3. The L^p spaces $(1 \leq p < \infty)$

In this section we discuss an important construction, which is extremely useful in virtually all branches of Analysis. In Section 1, we have already introduced the space $\mathfrak{L}^1$. The first construction deals with a generalization of this space.

DEFINITIONS. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$.

A. For a number $p \in (1, \infty)$, we define the space

$$\mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu) = \big\{ f : X \to \mathbb{K} \, : \, f \text{ measurable, and } \int_X |f|^p \in \, d\mu < \infty \big\}.$$

Here we use the convention introduced in Section 1, which defines $\int_X h \, d\mu = \infty$, for those measurable functions $h : X \to [0, \infty]$, that are not integrable.

Of course, in this definition we can allow also the value $p = 1$, and in this case we get the familiar definition of $\mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$.

B. For $p \in [1, \infty)$, we define the map $Q_p : \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \to [0, \infty)$ by

$$Q_p(f) = \int_X |f|^p \, d\mu, \ \ \forall \, f \in \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu).$$

REMARK 3.1. The space $\mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ was studied earlier (see Section 1). It has the following features:

(i) $\mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is a $\mathbb{K}$-vector space.

(ii) The map $Q_1 : \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \to [0, \infty)$ is a *seminorm*, i.e.

  (*a*) $Q_1(f + g) \leq Q_1(f) + Q_1(g)$, $\forall \, f, g \in \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$;

  (*b*) $Q_1(\alpha f) = |\alpha| \cdot Q_1(f)$, $\forall \, f \in \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, $\alpha \in \mathbb{K}$.

(iii) $\big| \int_X f \, d\mu \big| \leq Q_1(f)$, $\forall \, f \in \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$.

Property (*b*) is clear. Property (*a*) immediately follows from the inequality $|f+g| \leq |f| + |g|$, which after integration gives

$$\int_X |f + g| \, d\mu \leq \int_X \big[|f| + |g| \big] \, d\mu = \int_X |f| \, d\mu + \int_X |g| \, d\mu.$$

In what follows, we aim at proving similar features for the spaces $\mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ and Q_p, $1 < p < \infty$.

The following will help us prove that $\mathfrak{L}^p$ is a vector space.

Exercise 1$^\diamondsuit$. Let $p \in (1, \infty)$. Then one has the inequality

$$(s + t)^p \leq 2^{p-1}(s^p + t^p), \ \ \forall \, s, t \in [0, \infty).$$

HINT: The inequality is trivial, when $s = t = 0$. If $s + t > 0$, reduce the problem to the case $t + s = 1$, and prove, using elementary calculus techniques that

$$\min_{t \in [0,1]} \big[t^p + (1 - t)^p \big] = 2^{1-p}.$$

PROPOSITION 3.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in (1, \infty)$. When equipped with pointwise addition and scalar multiplication, $\mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is a $\mathbb{K}$-vector space.*

PROOF. It $f, g \in \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, then by Exercise 1 we have

$$\int_X |f+g|^p \, d\mu \le \int_X \big(|f|+|g|\big)^p \, d\mu \le 2^{p-1} \left[\int_X |f|^p \, d\mu + \int_X |g|^p \, d\mu \right] < \infty,$$

so $f+g$ indeed belongs to $\mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$.

It $f \in \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, and $\alpha \in \mathbb{K}$, then the equalities

$$\int_X |\alpha f|^p \, d\mu = \int_X |\alpha|^p \cdot |f|^p \, d\mu = |\alpha|^p \cdot \int_X |f|^p \, d\mu$$

clearly prove that αf also belongs to $\mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. $\square$

Our next task will be to prove that Q_p is a seminorm, for all $p > 1$. In this direction, the following is a key result. (The above mentioned convention will be used throughout this entire section.)

THEOREM 3.1 (Hölder's Inequality for integrals). *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f, g : X \to [0, \infty]$ be measurable functions, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then one has the inequality*[1]

$$\int_X fg \, d\mu \le \left[\int_X f^p \, d\mu \right]^{1/p} \cdot \left[\int_X g^q \, d\mu \right]^{1/q}. \tag{1}$$

PROOF. If either $\int_X f^p \, d\mu = \infty$, or $\int_X g^p \, d\mu = \infty$, then the inequality (1) is trivial, because in this case, the right hand side is ∞. For the remainder of the proof we will assume that $\int_X f^p \, d\mu < \infty$ and $\int_X g^q \, d\mu < \infty$.

Use Corollary 2.1 to find two sequences $(\varphi_n)_{n=1}^{\infty}$, $(\psi_n)_{n=1}^{\infty} \subset \mathfrak{L}^1_{\mathbb{R},elem}(X, \mathcal{A}, \mu)$, such that

- $0 \le \varphi_1 \le \varphi_2 \le \dots$ and $0 \le \psi_1 \le \psi_2 \le \dots$;
- $\lim_{n\to\infty} \varphi_n(x) = f(x)^p$ and $\lim_{n\to\infty} \psi_n(x) = g(x)^q$, $\forall\, x \in X$.

