at proving similar features for the spaces $L^p(X, \mathcal{A}, \mu)$ and $Q_p, 1 < p < \infty$.

The following will help us prove that $L^p$ is a vector space.

**Exercise 1.** Let $p \in (1, \infty)$. Then one has the inequality

$$(s + t)^p \leq 2^{p-1} (s^p + t^p), \quad \forall s, t \in [0, \infty).$$

**Hint:** The inequality is trivial, when $s = t = 0$. If $s + t > 0$, reduce the problem to the case $t + s = 1$, and prove, using elementary calculus techniques that

$$\min_{t \in [0,1]} \left[ t^p + (1 - t)^p \right] = 2^{1-p}.$$ 

**Proposition 3.1.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in (1, \infty)$. When equipped with pointwise addition and scalar multiplication, $L^p(X, \mathcal{A}, \mu)$ is a $\mathbb{K}$-vector space.
PROOF. It \( f, g \in L^p(X, \mathcal{A}, \mu) \), then by Exercise 1 we have
\[
\int_X |f + g|^p \, d\mu \leq \int_X (|f| + |g|)^p \, d\mu \leq 2^{p-1} \left[ \int_X |f|^p \, d\mu + \int_X |g|^p \, d\mu \right] < \infty,
\]
so \( f + g \) indeed belongs to \( L^p(X, \mathcal{A}, \mu) \).

Using the General Lebesgue Monotone Convergence Theorem, we then get
\[
(2)
\]
By the Lebesgue Dominated Convergence Theorem, we will also get the equalities
\[
(1)
\]
so
\[
\int_X |\alpha f|^p \, d\mu = \int_X |\alpha|^p \cdot |f|^p \, d\mu = |\alpha|^p \cdot \int_X |f|^p \, d\mu
\]
clearly prove that \( \alpha f \) also belongs to \( L^p(X, \mathcal{A}, \mu) \). \( \Box \)

Our next task will be to prove that \( Q_p \) is a seminorm, for all \( p > 1 \). In this
direction, the following is a key result. (The above mentioned convention will be
used throughout this entire section.)

**Theorem 3.1 (Hölder’s Inequality for integrals).** Let \( (X, \mathcal{A}, \mu) \) be a measure
space, let \( f, g : X \to [0, \infty) \) be measurable functions, and let \( p, q \in (1, \infty) \) be such
that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then one has the inequality\(^1\)
\[
(1) \quad \int_X fg \, d\mu \leq \left[ \int_X f^p \, d\mu \right]^{1/p} \cdot \left[ \int_X g^q \, d\mu \right]^{1/q}.
\]

**Proof.** If either \( \int_X f^p \, d\mu = \infty \), or \( \int_X g^q \, d\mu = \infty \), then the inequality (1) is
trivial, because in this case, the right hand side is \( \infty \). For the remainder of the
proof we will assume that \( \int_X f^p \, d\mu < \infty \) and \( \int_X g^q \, d\mu < \infty \).

Use Corollary 2.1 to find two sequences \((\varphi_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty \subset L^1_{\operatorname{elem}}(X, \mathcal{A}, \mu)\), such
that
- \( 0 \leq \varphi_1 \leq \varphi_2 \leq \ldots \) and \( 0 \leq \psi_1 \leq \psi_2 \leq \ldots \);
- \( \lim_{n \to \infty} \varphi_n(x) = f(x)^p \) and \( \lim_{n \to \infty} \psi_n(x) = g(x)^q, \forall x \in X \).

By the Lebesgue Dominated Convergence Theorem, we will also get the equalities
\[
(2) \quad \int_X f^p \, d\mu = \lim_{n \to \infty} \int_X \varphi_n \, d\mu \quad \text{and} \quad \int_X g^q \, d\mu = \lim_{n \to \infty} \int_X \psi_n \, d\mu.
\]
Remark that the functions \( f_n = \varphi_n^{1/p}, g_n \psi_n^{1/q}, n \geq 1 \) are also elementary (because
they obviously have finite range). It is obvious that we have
- \( 0 \leq f_1 \leq f_2 \leq \ldots \), and \( 0 \leq g_1 \leq g_2 \leq \ldots \);
- \( \lim_{n \to \infty} f_n(x) = f(x), \text{ and } \lim_{n \to \infty} g_n(x) = g(x), \forall x \in X \).

With these notations, the equalities (2) read
\[
(3) \quad \int_X f^p \, d\mu = \lim_{n \to \infty} \int_X (f_n)^p \, d\mu \quad \text{and} \quad \int_X g^q \, d\mu = \lim_{n \to \infty} \int_X (g_n)^q \, d\mu.
\]
Of course, the products \( f_n g_n, n \geq 1 \) are again elementary, and satisfy
- \( 0 \leq f_1 g_1 \leq f_2 g_2 \leq \ldots \);
- \( \lim_{n \to \infty} [f_n(x) g_n(x)] = f(x) g(x), \forall x \in X \).

