7. Signed measures and complex measures

In this section we discuss a generalization of the notion of a measure, to the case where the values are allowed to be outside $[0, \infty]$. The first notion is described by the following.

**Definition.** Suppose $A$ is a $\sigma$-algebra on a non-empty set $X$. A function $\mu : A \to [-\infty, \infty]$ is called a signed measure on $A$, if it has the properties below.

(i) Either one of the following is true
   • $\mu(A) < \infty$, $\forall A \in A$;
   • $\mu(A) > -\infty$, $\forall A \in A$.
(ii) $\mu(\emptyset) = 0$.
(iii) For any pairwise disjoint sequence $(A_n)_{n=1}^{\infty} \subset A$, one has the equality

$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Here we adopt the convention that if one term in the right hand side of (1) is equal to $\pm\infty$, then the entire sum is equal to $\pm\infty$. It is important to use condition (i), which avoids situations when one term is $\infty$ and another term is $-\infty$.

**Examples 7.1.** Let us agree, in this section only, to use the term “honest” measure, for a measure in the usual sense.

A. Any “honest” measure is of course a signed measure.
B. If $\mu$ is a signed measure, then $-\mu$ is again a signed measure.
C. If $\mu_1$ and $\mu_2$ are “honest” measures, one of which is finite, then $\mu_1 - \mu_2$ is a signed measure. Eventually (see Theorem 8.2) we are going to show that any signed measure can be written in this form.

One key technical result about signed measures is the following.

**Theorem 7.1.** Let $A$ be a $\sigma$-algebra on a non-empty set $X$, and let $\mu$ be a signed measure on $X$. Then there exist sets $L, U \in A$, such that

$\mu(L) = \inf \{\mu(A) : A \in A\}$;  
$\mu(U) = \sup \{\mu(A) : A \in A\}$.

**Proof.** Since $-\mu$ is also a signed measure, it suffices to prove only the existence of $M$ satisfying (3). Denote the right hand side of (3) by $\alpha$, and choose a sequence $(\alpha_n)_{n \geq 1} \subset \mathbb{R}$, such that $\lim_{n \to \infty} \alpha_n = \alpha$, and $\alpha_n < \alpha$, $\forall n \geq 1$. The key construction we need is contained in the following.

**Claim 1:** There exists a family of sets $\{B^k_n : k, n \in \mathbb{N}, 1 \leq k \leq n\} \subset A$, with the following properties:

(i) for every $n \geq 1$, one has the inclusions

$B^k_1 \subset B^k_2 \subset \ldots \subset B^k_n$ \quad $\cup$ \quad $\cup$ \quad $\cup$

$B^{k+1}_1 \subset B^{k+1}_2 \subset \ldots \subset B^{k+1}_n \subset B^{k+1}_{n+1}$

(ii) for every $k \geq 1$ one has the inequalities

$\mu(B^k_n \setminus B^{k+1}_n) \leq 0$, $\forall n \geq k$. 

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The sets $E_n$.

Notice that we have

$$E_n \in \{N_n\}$$

We construct this sequence inductively, one row at a time (the rows are indexed by the upper index $n$). Choose $B_1 \in \mathcal{A}$ to be any set with $\mu(B_1) \geq \alpha_1$. Suppose we have constructed the first $N$ rows, i.e., we have defined the sets $B_k$, $1 \leq k \leq n \leq m$, so that property (i) holds for all $n = 1, \ldots, m - 1$, property (ii) holds in the form

$$\alpha_k \leq \mu(B^k) \leq \mu(B^{k+1}) \leq \ldots \leq \mu(B^m), \quad \forall k = 1, \ldots, m,$$

and property (iii) holds for all $n = 1, \ldots, m$. Let us explain now how the next row $B^{m+1}_1 \subset B^{m+1}_2 \subset \ldots \subset B^{m+1}_m \subset B^{m+1}_{m+1}$ is constructed. Define the sets $E_1, E_2, \ldots, E_m \in \mathcal{A}$ by

$$E_1 = B^1_m, \text{ and } E_k = B^m_k \setminus B^m_{k-1}, \quad \forall k = 2, \ldots, m.$$}