By the Lebesgue Dominated Convergence Theorem, we will also get the equalities

$$\int_X f^p \, d\mu = \lim_{n\to\infty} \int_X \varphi_n \, d\mu \text{ and } \int_X g^q \, d\mu = \lim_{n\to\infty} \int_X \psi_n \, d\mu. \tag{2}$$

Remark that the functions $f_n = \varphi_n^{1/p}$, $g_n \psi_n^{1/q}$, $n \ge 1$ are also elementary (because they obviously have finite range). It is obvious that we have

- $0 \le f_1 \le f_2 \le \dots$, and $0 \le g_1 \le g_2 \le \dots$;
- $\lim_{n\to\infty} f_n(x) = f(x)$, and $\lim_{n\to\infty} g_n(x)] = g(x)$, $\forall\, x \in X$.

With these notations, the equalities (2) read

$$\int_X f^p \, d\mu = \lim_{n\to\infty} \int_X (f_n)^p \, d\mu \text{ and } \int_X g^q \, d\mu = \lim_{n\to\infty} \int_X (g_n)^q \, d\mu. \tag{3}$$

Of course, the products $f_n g_n$, $n \ge 1$ are again elementary, and satisfy

- $0 \le f_1 g_1 \le f_2 g_2 \le \dots$;
- $\lim_{n\to\infty} [f_n(x) g_n(x)] = f(x) g(x)$, $\forall\, x \in X$.

Using the General Lebesgue Monotone Convergence Theorem, we then get

$$\int_X fg \, d\mu = \lim_{n\to\infty} \int_X f_n g_n \, d\mu.$$

[1] Here we use the convention $\infty^{1/p} = \infty^{1/q} = \infty$.

Using (3) we now see that, in order to prove (1), it suffices to prove the inequalities

$$\int_X f_n g_n \, d\mu \leq \left[\int_X (f_n)^p \, d\mu \right]^{1/p} \cdot \left[\int_X (g_n)^q \, d\mu \right]^{1/q}, \ \forall \, n \geq 1.$$

In other words, it suffices to prove (1), under the extra assumption that *both* f *and* g *are elementary integrable.*

Suppose f and g are elementary integrable. Then (see III.1) there exist pairwise disjoint sets $(D_j)_{j=1}^m \subset \mathcal{A}$, with $\mu(D_j) < \infty$, $\forall \, j = 1, \dots, m$, and numbers $\alpha_1, \beta_1, \dots, \alpha_m, \beta_m \in [0, \infty)$, such that

$$\begin{aligned} f &= \alpha_1 \varkappa_{D_1} + \dots + \alpha_m \varkappa_{D_m} \\ g &= \beta_1 \varkappa_{D_1} + \dots + \beta_m \varkappa_{D_m} \end{aligned}$$

Notice that we have

$$fg = \alpha_1 \beta_1 \varkappa_{D_1} + \dots + \alpha_m \beta_m \varkappa_{D_m},$$

so the left hand side of (1) is the given by

$$\int_X fg \, d\mu = \sum_{j=1}^m \alpha_j \beta_j \mu(D_j).$$

Define the numbers $x_j = \alpha_j \mu(D_j)^{1/p}$, $y_j = \beta_j \mu(D_j)^{1/q}$, $j = 1, \dots, m$. Using these numbers, combined with $\frac{1}{p} + \frac{1}{q} = 1$, we clearly have

$$\int_X fg \, d\mu = \sum_{j=1}^m (x_j y_j). \tag{4}$$

At this point we are going to use the classical Hölder inequality (see Appendix D), which gives

$$\sum_{j=1}^m (x_j y_j) \leq \left[\sum_{j=1}^m (x_j)^p \right]^{1/p} \cdot \left[\sum_{j=1}^m (y_j)^q \right]^{1/q},$$

so the equality (4) continues with

$$\begin{aligned} \int_X fg \, d\mu &\leq \left[\sum_{j=1}^m (x_j)^p \right]^{1/p} \cdot \left[\sum_{j=1}^m (y_j)^q \right]^{1/q} = \\ &= \left[\sum_{j=1}^m (\alpha_j)^p \mu(D_j) \right]^{1/p} \cdot \left[\sum_{j=1}^m (\beta_j)^q \mu(D_j) \right]^{1/q} = \\ &= \left[\int_X f^p \, d\mu \right]^{1/p} \cdot \left[\int_X g^q \, d\mu \right]^{1/q}. \end{aligned}$$

□

COROLLARY 3.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For any two functions $f \in \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}^q_{\mathbb{K}}(X, \mathcal{A}, \mu)$, the product fg belongs to $\mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ and one has the inequality*

$$\left| \int_X fg \, d\mu \right| \leq Q_p(f) \cdot Q_q(g).$$

PROOF. By Hölder's inequality, applied to $|f|$ and $|g|$, we get

$$\int_X |fg|\,d\mu \le Q_p(f)\cdot Q_q(g) < \infty,$$

so $|fg|$ belongs to $\mathfrak{L}^1_+(X,\mathcal{A},\mu)$, i.e. fg belongs to $\mathfrak{L}^1_{\mathbb{K}}(X,\mathcal{A},\mu)$. The desired inequality then follows from the inequality $\left|\int_X fg\,d\mu\right| \le \int_X |fg|\,d\mu$. $\square$