Using the General Lebesgue Monotone Convergence Theorem, we then get
\[
\int_X fg \, d\mu = \lim_{n \to \infty} \int_X f_n g_n \, d\mu.
\]

\(^1\) Here we use the convention \( \infty^{1/p} = \infty^{1/q} = \infty \).
Using (3) we now see that, in order to prove (1), it suffices to prove the inequalities
\[
\int_X f_n g_n \, d\mu \leq \left( \int_X (f_n)^p \, d\mu \right)^{1/p} \cdot \left( \int_X (g_n)^q \, d\mu \right)^{1/q}, \quad \forall \, n \geq 1.
\]
In other words, it suffices to prove (1), under the extra assumption that both \( f \) and \( g \) are elementary integrable.

Suppose \( f \) and \( g \) are elementary integrable. Then (see III.1) there exist pairwise disjoint sets \( (D_j)_{j=1}^m \subset \mathcal{A} \), with \( \mu(D_j) < \infty \), \( \forall \, j = 1, \ldots, m \), and numbers \( \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m \in [0, \infty) \), such that
\[
\begin{align*}
f &= \alpha_1 \chi_{D_1} + \cdots + \alpha_m \chi_{D_m} \\
g &= \beta_1 \chi_{D_1} + \cdots + \beta_m \chi_{D_m}
\end{align*}
\]
Notice that we have
\[
f g = \alpha_1 \beta_1 \chi_{D_1} + \cdots + \alpha_m \beta_m \chi_{D_m},
\]
so the left hand side of (1) is the given by
\[
\int_X f g \, d\mu = \sum_{j=1}^m \alpha_j \beta_j \mu(D_j).
\]
Define the numbers \( x_j = \alpha_j \mu(D_j)^{1/p}, \ y_j = \beta_j \mu(D_j)^{1/q}, \ j = 1, \ldots, m \). Using these numbers, combined with \( \frac{1}{p} + \frac{1}{q} = 1 \), we clearly have
\[
\int_X f g \, d\mu = \sum_{j=1}^m (x_j y_j).
\]
At this point we are going to use the classical Hölder inequality (see Appendix D), which gives
\[
\sum_{j=1}^m (x_j y_j) \leq \left[ \sum_{j=1}^m (x_j)^p \right]^{1/p} \cdot \left[ \sum_{j=1}^m (y_j)^q \right]^{1/q},
\]
so the equality (4) continues with
\[
\int_X f g \, d\mu \leq \left[ \sum_{j=1}^m (x_j)^p \right]^{1/p} \cdot \left[ \sum_{j=1}^m (y_j)^q \right]^{1/q} = \left[ \sum_{j=1}^m (\alpha_j)^p \mu(D_j) \right]^{1/p} \cdot \left[ \sum_{j=1}^m (\beta_j)^q \mu(D_j) \right]^{1/q} = \left[ \int_X f^p \, d\mu \right]^{1/p} \cdot \left[ \int_X g^q \, d\mu \right]^{1/q}.
\]

**Corollary 3.1.** Let \( (X, \mathcal{A}, \mu) \) be a measure space, let \( \mathbb{K} \) be one of the fields \( \mathbb{R} \) or \( \mathbb{C} \), and let \( p, q \in (1, \infty) \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). For any two functions \( f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \) and \( g \in L^q_{\mathbb{K}}(X, \mathcal{A}, \mu) \), the product \( fg \) belongs to \( L^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \) and one has the inequality
\[
\left| \int_X f g \, d\mu \right| \leq Q_p(f) \cdot Q_q(g).
\]
PROOF. By Hölder’s inequality, applied to $|f|$ and $|g|$, we get
\[ \int_X |f g| \, d\mu \leq Q_p(f) \cdot Q_q(g) < \infty, \]
so $|f g|$ belongs to $L^1_p(X, \mathcal{A}, \mu)$, i.e. $f g$ belongs to $L^q_p(X, \mathcal{A}, \mu)$. The desired inequality then follows from the inequality $\left| \int_X f g \, d\mu \right| \leq \int_X |f g| \, d\mu$. \hfill \Box

NOTATION. Suppose $(X, \mathcal{A}, \mu)$ is a measure space, $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For any pair of functions $f \in L^p_\mathbb{K}(X, \mathcal{A}, \mu)$, $g \in L^q_\mathbb{K}(X, \mathcal{A}, \mu)$, we shall denote the number $\int_X f g \, d\mu \in \mathbb{K}$ simply by $\langle f, g \rangle$. With this notation, Corollary 3.1 reads:

\[ \left| \langle f, g \rangle \right| \leq Q_p(f) \cdot Q_q(g), \forall f \in L^p_\mathbb{K}(X, \mathcal{A}, \mu), \ g \in L^q_\mathbb{K}(X, \mathcal{A}, \mu). \]

The following result gives an alternative description of the maps $P_p, p \in (1, \infty)$.