The sets $E_k$, $k = 1, \ldots, m$ are pairwise disjoint, and we have

$$B^m_k = \bigcup_{j=1}^k E_j, \quad \forall k = 1, \ldots, m.$$}

Choose now an arbitrary set $D \in \mathcal{A}$, with $\mu(D) \geq \alpha_{m+1}$, and define, for each $j \in \{1, \ldots, m\}$, the set

$$G_j = \begin{cases} 
E_j & \text{if } \mu(E_j \setminus D) > 0 \\
E_j \cap D & \text{if } \mu(E_j \setminus D) \leq 0
\end{cases}$$

Notice that we have $E_j \supset G_j$, and using the equality $\mu(E_j) = \mu(E_j \setminus D) + \mu(E_j \setminus D)$, we also have

$$\mu(E_j \setminus G_j) \leq 0 \text{ and } \mu(G_j) \geq \mu(E_j \cap D), \quad \forall j = 1, \ldots, m.$$}

Define also the set $G_{m+1} = D \setminus B^m_m$. It is clear that the sets $G_1, G_2, \ldots, G_{m+1}$ are pairwise disjoint. Construct now the $m + 1$ row by taking

$$B^{m+1}_k = \bigcup_{j=1}^k G_j, \quad \forall k = 1, 2, \ldots, m + 1.$$}

It is obvious that one has the inclusions

$$B^{m+1}_1 \subset B^{m+1}_2 \subset \ldots \subset B^{m+1}_{m+1}.$$}

Since $E_k \supset G_k, \forall k = 1, \ldots, m$, it is also clear that we have the vertical inclusions $B^m_k \supset B^m_{k-1}, \forall k = 1, \ldots, m$. Using (4), for each $k = 1, \ldots, m$, we have

$$\mu(B^m_k \setminus B^m_{k+1}) = \mu\left(\bigcup_{j=1}^k [E_j \setminus G_j]\right) = \sum_{j=1}^k \mu(E_j \setminus G_j) \leq 0.$$}

Finally, again by (4), we have

$$\mu(B^{m+1}_{m+1}) = \mu\left(\bigcup_{k=1}^{m+1} G_k\right) = \sum_{k=1}^{m+1} \mu(G_k) \geq \mu(G_{m+1}) + \sum_{k=1}^{m} \mu(E_k \cap D) = \mu(G_{m+1}) + \mu\left(\bigcup_{k=1}^{m} [E_k \cap D]\right) = \mu(D \setminus B^m_m) + \mu(B^m_m \cap D) = \mu(D) \geq \alpha_{m+1}.$$}

Claim 2: There exists a sequence $(A_k)_{k=1}^\infty \subset \mathcal{A}$, such that

(i) $A_1 \subset A_2 \subset A_3 \subset \ldots$;
(ii) $\mu(A_k) \geq \alpha_k, \forall k \geq 1$. 

We fix a family \( \{ B^n_k : k, n \in \mathbb{N}, 1 \leq k \leq n \} \) satisfying the properties in Claim 1. For every \( k \geq 1 \), we define \( A_k = \bigcap_{n=k}^\infty B^n_k \). Notice that, using property (i) from Claim 1 (the vertical inclusions), we have

\[
B^n_k = A_k \cup \left( \bigcup_{n=k}^\infty (B^n_k \setminus B^{n+1}_k) \right),
\]

and the sets \( A_k, B^n_k \setminus B^{n+1}_k, n \geq k \), are pairwise disjoint, so using property (ii) from Claim 1, we have

\[
\mu(B^n_k) = \mu(A_k) + \sum_{n=k}^\infty \mu(B^n_k \setminus B^{n+1}_k) \leq \mu(A_k).
\]

Using property (iii) from Claim 1, we then get \( \mu(A_k) \geq \alpha_k \). The fact that we have the inclusions \( A_1 \subset A_2 \subset \ldots \) is clear, from property (i) in Claim 1 (the horizontal inclusions).