NOTATION. Suppose $(X,\mathcal{A},\mu)$ is a measure space, $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $p,q\in(1,\infty)$ are such that $\frac{1}{p}+\frac{1}{q}=1$. For any pair of functions $f\in\mathfrak{L}^p_{\mathbb{K}}(X,\mathcal{A},\mu)$, $g\in\mathfrak{L}^q_{\mathbb{K}}(X,\mathcal{A},\mu)$, we shall denote the number $\int_X fg\,d\mu\in\mathbb{K}$ simply by $\langle f,g\rangle$. With this notation, Corollary 3.1 reads:

$$\big|\langle f,g\rangle\big| \le Q_p(f)\cdot Q_q(g), \forall\, f\in\mathfrak{L}^p_{\mathbb{K}}(X,\mathcal{A},\mu),\ g\in\mathfrak{L}^q_{\mathbb{K}}(X,\mathcal{A},\mu).$$

The following result gives an alternative description of the maps Q_p, $p\in(1,\infty)$.

PROPOSITION 3.2. *Let $(X,\mathcal{A},\mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p,q\in(1,\infty)$ be such that $\frac{1}{p}+\frac{1}{q}=1$. and let $f\in\mathfrak{L}^p_{\mathbb{K}}(X,\mathcal{A},\mu)$. Then one has the equality*

$$Q_p(f)=\sup\big\{\big|\langle f,g\rangle\big|\ :\ g\in\mathfrak{L}^q_{\mathbb{K}}(X,\mathcal{A},\mu),\ Q_q(g)\le 1\big\}. \tag{5}$$

PROOF. Let us denote the right hand side of (5) simply by $P(f)$. By Corollary 3.1, we clearly have the inequality

$$P(f)\le Q_p(f).$$

To prove the other inequality, let us first observe that in the case when $Q_p(f)=0$, there is nothing to prove, because the above inequality already forces $P(f)=0$. Assume then $Q_p(f)>0$, and define the function $h: x\to\mathbb{K}$ by

$$h(x)=\begin{cases}\dfrac{|f(x)|^p}{f(x)} & \text{if } f(x)\ne 0\\[2ex] 0 & \text{if } f(x)=0\end{cases}$$

It is obvious that h is measurable. Moreover, one has the equality $|h|=|f|^{p-1}$, which using the equality $qp=p+q$ gives $|h|^q=|f|^{qp-q}=|f|^p$. This proves that $h\in\mathfrak{L}^q_{\mathbb{K}}(X,\mathcal{A},\mu)$, as well as the equality

$$Q_q(h)=\left[\int_X |h|^q\,d\mu\right]^{1/q}=\left[\int_X |f|^p\,d\mu\right]^{1/q}=Q_p(f)^{p/q}.$$

If we define the number $\alpha=Q_p(f)^{-p/q}$, then the function $g=\alpha h$ has $Q_q(g)=1$, so we get

$$P(f)\ge\left|\int_X fg\,d\mu\right|=\frac{1}{Q_p(f)^{p/q}}\left|\int_X fh\,d\mu\right|.$$

Notice that $fh=|f|^p$, so the above inequality can be continued with

$$P(f)\ge\frac{1}{Q_p(f)^{p/q}}\int_X |f|^p\,d\mu=\frac{Q_p(f)^p}{Q_p(f)^{p/q}}=Q_p(f).\quad\square$$

COROLLARY 3.2. *Let $(X,\mathcal{A},\mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p\in(1,\infty)$. Then the Q_p is a seminorm on $\mathfrak{L}^p_{\mathbb{K}}(X,\mathcal{A},\mu)$, i.e.*

(a) *$Q_p(f_1+f_2)\le Q_p(f_1)+Q_p(f_2)$, $\forall\, f_1,f_2\in\mathfrak{L}^p_{\mathbb{K}}(X,\mathcal{A},\mu)$;*
(b) *$Q_p(\alpha f)=|\alpha|\cdot Q_p(f)$, $\forall\, f\in\mathfrak{L}^p_{\mathbb{K}}(X,\mathcal{A},\mu)$, $\alpha\in\mathbb{K}$.*

PROOF. (a). Take $q = \frac{p}{p-1}$, so that $\frac{1}{p} + \frac{1}{q} = 1$. Start with some arbitrary $g \in \mathfrak{L}^q_{\mathbb{K}}(X, \mathcal{A}, \mu)$, with $Q_q(g) \leq 1$. Then the functions $f_1 g$ and $f_2 g$ belong to $\mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, and so $f_1 g + f_2 g$ also belongs to $\mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$. We then get

$$\begin{aligned}\left|\langle f_1 + f_2\, ,\, g\rangle\right| &= \left|\int_X (f_1 g + f_2 g)\, d\mu\right| = \left|\int_X f_1 g\, d\mu + \int_X f_2 g\, d\mu\right| \leq \\ &\leq \left|\int_X f_1 g\, d\mu\right| + \left|\int_X f_2 g\, d\mu\right| = \left|\langle f_1, g\rangle\right| + \left|\langle f_2, g\rangle\right|.\end{aligned}$$

Using Proposition 3.2, the above inequality gives

$$\left|\langle f_1 + f_2\, ,\, g\rangle\right| \leq Q_p(f_1) + Q_p(f_2).$$

Since the above inequality holds for all $g \in \mathfrak{L}^q_{\mathbb{K}}(X, \mathcal{A}, \mu)$, with $Q_q(g) \leq 1$, again by Proposition 3.2, we get

$$Q_p(f_1 + f_2) \leq Q_p(f_1) + Q_p(f_2).$$

Property (b) is obvious. □

REMARKS 3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in [1, \infty)$.