**Proposition 3.2.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $\int_X f \, d\mu \in L^p_\mathbb{K}(X, \mathcal{A}, \mu)$. Then one has the equality

\[ Q_p(f) = \sup \{ \left| \langle f, g \rangle \right| : g \in L^q_\mathbb{K}(X, \mathcal{A}, \mu), \ Q_q(g) \leq 1 \}. \tag{5} \]

**Proof.** Let us denote the right hand side of (5) simply by $P(f)$. By Corollary 3.1, we clearly have the inequality

\[ P(f) \leq Q_p(f). \]

To prove the other inequality, let us first observe that in the case when $Q_p(f) = 0$, there is nothing to prove, because the above inequality already forces $P(f) = 0$. Assume then $Q_p(f) > 0$, and define the function $h : x \to \mathbb{K}$ by

\[ h(x) = \begin{cases} 
\frac{|f(x)|^p}{f(x)} & \text{if } f(x) \neq 0 \\
0 & \text{if } f(x) = 0
\end{cases} \]

It is obvious that $h$ is measurable. Moreover, one has the equality $|h| = |f|^{p-1}$, which using the equality $qp = p + q$ gives $|h|^q = |f|^{p-1}q = |f|^p$. This proves that $h \in L^q_\mathbb{K}(X, \mathcal{A}, \mu)$, as well as the equality

\[ Q_q(h) = \left[ \int_X |h|^q \, d\mu \right]^{1/q} = \left[ \int_X |f|^p \, d\mu \right]^{1/q} = Q_p(f)^{p/q}. \]

If we define the number $\alpha = Q_p(f)^{-p/q}$, then the function $g = \alpha h$ has $Q_q(g) = 1$, so we get

\[ P(f) \geq \int_X f g \, d\mu = \frac{1}{Q_p(f)^{p/q}} \left| \int_X f h \, d\mu \right|. \]

Notice that $fh = |f|^p$, so the above inequality can be continued with

\[ P(f) \geq \frac{1}{Q_p(f)^{p/q}} \int_X |f|^p \, d\mu = Q_p(f)^p = Q_p(f). \hfill \Box \]

**Corollary 3.2.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in (1, \infty)$. Then the $Q_p$ is a seminorm on $L^p_\mathbb{K}(X, \mathcal{A}, \mu)$, i.e.

(a) $Q_p(f_1 + f_2) \leq Q_p(f_1) + Q_p(f_2), \forall f_1, f_2 \in L^p_\mathbb{K}(X, \mathcal{A}, \mu)$;
(b) $Q_p(\alpha f) = |\alpha| \cdot Q_p(f), \forall f \in L^p_\mathbb{K}(X, \mathcal{A}, \mu), \alpha \in \mathbb{K}$. 
Proposition 3.2, we get

Since the above inequality holds for all $g \in \mathcal{L}^q(X, \mathcal{A}, \mu)$, with $Q_g(g) \leq 1$. Then the functions $f_1g$ and $f_2g$ belong to $\mathcal{L}^1(X, \mathcal{A}, \mu)$, and so $f_1g + f_2g$ also belongs to $\mathcal{L}^1(X, \mathcal{A}, \mu)$. We then get

$$\langle f_1 + f_2, g \rangle = \left| \int_X (f_1g + f_2g) \, d\mu \right| = \left| \int_X f_1g \, d\mu + \int_X f_2g \, d\mu \right| \leq \left| \int_X f_1g \, d\mu \right| + \left| \int_X f_2g \, d\mu \right| = \langle f_1, g \rangle + \langle f_2, g \rangle.$$

Using Proposition 3.2, the above inequality gives

$$\langle f_1 + f_2, g \rangle \leq Q_p(f_1) + Q_p(f_2).$$

Since the above inequality holds for all $g \in \mathcal{L}^q(X, \mathcal{A}, \mu)$, with $Q_g(g) \leq 1$, again by Proposition 3.2, we get

$$Q_p(f_1 + f_2) \leq Q_p(f_1) + Q_p(f_2).$$

Property (b) is obvious. \hfill \Box

Remarks 3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in [1, \infty)$.

A. If $f \in \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ and if $g : X \to \mathbb{K}$ is a measurable function, with $g = f$, $\mu$-a.e., then $g \in \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, and $Q_p(g) = Q_p(f)$.

B. If we define the space

$$\mathcal{N}_\mathbb{K}(X, \mathcal{A}, \mu) = \{ f : X \to \mathbb{K} : f \text{ measurable, } f = 0, \mu\text{-a.e.} \},$$

then $\mathcal{N}_\mathbb{K}(X, \mathcal{A}, \mu)$ is a linear subspace of $\mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. In fact one has the equality

$$\mathcal{N}_\mathbb{K}(X, \mathcal{A}, \mu) = \{ f \in \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu) : Q_p(f) = 0 \}.$$

The inclusion "$\subset$" is trivial. Conversely, $f \in \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ has $Q_p(f) = 0$, then the measurable function $g : X \to [0, \infty)$ defined by $g = |f|^p$ will have $\int_X g \, d\mu = 0$. By Exercise 2.3 this forces $g = 0$, $\mu$-a.e., which clearly gives $f = 0$, $\mu$-a.e.

Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in [1, \infty)$. We define

$$L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) = \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)/\mathcal{N}_\mathbb{K}(X, \mathcal{A}, \mu).$$

In other words, $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is the collection of equivalence classes associated with the relation "$\equiv$, $\mu$-a.e." For a function $f \in \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ we denote by $[f]$ its equivalence class in $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. So the equality $[f] = [g]$ is equivalent to $f = g$, $\mu$-a.e. By the above Remark, there exists a (unique) map $\| \cdot \|_p : L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \to [0, \infty)$, such that

$$\| [f] \|_p = Q_p(f), \quad \forall f \in \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu).$$

By the above Remark, it follows that $\| \cdot \|_p$ is a norm on $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. When $\mathbb{K} = \mathbb{C}$ the subscript $\mathbb{C}$ will be omitted.

Conventions. Let $(X, \mathcal{A}, \mu)$, $\mathbb{K}$, and $p$ be as above. We are going to abuse a bit the notation, by writing

$$f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu),$$

if $f$ belongs to $\mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. (We will always have in mind the fact that this notation signifies that $f$ is almost uniquely determined.) Likewise, we are going to replace $Q_p(f)$ with $\| f \|_p$. 

Given $p, q \in (1, \infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$, we use the same notation for the (correctly defined) map
\[(\cdot, \cdot) : L^p_K(X, A, \mu) \times L^q_K(X, A, \mu) \to K.\]

**Remark 3.3.** Let $(X, A, \mu)$ be a measure space, let $K$ be either $\mathbb{R}$ or $\mathbb{C}$, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Given $f \in L^p_K(X, A, \mu)$, we define the map
\[
\Lambda_f : L^q_K(X, A, \mu) \ni g \mapsto \langle f, g \rangle \in K.
\]
According to Proposition 3.2, the map $\Lambda_f$ is linear, continuous, and has norm $\|\Lambda_f\| = \|f\|_p$. If we denote by $L^q_K(X, A, \mu)^*$ the Banach space of all linear continuous maps $L^q_K(X, A, \mu) \to K$, then we have a correspondence
\[
L^p_K(X, A, \mu) \ni f \mapsto \Lambda_f \in L^q_K(X, A, \mu)^*
\]
which is linear and isometric. This correspondence will be analyzed later in Section 5.

**Notation.** Given a sequence $(f_n)_{n=1}^\infty$, and a function $f$, in $L^p_K(X, A, \mu)$, we are going to write
\[f = L^p\text{-}\lim_{n \to \infty} f_n,
\]
if $(f_n)_{n=1}^\infty$ converges to $f$ in the norm topology, i.e. $\lim_{n \to \infty} \|f_n - f\|_p = 0$.

The following technical result is very useful in the study of $L^p$ spaces.

**Theorem 3.2 (L^p Dominated Convergence Theorem).** Let $(X, A, \mu)$ be a measure space, let $K$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty)$ and let $(f_n)_{n=1}^\infty$ be a sequence in $L^p_K(X, A, \mu)$. Assume $f : X \to K$ is a measurable function, such that
\[\text{(i)} \quad f = \mu\text{-}a.e.-\lim_{n \to \infty} f_n;
\]
\[\text{(ii)} \quad \text{there exists some function } g \in L^p_K(X, A, \mu), \text{ such that}
\]
\[|f_n| \leq |g|, \mu\text{-}a.e., \forall n \geq 1.
\]
Then $f \in L^p_K(X, A, \mu)$, and one has the equality
\[f = L^p\text{-}\lim_{n \to \infty} f_n.
\]

**Proof.** Consider the functions $\varphi_n = |f_n|^p, n \geq 1$, and $\varphi = |f|^p$, and $\psi = |g|^p$. Notice that
\[\varphi = \mu\text{-}a.e.-\lim_{n \to \infty} \varphi_n;
\]
\[|\varphi_n| \leq \psi, \mu\text{-}a.e., \forall n \geq 1;
\]
\[\psi \in \mathcal{L}^1_+(X, A, \mu).
\]
We can apply the Lebesgue Dominated Convergence Theorem, so we get the fact that $\varphi \in \mathcal{L}^1_+(X, A, \mu)$, which gives the fact that $f \in L^p_K(X, A, \mu)$. Now if we consider the functions $\eta_n = |f_n - f|^p$, and $\eta = 2^{p-1}(|g|^p + |f|^p)$, then using Exercise 1, we have:
\[0 = \mu\text{-}a.e.-\lim_{n \to \infty} \eta_n;
\]
\[|\eta_n| \leq \eta, \mu\text{-}a.e., \forall n \geq 1;
\]
\[\eta \in \mathcal{L}^1_+(X, A, \mu).
\]
Again using the Lebesgue Dominated Convergence Theorem, we get
\[
\lim_{n \to \infty} \int_X \eta_n \, d\mu = 0,
\]
which means that
\[ \lim_{n \to \infty} |f_n - f|^p \, d\mu, \]
which reads \( \lim_{n \to \infty} (\|f_n - f\|_p)^p = 0 \), so we clearly have \( f = L^p - \lim_{n \to \infty} f_n \). \( \square \)

Our main goal is to prove that the \( L^p \) spaces are Banach spaces. The key result which gives this, but also has some other interesting consequences, is the following.