Fix now the sequence \( (A_k)_{k=1}^\infty \subset A \) as in Claim 2, and let us consider the set \( M = \bigcup_{k=1}^\infty A_k \). If we define the sets \( M_1 = A_1 \) and \( M_k = A_k \setminus A_{k-1}, \forall k \geq 2 \), then we have \( M = \bigcup_{k=1}^\infty M_k \), and the sets \( M_1, M_2, M_3, \ldots \) are pairwise disjoint. In particular, this gives

\[
\mu(M) = \sum_{k=1}^\infty \mu(M_k) = \lim_{k \to \infty} \left[ \sum_{j=1}^k \mu(M_j) \right] = \lim_{k \to \infty} \mu \left( \bigcup_{j=1}^k M_j \right).
\]

Since we obviously have \( \bigcup_{j=1}^k M_j = A_k, \forall k \geq 1 \), the above equality proves that (5)

\[
\mu(M) = \lim_{k \to \infty} \mu(A_k).
\]

Since we have \( \alpha_k \leq \mu(A_k) \leq \alpha, \forall k \geq 1 \), as well as \( \lim_{k \to \infty} \alpha_k = \alpha \), the equality (5) forces \( \mu(M) = \alpha \).

**Remark 7.1.** One interesting application of the above result is the fact that, whenever \( \mu \) is a signed measure on \( A \), such that

\[
-\infty < \mu(A) < \infty, \forall A \in A,
\]

then

\[
-\infty < \inf \{ \mu(A) : A \in A \} \leq \sup \{ \mu(A) : A \in A \} < \infty.
\]

A signed measure with property (6) is called **finite**.

We are now in position to prove the statement made in Example 8.1.C.

**Theorem 7.2.** Let \( X \) be a non-empty set, let \( A \) be a \( \sigma \)-algebra on \( X \), and let \( \mu \) be a signed measure on \( A \). Then there exist subsets \( X^+, X^- \in A \), with the following properties:

(i) \( X^+ \cap X^- = \emptyset \), and \( X^+ \cup X^- = X \);

(ii) the maps \( \mu^\pm : A \to [-\infty, \infty] \), defined by

\[
\mu^\pm(A) = \pm \mu(A \cap X^\pm), \forall A \in A,
\]

are “honest” measures on \( A \);

(iii) one of the measures \( \mu^\pm \) is finite, and one has the equality \( \mu = \mu^+ - \mu^- \).
7. Signed measures and complex measures

PROOF. Without any loss of generality, we can assume that

\[ \mu(A) < \infty, \quad \forall A \in \mathcal{A}. \]

(Otherwise, we replace \( \mu \) with \(-\mu\), and the conclusion does not essentially change.)

Put \( \alpha = \sup \{ \mu(A) : A \in \mathcal{A} \} \). By Theorem 8.1, it follows that \( 0 \leq \alpha < \infty \), and there exists a set \( X^+ \in \mathcal{A} \), such that \( \mu(X^+) = \alpha \). Define \( X^- = X \setminus X^+ \).

Claim: The sets \( X^\pm \) have the following properties:

(a) \( 0 \leq \mu(A) \leq \alpha \), for all \( A \in \mathcal{A} \), with \( A \subset X^+ \);
(b) \( 0 \geq \mu(B) \), for all \( B \in \mathcal{A} \), with \( B \subset X^- \).