A. If $f \in \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ and if $g : X \to \mathbb{K}$ is a measurable function, with $g = f$, μ-a.e., then $g \in \mathfrak{L}^p_{\mathbb{K}}(x, \mathcal{A}, \mu)$, and $Q_p(g) = Q_p(f)$.

B. If we define the space

$$\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) = \big\{f : X \to \mathbb{K} \,:\, f \text{ measurable},\ f = 0,\ \mu\text{-a.e.}\big\},$$

then $\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is a linear subspace of $\mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. In fact one has the equality

$$\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) = \big\{f \in \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \,:\, Q_p(f) = 0\big\}.$$

The inclusion "$\subset$" is trivial. Conversely, $f \in \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ has $Q_p(f) = 0$, then the measurable function $g : X \to [0, \infty)$ defined by $g = |f|^p$ will have $\int_X g\, d\mu = 0$. By Exercise 2.3 this forces $g = 0$, μ-a.e., which clearly gives $f = 0$, μ-a.e.

DEFINITION. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $p \in [1, \infty)$. We define

$$L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) = \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)/\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu).$$

In other words, $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is the collection of equivalence classes associated with the relation "$=$, μ-a.e." For a function $f \in \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ we denote by $[f]$ its equivalence class in $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. So the equality $[f] = [g]$ is equivalent to $f = g$, μ-a.e. By the above Remark, there exists a (unique) map $\|\,.\,\|_p : L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \to [0, \infty)$, such that

$$\|[f]\|_p = Q_p(f),\ \ \forall\, f \in \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu).$$

By the above Remark, it follows that $\|\,.\,\|_p$ is a *norm* on $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. When $\mathbb{K} = \mathbb{C}$ the subscript $\mathbb{C}$ will be ommitted.

CONVENTIONS. Let $(X, \mathcal{A}, \mu)$, $\mathbb{K}$, and p be as above We are going to abuse a bit the notation, by writing

$$f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu),$$

if f belongs to $\mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. (We will always have in mind the fact that this notation signifies that f is almost uniquely determined.) Likewise, we are going to replace $Q_p(f)$ with $\|f\|_p$.

Given $p, q \in (1, \infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$, we use the same notation for the (correctly defined) map

$$\langle\, .\, ,\, .\, \rangle : L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \times L^q_{\mathbb{K}}(X, \mathcal{A}, \mu) \to \mathbb{K}.$$

REMARK 3.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Given $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, we define the map

$$\Lambda_f : L^q_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni g \longmapsto \langle f, g \rangle \in \mathbb{K}.$$

According to Proposition 3.2, *the map Λ_f is linear, continuous, and has norm $\|\Lambda_f\| = \|f\|_p$.* If we denote by $L^q_{\mathbb{K}}(X, \mathcal{A}, \mu)^*$ the Banach space of all linear continuous maps $L^q_{\mathbb{K}}(X, \mathcal{A}, \mu) \to \mathbb{K}$, then we have a correspondence

$$L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni f \longmapsto \Lambda_f \in L^q_{\mathbb{K}}(X, \mathcal{A}, \mu)^* \tag{6}$$

which is *linear and isometric*. This correspondence will be analyzed later in Section 5.

NOTATION. Given a sequence $(f_n)_{n=1}^\infty$, and a function f, in $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, we are going to write

$$f = L^p\text{-}\lim_{n\to\infty} f_n,$$

if $(f_n)_{n=1}^\infty$ converges to f in the norm topology, i.e. $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

The following technical result is very useful in the study of L^p spaces.

THEOREM 3.2 (L^p Dominated Convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty)$ and let $(f_n)_{n=1}^\infty$ be a sequence in $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. Assume $f : X \to \mathbf{K}$ is a measurable function, such that*

(i) $f = \mu$-a.e.-$\lim_{n\to\infty} f_n$;
(ii) *there exists some function $g \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, such that*

$$|f_n| \leq |g|, \ \ \mu\text{-a.e.}, \ \ \forall\, n \geq 1.$$

Then $f \in L^p_{\mathbf{K}}(X, \mathcal{A}, \mu)$, and one has the equality

$$f = L^p\text{-}\lim_{n\to\infty} f_n.$$

PROOF. Consider the functions $\varphi_n = |f_n|^p$, $n \geq 1$, and $\varphi = |f|^p$, and $\psi = |g|^p$. Notice that

- $\varphi = \mu$-a.e.-$\lim_{n\to\infty} \varphi_n$;
- $|\varphi_n| \leq \psi$, μ-a.e., $\forall\, n \geq 1$;
- $\psi \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$.