**Theorem 3.3.** Let \((X, \mathcal{A}, \mu)\) be a measure space, let \( \mathbb{K} \) be one of the fields \( \mathbb{R} \) or \( \mathbb{C} \), let \( p \in [1, \infty) \) and let \((f_k)_{k=1}^\infty\) be a sequence in \( L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \), such that
\[ \sum_{k=1}^{\infty} \|f_k\|_p < \infty. \]

Consider the sequence \((g_n)_{n=1}^\infty \subset L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)\) of partial sums:
\[ g_n = \sum_{k=1}^{n} f_k, \quad n \geq 1. \]

Then there exists a function \( g \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \), such that
\begin{enumerate}
  \item \( g = \mu\text{-a.e.}-\lim_{n \to \infty} g_n; \)
  \item \( g = L^p\text{-lim}_{n \to \infty} g_n. \)
\end{enumerate}

**Proof.** Denote the sum \( \sum_{k=1}^{\infty} \|f_n\|_p \) simply by \( S \). For each integer \( n \geq 1 \), define the function \( h_n : X \to [0, \infty] \), by
\[ h_n(x) = \sum_{k=1}^{n} |f_n(x)|, \quad \forall x \in X. \]
It is clear that \( h_n \in L^p(X, \mathcal{A}, \mu) \), and we also have
\[ \|h_n\|_p \leq \sum_{k=1}^{n} \|f_k\|_p \leq S, \quad \forall n \geq 1. \] \( (7) \)

Notice also that \( 0 \leq h_1 \leq h_2 \leq \ldots \). Define then the function \( h : X \to [0, \infty] \) by
\[ h(x) = \lim_{n \to \infty} h_n(x), \quad \forall x \in X. \]

**Claim:** \( h \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \).

To prove this fact, we define the functions \( \varphi = h^p \) and \( \varphi_n = (h_n)^p, \ n \geq 1 \). Notice that, we have
\begin{itemize}
  \item \( 0 \leq \varphi_1 \leq \varphi_2 \leq \ldots \);
  \item \( \varphi_n \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu), \forall n \geq 1; \)
  \item \( \lim_{n \to \infty} \varphi_n(x) = \varphi(x), \forall x \in X; \)
  \item \( \sup \left\{ \int_X \varphi_n \, d\mu : n \geq 1 \right\} \leq M^p. \)
\end{itemize}

Using the Lebesgue Monotone Convergence Theorem, it then follows that \( h^p = \varphi \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \), so \( h \) indeed belongs to \( L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \). \( (7) \) gives

Let us consider now the set \( N = \{ x \in X : h(x) = \infty \} \). On the one hand, since we also have
\[ N = \{ x \in X : \varphi(x) < \infty \}, \]
and \( \varphi \) is integrable, it follows that \( N \in \mathcal{A} \), and \( \mu(N) = 0 \). On the other hand, since
\[ \sum_{k=1}^{\infty} |f_n(x)| = h(x) < \infty, \quad \forall x \in X \setminus N, \]
it follows that, for each $x \in X \setminus N$, the series $\sum_{k=1}^{\infty} f_k(x)$ is convergent. Let us define then $g : X \to \mathbb{K}$ by

$$
g(x) = \begin{cases} 
\sum_{k=1}^{\infty} f_k(x) & \text{if } x \in X \setminus N \\
0 & \text{if } x \in N
\end{cases}
$$

It is obvious that $g$ is measurable, and we have

$$
g = \mu\text{-a.e.- } \lim_{n \to \infty} g_n.
$$

Since we have

$$
|g_n| = \left| \sum_{k=1}^{n} f_k \right| \leq \sum_{k=1}^{n} |f_k| = h_n \leq h, \ \forall n \geq 1,
$$

using the Claim, and Theorem 3.2, it follows that $g$ indeed belongs to $L^p_K(X, A, \mu)$ and we also have the equality $g = L^p\text{-lim}_{n \to \infty} g_n$. □

**Corollary 3.3.** Let $(X, A, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Then $L^p_K(X, A, \mu)$ is a Banach space, for each $p \in [1, \infty)$.

**Proof.** This is immediate from the above result, combined with the completeness criterion given by Remark II.3.1. □

Another interesting consequence of Theorem 3.3 is the following.