To prove (a) start with some arbitrary subset \( A \subset X^+ \). First of all, by the definition of \( \alpha \), it is clear that \( \mu(A) \leq \alpha \). Second, using the equality \( \mu(X^+) = \mu(A) + \mu(X^+ \setminus A) \), it is clear that \( \mu(A), \mu(X^+ \setminus A) > -\infty \), so we have

\[ \alpha \geq \mu(X^+ \setminus A) = \mu(X^+) - \mu(A) = \alpha - \mu(A), \]

which clearly forces \( \mu(A) \geq 0 \). To prove (b), we start with some set \( B \in \mathcal{A} \) with \( B \subset X^- \). Using the fact that \( X^+ \cap B = \emptyset \), we have

\[ \alpha \geq \mu(X^+ \cup B) = \mu(X^+) + \mu(B) = \alpha + \mu(B), \]

which clearly forces \( \mu(B) \leq 0 \).

Having proven the Claim, we define the maps \( \mu^\pm : \mathcal{A} \to [-\infty, \infty] \) as in the statement of the Theorem. By the Claim, we get \( \mu^\pm(A) \geq 0 \), \( \forall A \in \mathcal{A} \). It is also pretty clear that both \( \mu^+ \) and \( \mu^- \) are \( \sigma \)-additive, so they define “honest” measures. Also, by the Claim, we have \( \mu^+(A) \leq \alpha, \forall A \in \mathcal{A} \), so \( \mu^+ \) is a finite measure. Finally, if we start with some arbitrary \( A \in \mathcal{A} \), and we write it as \( A = (A \cap X^+) \cup (A \cap X^-) \), then using the fact that \( (A \cap X^+) \cap (A \cap X^-) = \emptyset \), we get

\[ \mu(A) = \mu(A \cap X^+) + \mu(A \cap X^-) = \mu^+(A) - \mu^-(A). \]

It will be helpful not only here, but also in some future discussions, to isolate a certain feature identified by the above result.

DEFINITION. Given a \( \sigma \)-algebra \( \mathcal{A} \) on a non-empty set \( X \), and two “honest” measures \( \mu \) and \( \nu \) on \( \mathcal{A} \), we say that \( \mu \) and \( \nu \) are mutually singular, if there exists sets \( M, N \in \mathcal{A} \), with \( M \cup N = X \) and \( M \cap N = \emptyset \), such that \( \mu(N) = \nu(M) = 0 \). Notice that this implies the equalities

\[ \mu(A) = \mu(A \cap M) \quad \text{and} \quad \nu(A) = \nu(A \cap N), \quad \forall A \in \mathcal{A}. \]

If this situation occurs, we write \( \mu \perp \nu \).

With this terminology, Theorem 8.2 states that any signed measure \( \mu \) can be written as \( \mu = \mu^+ - \mu^- \), with \( \mu^+ \) and \( \mu^- \) “honest” mutually singular measures, and one of them finite.

Although the sets \( X^\pm \) may not be uniquely determined, the decomposition \( \mu = \mu^+ - \mu^- \) is unique, as indicated by the following result.

THEOREM 7.3 (Minimality). Let \( X \) be a non-empty set, let \( \mathcal{A} \) be a \( \sigma \)-algebra on \( X \), and let \( \mu \) be a signed measure on \( \mathcal{A} \). Suppose \( \mu^+ \) and \( \mu^- \) are mutually singular “honest” measures on \( \mathcal{A} \), one of them being finite, such that \( \mu = \mu^+ - \mu^- \). Suppose \( \nu \) and \( \eta \) are two “honest” measures on \( \mathcal{A} \), one of which being finite, such that \( \mu = \nu - \eta \). Then one has the inequalities \( \mu^+ \leq \nu \) and \( \mu^- \leq \eta \).
Proof. Fix sets $X^+, X^- \in \mathcal{A}$, such that $X^+ \cup X^- = X$, $X^+ \cap X^- = \emptyset$, and $\mu^+(X^-) = \mu^-(X^+) = 0$.