We can apply the Lebsgue Dominated Convergence Theorem, so we get the fact that $\varphi \in \mathcal{L}^1_+(X, \mathcal{A}, \mu)$, which gives the fact that $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. Now if we consider the functions $\eta_n = |f_n - f|^p$, and $\eta = 2^{p-1}\big(|g|^p + |f|^p\big)$, then using Exercise 1, we have:

- $0 = \mu$-a.e.-$\lim_{n\to\infty} \eta_n$;
- $|\eta_n| \leq \eta$, μ-a.e., $\forall\, n \geq 1$;
- $\eta \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$.

Again using the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_X \eta_n \, d\mu = 0,$$

which means that

$$\lim_{n\to\infty} |f_n - f|^p \, d\mu,$$

which reads $\lim_{n\to\infty} \big(\|f_n - f\|_p\big)^p = 0$, so we clearly have $f = L^p\text{-}\lim_{n\to\infty} f_n$. $\square$

Our main goal is to prove that the L^p spaces are Banach spaces. The key result which gives this, but also has some other interesting consequences, is the following.

THEOREM 3.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty)$ and let $(f_k)_{k=1}^\infty$ be a sequence in $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, such that*

$$\sum_{k=1}^\infty \|f_k\|_p < \infty.$$

Consider the sequence $(g_n)_{n=1}^\infty \subset L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ of partial sums:

$$g_n = \sum_{k=1}^n f_k, \ \ n \geq 1.$$

Then there exists a function $g \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, such that

(a) $g = \mu$-a.e.-$\lim_{n\to\infty} g_n$;
(b) $g = L^p\text{-}\lim_{n\to\infty} g_n$.

PROOF. Denote the sum $\sum_{k=1}^\infty \|f_n\|_p$ simply by S. For each integer $n \geq 1$, define the function $h_n : X \to [0, \infty]$, by

$$h_n(x) = \sum_{k=1}^n |f_n(x)|, \ \ \forall\, x \in X.$$

It is clear that $h_n \in L^p_{\mathbb{R}}(X, \mathcal{A}, \mu)$, and we also have

$$\|h_n\|_p \leq \sum_{k=1}^n \|f_k\|_p \leq S, \ \ \forall\, n \geq 1. \tag{7}$$

Notice also that $0 \leq h_1 \leq h_2 \leq \dots$. Define then the function $h : X \to [0, \infty]$ by

$$h(x) = \lim_{n\to\infty} h_n(x), \ \ \forall\, x \in X.$$

Claim: $h \in L^p_{\mathbb{R}}(X, \mathcal{A}, \mu)$.

To prove this fact, we define the functions $\varphi = h^p$ and $\varphi_n = (h_n)^p$, $n \geq 1$. Notice that, we have

- $0 \leq \varphi_1 \leq \varphi_2 \leq \dots$;
- $\varphi_n \in L^1_{\mathbb{R}}(X, \mathcal{A}, \mu)$, $\forall\, n \geq 1$;
- $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, $\forall\, x \in X$;
- $\sup\big\{ \int_X \varphi_n \, d\mu \, : \, n \geq 1 \big\} \leq M^p$.

Using the Lebesgue Monotone Convergence Theorem, it then follows that $h^p = \varphi \in L^1_{\mathbb{R}}(X, \mathcal{A}, \mu)$, so h indeed belongs to $L^p_{\mathbb{R}}(X, \mathcal{A}, \mu)$.(7) gives

Let us consider now the set $N = \{x \in X \, : \, h(x) = \infty\}$. On the one hand, since we also have

$$N = \{x \in X \, : \, \varphi(x) < \infty\},$$

and φ is integrable, it follows that $N \in \mathcal{A}$, and $\mu(N) = 0$. On the other hand, since

$$\sum_{k=1}^\infty |f_n(x)| = h(x) < \infty, \ \ \forall\, x \in X \smallsetminus N,$$

it follows that, for each $x \in X \smallsetminus N$, the series $\sum_{k=1}^\infty f_k(x)$ is convergent. Let us define then $g : X \to \mathbb{K}$ by

$$g(x) = \begin{cases} \sum_{k=1}^\infty f_k(x) & \text{if } x \in X \smallsetminus N \\ 0 & \text{if } x \in N \end{cases}$$

It is obvious that g is measurable, and we have

$$g = \mu\text{-a.e.-}\lim_{n\to\infty} g_n.$$

Since we have

$$|g_n| = \left|\sum_{k=1}^n f_k\right| \le \sum_{k=1}^n |f_k| = h_n \le h, \ \ \forall\, n \ge 1,$$

using the Claim, and Theorem 3.2, it follows that g indeed belongs to $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ and we also have the equality $g = L^p\text{-}\lim_{n\to\infty} g_n$. □

COROLLARY 3.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Then $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is a Banach space, for each $p \in [1, \infty)$.*

PROOF. This is immediate from the above result, combined with the completeness criterion given by Remark II.3.1. □

Another interesting consequence of Theorem 3.3 is the following.