**Corollary 3.4.** Let $(X, A, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty)$, and let $f \in L^p_K(X, A, \mu)$. Any sequence $(f_n)_{n=1}^{\infty} \subset L^p_K(X, A, \mu)$, with $f = L^p\text{-lim}_{n \to \infty} f_n$, has a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $f = \mu\text{-a.e.- } \lim_{k \to \infty} f_{n_k}$. □

**Proof.** Without any loss of generality, we can assume that $f = 0$, so that we have

$$
\lim_{n \to \infty} \|f_n\|_p = 0.
$$

Choose then integers $1 \leq n_1 < n_2 < \ldots$, such that

$$
\|f_{n_k}\|_p \leq \frac{1}{2^k}, \ \forall k \geq 1.
$$

If we define the functions

$$
g_m = \sum_{k=1}^{m} f_{n_k},
$$

then by Theorem 3.3, it follows that there exists some $g \in L^p_K(X, A, \mu)$, such that

$$
g = \mu\text{-a.e.- } \lim_{m \to \infty} g_m.
$$

This means that there exists some $N \in A$, with $\mu(N) = 0$, such that

$$
\lim_{m \to \infty} g_m(x) = g(x), \ \forall x \in X \setminus N.
$$

In other words, for each $x \in X \setminus N$, the series $\sum_{k=1}^{\infty} f_{n_k}(x)$ is convergent (to some number $g(x) \in \mathbb{K}$). In particular, it follows that

$$
\lim_{k \to \infty} f_{n_k}(x) = 0, \ \forall x \in X \setminus N,
$$

so we indeed have $0 = \mu\text{-a.e.- } \lim_{k \to \infty} f_{n_k}$. □

The following result collects some properties of $L^p$ spaces in the case when the underlying measure space is finite.
PROPOSITION 3.3. Suppose \((X, \mathcal{A}, \mu)\) is a finite measure space, and \(\mathbb{K}\) is one of the fields \(\mathbb{R}\) or \(\mathbb{C}\).

(i) If \(f : X \to \mathbb{K}\) is a bounded measurable function, then \(f \in L^p_\mathbb{K}(X, \mathcal{A}, \mu), \forall p \in [1, \infty)\).

(ii) For any \(p, q \in [1, \infty)\), with \(p < q\), one has the inclusion \(L^q_\mathbb{K}(X, \mathcal{A}, \mu) \subset L^p_\mathbb{K}(X, \mathcal{A}, \mu)\). So taking quotients by \(\mathbb{K}(X, \mathcal{A}, \mu)\), one gets an inclusion of vector spaces

\[
L^q_\mathbb{K}(X, \mathcal{A}, \mu) \hookrightarrow L^p_\mathbb{K}(X, \mathcal{A}, \mu).
\]

Moreover the above inclusion is a continuous linear map.

PROOF. The key property that we are going to use here is the fact that the constant function 1 = \(\kappa\) is \(\mu\)-integrable (being elementary \(\mu\)-integrable).

(i). This part is pretty clear, because if we start with a bounded measurable function \(f : X \to \mathbb{K}\) and we take \(M = \sup_{x \in X} |f(x)|\), then the inequality \(|f|^p \leq M^p \cdot 1\), combined with the integrability of 1, will force the integrability of \(|f|^p\), i.e. \(f \in L^p_\mathbb{K}(X, \mathcal{A}, \mu)\).

(ii). Fix \(1 \leq p < q < \infty\), as well as a function \(f \in L^q_\mathbb{K}(X, \mathcal{A}, \mu)\). Consider the number \(r = \frac{q}{p} > 1\), and \(s = \frac{r}{r-1}\), so that we have \(\frac{1}{r} + \frac{1}{s} = 1\). Since \(f \in L^q_\mathbb{K}(X, \mathcal{A}, \mu)\), the function \(g = |f|^q\) belongs to \(L^q_\mathbb{K}(X, \mathcal{A}, \mu)\). If we define then the function \(h = |f|^p\), then we obviously have \(g = h^r\), so we get the fact that \(h\) belongs to \(L^q_\mathbb{K}(X, \mathcal{A}, \mu)\).

Using part (i), we get the fact that \(1 \in L^q_\mathbb{K}(X, \mathcal{A}, \mu)\), so by Corollary 3.1, it follows that \(h = 1 \cdot h\) belongs to \(L^1_\mathbb{K}(X, \mathcal{A}, \mu)\), and moreover, one has the inequality

\[
\int_X |f|^p \, d\mu = \int_X h \, d\mu \leq \|1\|_s \cdot \|h\|_r = \left[ \int_X 1 \, d\mu \right]^{1/s} \cdot \left[ \int_X h^r \, d\mu \right]^{1/r} = \mu(X)^{1/s} \cdot \left[ \int_X |f|^q \, d\mu \right]^{1/r} = \mu(X)^{1/s} \cdot (\|f\|_q)^{q/r}.
\]

On the one hand, this inequality proves that \(f \in L^q_\mathbb{K}(X, \mathcal{A}, \mu)\). On the other hand, this also gives the inequality

\[
\left(\|f\|_p\right)^p \leq \mu(X)^{1/s} \cdot (\|f\|_q)^{q/r} = \mu(X)^{1-\frac{p}{r}} \cdot (\|f\|_q)^p,
\]

which yields

\[
\|f\|_p \leq \mu(X)^{\frac{1}{s} - \frac{1}{r}} \cdot \|f\|_q.
\]

This proves that the linear map (8) is continuous (and has norm no greater than \(\mu(X)^{\frac{1}{s} - \frac{1}{r}}\)).

\[\square\]

Exercise 2. Give an example of a sequence of continuous functions \(f_n : [0, 1] \to [0, \infty), \ n \geq 1\), such that \(L^p-\lim_{n \to \infty} f_n = 0, \ \forall p \in [1, \infty)\), but for which it is not true that \(0 = \mu-a.e.-\lim_{n \to \infty} f_n\). (Here we work on the measure space \(([0, 1], \mathfrak{M}_\lambda([0, 1]), \lambda)\).)