Start with some arbitrary set $A \in \mathcal{A}$. On the one hand, since $A = (A \cap X^+) \cup (A \cap X^-)$, with $A = (A \cap X^+) \cap (A \cap X^-) = \emptyset$, we see that if $\lambda$ is either one of the measures $\mu$, $\mu^+$, or $\mu^-$, we have the equality

\[ \lambda(A) = \lambda(A \cap X^+) + \lambda(A \cap X^-), \quad \forall A \in \mathcal{A}. \]

On the other hand, since $\mu^+$ is an "honest" measure, and $\mu^+(X^-) = 0$, the inclusion $A \cap X^- \subset X^-$ will force

\[ \mu^+(A \cap X^-) = 0, \quad \forall A \in \mathcal{A}. \]

Likewise, we have the equality

\[ \mu^-(A \cap X^+) = 0, \quad \forall A \in \mathcal{A}. \]

These equalities, combined with $\mu = \mu^+ - \mu^-$, and with (7), give the equalities

\[ \mu^+(A) = \mu^+(A \cap X^+) = \mu^+(A \cap X^+) - \mu^-(A \cap X^+) = \mu(A \cap X^+), \]

\[ \mu^-(A) = \mu^-(A \cap X^-) = -\mu^+(A \cap X^-) + \mu^-(A \cap X^-) = -\mu(A \cap X^-), \]

for all $A \in \mathcal{A}$. Fix now some set $A \in \mathcal{A}$. Since $\nu$ is an "honest" measure, and $\eta(A \cap X^+) \geq 0$, using (8) we get

\[ \nu(A) \geq \nu(A \cap X^+) \geq \nu(A \cap X^+) - \eta(A \cap X^-) = \mu(A \cap X^+) = \mu^+(A). \]

Likewise, we have

\[ \eta(A) \geq \eta(A \cap X^-) \geq \eta(A \cap X^-) - \nu(A \cap X^-) = -\mu(A \cap X^-) = \mu^-(A). \]

\[ \square \]

Corollary 7.1. Let $\mathcal{A}$ be a $\sigma$-algebra on $X$, let $\mu$ be a signed measure on $\mathcal{A}$, and let $\mu^+$, $\mu^-$, $\nu^+$ and $\nu^-$ be "honest" measures on $\mathcal{A}$ with

- $\mu^+ \perp \mu^-$, and one of the measures $\mu^+$ and $\mu^-$ is finite;
- $\nu^+ \perp \nu^-$, and one of the measures $\nu^+$ and $\nu^-$ is finite;
- $\mu = \mu^+ - \mu^- = \nu^+ - \nu^-$.  

Then one has the equalities $\mu^+ = \nu^+$ and $\mu^- = \nu^-$. \[ \square \]

Proof. Apply Theorem 8.3 "both ways" to get $\mu^+ \leq \nu^+$ and $\mu^- \leq \nu^-$, as well as $\nu^+ \leq \mu^+$ and $\nu^- \leq \mu^-$. \[ \square \]

Definition. Given a signed measure $\mu$, the decomposition $\mu = \mu^+ - \mu^-$, whose existence is shown in Theorem 8.2, and whose uniqueness is shown above, is called the Hahn-Jordan decomposition of $\mu$. A pair of sets $(X^+, X^-)$, with $X^+ \in \mathcal{A}$, $X^+ \cup X^- = X$, $X^+ \cap X^- = \emptyset$, and $\mu^+(X^-) = \mu^-(X^+) = 0$, is called a Hahn-Jordan set decomposition of $X$ relative to $\mu$.

Exercise 1. Let $\mu$ be a signed measure, and let $\mu = \mu^+ - \mu^-$ be the Hahn-Jordan decomposition. Prove that the following are equivalent

- $\mu$ is finite, i.e. $-\infty < \mu(A) < \infty$, $\forall A \in \mathcal{A}$;
- both "honest" measures $\mu^+$ and $\mu^-$ are finite.

The following result characterizes mutual singularity in an approximate fashion.