COROLLARY 3.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty)$, and let $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. Any sequence $(f_n)_{n=1}^\infty \subset\in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, with $f = L^p\text{-}\lim_{n\to\infty} f_n$, has a subsequence $(f_{n_k})_{k=1}^\infty$ such that $f = \mu$-a.e.-$\lim_{k\to\infty} f_{n_k}$.*

PROOF. Without any loss of generality, we can assume that $f = 0$, so that we have

$$\lim_{n\to\infty} \|f_n\|_p = 0.$$

Choose then integers $1 \le n_1 < n_2 < \dots$, such that

$$\|f_{n_k}\|_p \le \frac{1}{2^k}, \ \ \forall\, k \ge 1.$$

If we define the functions

$$g_m = \sum_{k=1}^m f_{n_k},$$

then by Theorem 3.3, it follows that there exists some $g \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, such that

$$g = \mu\text{-a.e.-}\lim_{m\to\infty} g_m.$$

This measn that there exists some $N \in \mathcal{A}$, with $\mu(N) = 0$, such that

$$\lim_{m\to\infty} g_m(x) = g(x), \ \ \forall\, x \in X \smallsetminus N.$$

In other words, *for each $x \in X \smallsetminus N$, the series $\sum_{k=1}^\infty f_{n_k}(x)$ is convergent (to some number $g(x) \in \mathbb{K}$)*. In particular, it follows that

$$\lim_{k\to\infty} f_{n_k}(x) = 0, \ \ \forall\, x \in X \smallsetminus N,$$

so we indeed have $0 = \mu$-a.e.-$\lim_{k\to\infty} f_{n_k}$. □

The following result collects some properties of L^p spaces in the case when the undelrying measure space is finite.

PROPOSITION 3.3. *Suppose $(X, \mathcal{A}, \mu)$ is a* finite *measure space, and $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$.*

(i) *If $f : X \to \mathbb{K}$ is a bounded measurable function, then $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, $\forall\, p \in [1, \infty)$.*

(ii) *For any $p, q \in [1, \infty)$, with $p < q$, one has the inclusion $\mathfrak{L}^q_{\mathbb{K}}(X, \mathcal{A}, \mu) \subset \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. So taking quotients by $\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu)$, one gets an inclusion of vector spaces*

$$L^q_{\mathbb{K}}(X, \mathcal{A}, \mu) \hookrightarrow L^p_{\mathbb{K}}(X, \mathcal{A}, \mu). \tag{8}$$

Moreover the above inclusion is a continuous linear map.

PROOF. The key property that we are going to use here is the fact that the constant function $1 = \varkappa_X$ is μ-integrable (being elementary μ-integrable).

(i). This part is pretty clear, because if we start with a bounded measurable function $f : X \to \mathbb{K}$ and we take $M = \sup_{x \in X} |f(x)|$, then the inequality $|f|^p \le M^p \cdot 1$, combined with the integrability of 1, will force the inetgrability of $|f|^p$, i.e. $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$.

(ii). Fix $1 \le p < q < \infty$, as well as a function $f \in \mathfrak{L}^q_{\mathbb{K}}(X, \mathcal{A}, \mu)$. Consider the number $r = \frac{q}{p} > 1$, and $s = \frac{r}{r-1}$, so that we have $\frac{1}{r} + \frac{1}{s} = 1$. Since $f \in \mathfrak{L}^q_{\mathbb{K}}(X, \mathcal{A}, \mu)$, the function $g = |f|^q$ belongs to $\mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$. If we define then the function $h = |f|^p$, then we obviously have $g = h^r$, so we get the fact that h belongs to $\mathfrak{L}^r_{\mathbb{K}}(X, \mathcal{A}, \mu)$. Using part (i), we get the fact that $1 \in \mathfrak{L}^s_{\mathbb{K}}(X, \mathcal{A}, \mu)$, so by Corollary 3.1, it follows that $h = 1 \cdot h$ belongs to $\mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, and moreover, one has the inequality

$$\begin{aligned}\int_X |f|^p \, d\mu = \int_X h \, d\mu \le \|1\|_s \cdot \|h\|_r = \left[\int_X 1 \, d\mu \right]^{1/s} \cdot \left[\int_X h^r \, d\mu \right]^{1/r} = \\ = \mu(X)^{1/s} \cdot \left[\int_X |f|^q \, d\mu \right]^{1/r} = \mu(X)^{1/s} \cdot \big(\|f\|_q \big)^{q/r}.\end{aligned}$$

On the one hand, this inequality proves that $f \in \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. On the other hand, this also gives the inequality

$$\big(\|f\|_p \big)^p \le \mu(X)^{1/s} \cdot \big(\|f\|_q \big)^{q/r} = \mu(X)^{1 - \frac{p}{q}} \cdot \big(\|f\|_q \big)^p,$$

which yields

$$\|f\|_p \le \mu(X)^{\frac{1}{p} - \frac{1}{q}} \cdot \|f\|_q.$$

This proves that the linear map (8) is continuous (and has norm no greater than $\mu(X)^{\frac{1}{p} - \frac{1}{q}}$). ☐

Exercise 2. Give an example of a sequence of continuous functions $f_n : [0,1] \to [0, \infty)$, $n \ge 1$, such that $L^p\text{-}\lim_{n \to \infty} f_n = 0$, $\forall\, p \in [1, \infty)$, but for which it is not true that $0 = \mu\text{-a.e.-}\lim_{n \to \infty} f_n$. (Here we work on the measure space $\big([0,1], \mathfrak{M}_\lambda([0,1]), \lambda\big)$.)