Exercise 3. Let \(\Omega \subset \mathbb{R}^n\) be an open set. Prove that \(C^\infty_\text{c}(\Omega)\) is dense in \(L^p_{\mathbb{K}}(\Omega, \mathfrak{M}_\lambda(\Omega), \lambda)\), for every \(p \in [1, \infty)\). (Here \(\lambda\) denotes the \(n\)-dimensional Lebesgue measure, and \(\mathfrak{M}_\lambda(\Omega)\) denotes the collection of all Lebesgue measurable subsets of \(\Omega\).)
§3. $L^p$ spaces ($1 \leq p < \infty$)

Notations. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. We define the space
$$\mathcal{N}_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu) = \mathcal{L}^1_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu) \cap \mathcal{N}_{\mathbb{K}}(X, \mathcal{A}, \mu),$$
and we define the quotient space
$$L^1_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu) = \mathcal{L}^1_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu) / \mathcal{N}_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu).$$
In other words, if one considers the quotient map
$$\Pi_1 : \mathcal{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \to L^1_{\mathbb{K}}(X, \mathcal{A}, \mu),$$
then
$$L^1_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu) = \Pi_1(\mathcal{L}^1_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu)).$$
Notice that we have the obvious inclusion
$$\mathcal{L}^1_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu) \subset \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu), \quad \forall p \in [1, \infty),$$
so we consider the quotient map
$$\Pi_p : \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \to L^p_{\mathbb{K}}(X, \mathcal{A}, \mu),$$
we can also define the subspace
$$L^p_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu) = \Pi_p(\mathcal{L}^p_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu)), \quad \forall p \in [1, \infty).$$
Remark that, as vector spaces, the spaces $L^p_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu)$ are identical, since
$$\text{Ker} \, \Pi_p = \mathcal{N}_{\mathbb{K}}(x, \mathcal{A}, \mu), \quad \forall p \in [1, \infty).$$
With these notations we have the following fact.

**Proposition 3.4.** $L^p_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu)$ is dense in $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, for each $p \in [1, \infty)$.

**Proof.** Fix $p \in [1, \infty)$, and start with some $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. What we need to prove is the existence of a sequence $(f_n)_{n=1}^{\infty} \subset \mathcal{L}^1_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu)$, such that $f = L^p-\lim_{n \to \infty} f_n$. Taking real and imaginary parts (in the case $\mathbb{K} = \mathbb{C}$), it suffices to consider the case when $f$ is real valued. Since $|f|$ also belongs to $L^p$, it follows that $f^+ = \max\{|f|, 0\} = \frac{1}{2}(|f| + f)$, and $f^- = \max\{-f, 0\} = \frac{1}{2}(|f| - f)$ both belong to $L^p$, so in fact we can assume that $f$ is non-negative. Consider the function $g = f^p \in \mathcal{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$. Use the definition of the integral, to find a sequence $(g_n)_{n=1}^{\infty} \subset \mathcal{L}^1_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu)$, such that
- $0 \leq g_n \leq g$, $\forall n \geq 1$;
- $\lim_{n \to \infty} \int_X g_n \, d\mu = \int_X g \, d\mu$.
This gives the fact that $g = L^1-\lim_{n \to \infty} g_n$. Using Corollary 3.4, after replacing $(g_n)_{n=1}^{\infty}$ with a subsequence, we can also assume that $g = \mu$-a.e.-$\lim_{n \to \infty} g_n$. If we put $f_n = (g_n)^{1/p}$, $\forall n \geq 1$, we now have
- $0 \leq f_n \leq f$, $\forall n \geq 1$;
- $f_n = \mu$-a.e.-$\lim_{n \to \infty} f_n$.
Obviously, the $f_n$'s are still elementary integrable, and by the $L^p$ Dominated Convergence Theorem, we indeed get $f = L^p\text{-}lim_{n \to \infty} f_n$. $$\square$$

**Comments.** A. The above result gives us the fact that $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is the completion of $L^p_{\mathbb{K}, \text{elem}}(X, \mathcal{A}, \mu)$. This allows for the following alternative construction of the $\mathcal{L}^p$ spaces.

B. For a measurable function $f : X \to \mathbb{K}$, by the (proof of the) above result, it follows that the condition $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is equivalent to the equality $f = \ldots$
\[ \mu\text{-a.e.-}\lim_{n \to \infty} f_n, \text{ for some sequence } (f_n)_{n=1}^{\infty} \text{ of elementary integrable functions, which is Cauchy in the } L^p \text{ norm, i.e.} \]

\[(c) \text{ for every } \varepsilon > 0, \text{ there exists } N_\varepsilon, \text{ such that} \]

\[ \|f_m - f_n\|_p < \varepsilon, \forall m, n \geq N_\varepsilon. \]

One key feature, which will be heavily exploited in the next section, deals with the Banach space \( p = 2 \), for which we have the following.