Lemma 7.1. Let $\mathcal{A}$ be a $\sigma$-algebra on $X$, and let $\mu$ and $\nu$ be "honest" measures on $\mathcal{A}$. The following are equivalent

- $\mu \perp \nu$;
(ii) for every \( \varepsilon > 0 \), there exist sets \( D, E \in \mathcal{A} \), such that \( \mu(D) < \varepsilon \), \( \nu(E) < \varepsilon \), and \( D \cup E = X \).

PROOF. The implication \((i) \Rightarrow (ii)\) is trivial.

To prove the implication \((ii) \Rightarrow (i)\) construct, for each \( \varepsilon > 0 \), two sequences \( (D^\varepsilon_n)_{n=1}^\infty \) and \( (E^\varepsilon_n)_{n=1}^\infty \) of sets in \( \mathcal{A} \), such that \( \mu(D^\varepsilon_n) < \varepsilon / 2^n \), \( \nu(E^\varepsilon_n) < \varepsilon / 2^n \), and \( D^\varepsilon_n \cup E^\varepsilon_n = X \). Put \( A_\varepsilon = \bigcap_{n=1}^\infty D^\varepsilon_n \) and \( B_\varepsilon = \bigcup_{n=1}^\infty E^\varepsilon_n \). Fix for the moment \( \varepsilon > 0 \). On the one hand, using the inclusion \( A_\varepsilon \subset D^\varepsilon_n \), \( \forall n \geq 1 \), we get \( \mu(A_\varepsilon) \leq \varepsilon / 2^n \), \( \forall n \geq 1 \), which clearly forces

\[
\mu(A_\varepsilon) = 0. \tag{10}
\]

On the other hand, using \( \sigma\)-subadditivity, we have

\[
\nu(B_\varepsilon) = \nu\left( \bigcup_{n=1}^\infty E^\varepsilon_n \right) \leq \sum_{n=1}^\infty \nu(E^\varepsilon_n) < \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon. \tag{11}
\]

Finally, since we have, \( X \setminus D^\varepsilon_n \subset E^\varepsilon_n \), \( \forall n \geq 1 \), we get

\[
X \setminus A_\varepsilon = \bigcup_{n=1}^\infty (X \setminus D^\varepsilon_n) \subset \bigcup_{n=1}^\infty E^\varepsilon_n = B_\varepsilon,
\]

which gives

\[
A_\varepsilon \supset X \setminus B_\varepsilon. \tag{12}
\]

Define now the sets \( N = \bigcup_{n=1}^\infty A_{1/n} \) and \( M = X \setminus N \). On the one hand, using \( \sigma\)-subadditivity, combined with (10), we get \( \mu(N) = 0 \). On the other hand, using (12), we have

\[
M = X \setminus N = X \setminus \left( \bigcup_{n=1}^\infty A_{1/n} \right) = \bigcap_{n=1}^\infty (X \setminus A_{1/n}) \subset \bigcap_{n=1}^\infty B_{1/n} \subset B_{1/k}, \quad \forall k \geq 1,
\]

which forces \( \nu(M) = 0 \). \( \square \)

Although the next technical result seems a bit out of context at this point, we prove it here, and record it for future use.

**Lemma 7.2.** Let \( \mathcal{A} \) be a \( \sigma\)-algebra on some non-empty set \( X \), and let \( \mu, \eta \) be signed measures on \( \mathcal{A} \). Assume there is an “honest” finite measure \( \nu \) on \( \mathcal{A} \), with \( \mu + \nu = \eta \).