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be an open set. Prove that $C^{\mathbb{K}}_c(\Omega)$ is dense in $L^p_{\mathbb{K}}(\Omega, \mathfrak{M}_\lambda(\Omega), \lambda)$, for every $p \in [1, \infty)$. (Here λ denotes the n-dimensional Lebesgue measure, and $\mathfrak{M}_\lambda(\Omega)$ denotes the collection of all Lebesgue measurable subsets of Ω.)

NOTATIONS. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. We define the space

$$\mathfrak{N}_{\mathbb{K},elem}(X, \mathcal{A}, \mu) = \mathfrak{L}^1_{\mathbb{K},elem}(X, \mathcal{A}, \mu) \cap \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu),$$

and we define the quotient space

$$L^1_{\mathbb{K},elem}(X, \mathcal{A}, \mu) = \mathfrak{L}^1_{\mathbb{K},elem}(X, \mathcal{A}, \mu) \,/\, \mathfrak{N}_{\mathbb{K},elem}(X, \mathcal{A}, \mu).$$

In other words, if one considers the quotient map

$$\Pi_1 : \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \to L^1_{\mathbb{K}}(X, \mathcal{A}, \mu),$$

then $L^1_{\mathbb{K},elem}(X, \mathcal{A}, \mu) = \Pi_1\big(\mathfrak{L}^1_{\mathbb{K},elem}(X, \mathcal{A}, \mu)\big)$. Notice that we have the obvious inclusion

$$\mathfrak{L}^1_{\mathbb{K},elem}(X, \mathcal{A}, \mu) \subset \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu), \ \ \forall\, p \in [1, \infty),$$

so we we consider the quotient map

$$\Pi_p : \mathfrak{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \to L^p_{\mathbb{K}}(X, \mathcal{A}, \mu),$$

we can also define the subspace

$$L^p_{\mathbb{K},elem}(X, \mathcal{A}, \mu) = \Pi_p\big(\mathfrak{L}^p_{\mathbb{K},elem}(X, \mathcal{A}, \mu)\big), \ \ \forall\, p \in [1, \infty).$$

Remark that, as vector spaces, the spaces $L^p_{\mathbb{K},elem}(X, \mathcal{A}, \mu)$ are identical, since

$$\operatorname{Ker} \Pi_p = \mathfrak{N}_{\mathbb{K}}(x, \mathcal{A}, \mu), \ \ \forall\, p \in [1, \infty).$$

With these notations we have the following fact.

PROPOSITION 3.4. *$L^p_{\mathbb{K},elem}(X, \mathcal{A}, \mu)$ is dense in $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, for each $p \in [1, \infty)$.*

PROOF. Fix $p \in [1, \infty)$, and start with some $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$. What we need to prove is the existence of a sequence $(f_n)_{n=1}^\infty \subset \mathfrak{L}^1_{\mathbb{K},elem}(X, \mathcal{A}, \mu)$, such that $f = L^p\text{-}\lim_{n\to\infty} f_n$. Taking real and imaginary parts (in the case $\mathbb{K} = \mathbb{C}$), it suffieces to consider the case when f is real valued. Since $|f|$ also belongs to L^p, it follows that $f^+ = \max\{f, 0\} = \frac{1}{2}\big(|f| + f\big)$, and $f^- = \max\{-f, 0\} = \frac{1}{2}\big(|f| - f\big)$ both belong to L^p, so in fact we can assume that f is non-negative. Consider the function $g = f^p \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$. Use the definition of the integral, to find a sequence $(g_n)_{n=1}^\infty \subset \mathfrak{L}^1_{\mathbb{R},elem}(X, \mathcal{A}, \mu)$, such that

- $0 \le g_n \le g$, $\forall\, n \ge 1$;
- $\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu$.

This gives the fact that $g = L^1\text{-}\lim_{n\to\infty} g_n$. Using Corollary 3.4, after replacing $(g_n)_{n=1}^\infty$ with a subsequence, we can also assume that $g = \mu\text{-a.e.-}\lim_{n\to\infty} g_n$. If we put $f_n = (g_n)^{1/p}$, $\forall\, n \ge 1$, we now have

- $0 \le f_n \le f$, $\forall\, n \ge 1$;
- $f = \mu\text{-a.e.-}\lim_{n\to\infty} f_n$.