**Proposition 3.5.** Let \((X, \mathcal{A}, \mu)\) be a measure space, and let \( \mathbb{K} \) be one of the fields \( \mathbb{R} \) or \( \mathbb{C} \).

(i) The map \((., . ) : L^2_\mathbb{K}(X, \mathcal{A}, \mu) \times L^2_\mathbb{K}(X, \mathcal{A}, \mu) \to \mathbb{K} \), given by

\[ (f | g) = \langle \bar{f}, g \rangle = \int_X \bar{f} g \, d\mu, \forall f, g \in L^2_\mathbb{K}(X, \mathcal{A}, \mu), \]

defines an inner product on \( L^2_\mathbb{K}(X, \mathcal{A}, \mu) \).

(ii) One has the equality

\[ \|f\|_2 = \sqrt{(f | f)}, \forall f \in L^2_\mathbb{K}(X, \mathcal{A}, \mu). \]

Consequently, \( L^2_\mathbb{K}(X, \mathcal{A}, \mu) \) is a Hilbert space.

**Proof.** The properties of the inner product are immediate, from the properties of integration. The second property is also clear. \( \square \)

**Remark 3.4.** The main biproduct of the above feature is the fact that the correspondence (6) is an isometric isomorphism, in the case \( p = q = 2 \). This follows from Riesz Theorem (only the surjectivity is the issue here; the rest has been discussed in Remark 3.3). If \( \phi : L^2_\mathbb{K}(X, \mathcal{A}, \mu) \to \mathbb{K} \) is a linear continuous map, then there exists some \( h \in L^2_\mathbb{K}(X, \mathcal{A}, \mu) \), such that

\[ \phi(g) = (h | g), \forall g \in L^2_\mathbb{K}(X, \mathcal{A}, \mu). \]

If we put \( f = \bar{h} \), then the above equality gives

\[ \phi(g) = (f, g), \forall g \in L^2_\mathbb{K}(X, \mathcal{A}, \mu). \]

i.e. \( \phi = \Lambda f \).

**Comments.** Eventually (see Section 5) we shall prove that the correspondence (6) is surjective also in the general case.

The correspondence (6) also has a version for \( q = 1 \). This would require the definition of an \( L^p \) space for the case \( p = \infty \). We shall postpone this until we reach Section 5. The next exercise hints towards such a construction.

**Exercise 4.** Let \((X, \mathcal{A}, \mu)\) be a measure space, let \( \mathbb{K} \) be one of the fields \( \mathbb{R} \) or \( \mathbb{C} \), and let \( f : X \to \mathbb{K} \) be a bounded measurable function. Define \( M = \sup_{x \in X} |f(x)| \).

Prove the following.

(i) Whenever \( g \in L^1_\mathbb{K}(X, \mathcal{A}, \mu) \), it follows that the function \( fg \) also belongs to \( L^1_\mathbb{K}(X, \mathcal{A}, \mu) \), and one has the inequality

\[ \|fg\|_1 \leq M \cdot \|g\|_1. \]

(ii) The map

\[ \Lambda_f : L^1_\mathbb{K}(X, \mathcal{A}, \mu) \ni g \mapsto \int_X fg \, d\mu \in \mathbb{K} \]

is linear and continuous. Moreover, one has the inequality \( \|\Lambda_f\| \leq M \).
**Remark 3.5.** If we apply the above Exercise to the constant function $f = 1$, we get the (already known) fact that the integration map

$$
(9) \quad \Lambda_1 : L^1_K(X, \mathcal{A}, \mu) \ni g \mapsto \int_X g \, d\mu \in K
$$

is linear and continuous, and has norm $\|\Lambda_1\| \leq 1$. The following exercise gives the exact value of the norm.

**Exercise 5.** With the notations above, prove that the following are equivalent:

(i) the measure space $(X, \mathcal{A}, \mu)$ is *non-degenerate*, i.e. there exists $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$;

(ii) $L^1_K(X, \mathcal{A}, \mu) \neq \{0\}$;

(iii) the integration map (9) has norm $\|\Lambda_1\| = 1$.

**Exercise 6.** Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathcal{B} \subset \mathcal{A}$ be a $\sigma$-algebra. Consider the measure space $(X, \mathcal{B}, \mu\upharpoonright\mathcal{B})$. Let $K$ be either $\mathbb{R}$ or $\mathbb{C}$. Prove that for every $p \in [1, \infty)$ one has the inclusion

$$
L^p_K(X, \mathcal{B}, \mu\upharpoonright\mathcal{B}) \subset L^p_K(X, \mathcal{A}, \mu).
$$

Prove that this inclusion gives rise to an isometric linear map

$$
T : L^p_K(X, \mathcal{B}, \mu\upharpoonright\mathcal{B}) \subset L^p_K(X, \mathcal{A}, \mu).
$$