Obviously, the f_n's are still elementary integrable, and by the L^p Dominated Convergence Theorem, we indeed get $f = L^p\text{-}\lim_{n\to\infty} f_n$. □

COMMENTS. A. The above result gives us the fact that $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is the *completion* of $L^p_{\mathbb{K},elem}(X, \mathcal{A}, \mu)$. This allows for the following alternative construction of the $\mathfrak{L}^p$ spaces.

B. For a measurable function $f : X \to \mathbb{K}$, by the (proof of the) above result, it follows that the condition $f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is equivalent to the equality $f =$

μ-a.e.-$\lim_{n\to\infty} f_n$, for some sequence $(f_n)_{n=1}^\infty$ of elementary integrable functions, which is *Cauchy in the L^p norm*, i.e.

(C) *for every $\varepsilon > 0$, there exists N_ε, such that*

$$\|f_m - f_n\|_p < \varepsilon, \ \forall\, m, n \geq N_\varepsilon.$$

One key feature, which will be heavily exploited in the next section, deals with the Banach space $p = 2$, for which we have the following.

PROPOSITION 3.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$.*

(i) *The map $(\,.\,|\,.\,) : L^2_{\mathbb{K}}(X, \mathcal{A}, \mu) \times L^2_{\mathbb{K}}(X, \mathcal{A}, \mu) \to \mathbb{K}$, given by*

$$(\, f \,|\, g\,) = \langle \bar{f}, g\rangle = \int_X \bar{f} g \, d\mu, \ \forall\, f, g \in L^2_{\mathbb{K}}(X, \mathcal{A}, \mu),$$

defines an inner product on $L^2_{\mathbb{K}}(X, \mathcal{A}, \mu)$.

(ii) *One has the equality*

$$\|f\|_2 = \sqrt{(\, f \,|\, f\,)}, \ \forall\, f \in L^2_{\mathbb{K}}(X, \mathcal{A}, \mu).$$

Consequently , $L^2_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is a Hilbert space.

PROOF. The properties of the inner product are immediate, from the properties of integration. The second property is also clear. $\square$

REMARK 3.4. The main biproduct of the above feature is the fact that the correspondence (6) is an *isometric isomorphism*, in the case $p = q = 2$. This follows from Riesz Theorem (only the surjectivity is the issue here; the rest has been discussed in Remark 3.3). If $\phi : L^2_{\mathbb{K}}(X, \mathcal{A}, \mu) \to \mathbb{K}$ is a linear continuous map, then there exists some $h \in L^2_{\mathbb{K}}(X, \mathcal{A}, \mu)$, such that

$$\phi(g) = (\, h \,|\, g\,), \ \forall\, g \in L^2_{\mathbb{K}}(X, \mathcal{A}, \mu).$$

If we put $f = \bar{h}$, then the above equality gives

$$\phi(g) = \langle f, g\rangle, \ \forall\, g \in L^2_{\mathbb{K}}(X, \mathcal{A}, \mu).$$

i.e. $\phi = \Lambda_f$.

COMMENTS. Eventually (see Section 5) we shall prove that the correspondence (6) is surjective also in the general case.

The correspondence (6) also has a version for $q = 1$. This would require the definition of an L^p space for the case $p = \infty$. We shall postpone this until we reach Section 5. The next exercise hints towards such a construction.

Exercise 4$^\diamond$. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $f : X \to \mathbb{K}$ be a *bounded* measurable function. Define $M = \sup_{x\in X} |f(x)|$. Prove the following.

(i) Whenever $g \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, it follows that the function fg also belongs to $L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, and one has the inequality

$$\|fg\|_1 \leq M \cdot \|g\|_1.$$

(ii) The map

$$\Lambda_f : L^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni g \longmapsto \int_X fg \, d\mu \in \mathbb{K}$$

is linear and continuous. Moreover, one has the inequality $\|\Lambda_f\| \leq M$.

REMARK 3.5. If we apply the above Exercise to the constant function $f = 1$, we get the (already known) fact that the integration map

$$\Lambda_1 : L^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni g \longmapsto \int_X g\, d\mu \in \mathbb{K} \tag{9}$$

is linear and continuous, and has norm $\|\Lambda_1\| \le 1$. The follwing exercise gives the exact value of the norm.

Exercise 5. With the notations above, prove that the following are equivalent:

(i) the measure space $(X, \mathcal{A}, \mu)$ is *non-degenerate*, i.e. there exists $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$;
(ii) $L^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \ne \{0\}$;
(ii) the integration map (9) has norm $\|\Lambda_1\| = 1$.

Exercise 6$^\diamondsuit$. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathcal{B} \subset \mathcal{A}$ be a σ-algebra. Consider the measure space $(X, \mathcal{B}\, \mu\big|_{\mathcal{B}})$. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Prove that for every $p \in [1, \infty)$ one has the inclusion

$$\mathcal{L}^p_{\mathbb{K}}(X, \mathcal{B}, \mu\big|_{\mathcal{B}}) \subset \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu).$$

Prove that this inclusion gives rise to an isometric linear map

$$T : L^p_{\mathbb{K}}(X, \mathcal{B}, \mu\big|_{\mathcal{B}}) \subset L^p_{\mathbb{K}}(X, \mathcal{A}, \mu).$$