(i) If \( \mu = \mu^+ - \mu^- \) and \( \eta = \eta^+ - \eta^- \) are the Hahn-Jordan decompositions of \( \mu \) and \( \eta \) respectively, then one has the inequalities

\[
\mu^+ \leq \eta^+ \leq \mu^+ + \nu \tag{13}
\]

\[
\eta^- \leq \mu^- \leq \eta^- + \nu. \tag{14}
\]

(ii) If \( (X^+, X^-) \) is a Hahn-Jordan set decomposition of \( X \) relative to \( \mu \), and if \( (Y^+, Y^-) \) is a Hahn-Jordan set decomposition of \( X \) relative to \( \eta \), then one has the relations \( X^+ \subset Y^+ \) and \( Y^- \subset X^- \).

PROOF. On the one hand, the signed measure \( \eta \) has a decomposition

\[
\eta = \mu + \nu = (\mu^+ + \nu) - \mu^-,
\]
with \( \mu^+ + \nu \) and \( \mu^- \) “honest” measures (one of them finite). Using the minimality Theorem 8.3, we get the inequalities
\[
\eta^+ \leq \mu^+ + \nu \quad \text{and} \quad \eta^- \leq \mu^-.
\]
On the other hand, we can also consider the signed measure \( \mu = \eta - \nu \), which has a decomposition
\[
\mu = \eta^+ - (\eta^- + \nu),
\]
with \( \eta^+ \) and \( \eta^- + \nu \) “honest” measures (one of them finite). Using again the minimality Theorem 8.3, we get the inequalities
\[
\mu^+ \leq \eta^+ \quad \text{and} \quad \mu^- \leq \eta^- + \nu.
\]
Clearly the inequalities (15) and (16) cover the desired inequalities (13) and (14) (ii).

Recall (see Section 4) that the relation \( A \subseteq B \) means that \( \nu(A \setminus B) = 0 \). In our case, we have to look at the set
\[
N = X^+ \setminus Y^+ = Y^- \setminus X^-,
\]
for which we have to show that \( \nu(N) = 0 \). On the one hand, since \( N \subseteq Y^- \), we get \( \eta^+(N) = 0 \). Using (13) this forces \( \mu^+(N) = 0 \). On the other hand, since \( N \subseteq X^+ \), we get \( \mu^-(N) = 0 \), and using (14) we also get \( \eta^-(N) = 0 \). In other words, we get the equalities
\[
\mu(N) = \mu^+(N) - \mu^-(N) = 0,
\eta(N) = \eta^+(N) - \eta^-(N) = 0,
\]
and then the equality \( \eta = \mu + \nu \) clearly forces \( \nu(N) = 0 \). \( \square \)

The Hahn-Jordan decomposition has the following interesting application to the properties of the natural order relation on “honest” measures. The result below gives the existence of a “infimum” and a “supremum” for a pair of finite “honest” measures.

**Proposition 7.1 (Lattice Property).** Let \( A \) be a \( \sigma \)-algebra on a non-empty set \( X \), and let \( \mu \) and \( \nu \) be “honest” measures on \( A \), with one of them finite.

(i) There exists a unique measure \( \mu \lor \nu \) with:
(a) \( \mu \lor \nu \geq \mu \) and \( \mu \lor \nu \geq \nu \);
(b) whenever \( \omega \) is an “honest” measure on \( A \), with \( \mu \leq \omega \) and \( \nu \leq \omega \), it follows that one has the inequality \( \mu \lor \nu \leq \omega \).

(ii) There exists a unique measure \( \mu \land \nu \) with:
(a) \( \mu \geq \mu \land \nu \) and \( \nu \geq \mu \land \nu \);
(b) whenever \( \lambda \) is an “honest” measure on \( A \), with \( \mu \geq \lambda \) and \( \nu \geq \lambda \), it follows that one has the inequality \( \mu \land \nu \geq \lambda \).

**Proof.** Since the statement of the Theorem is “symmetric,” without any loss of generality we can assume that \( \mu \) is finite.

Consider the signed measure \( \eta = \mu - \nu \), and its Hahn-Jordan decomposition \( \eta = \eta^+ - \eta^- \). Let \( (X^+, X^-) \) be a Hahn-Jordan set decomposition of \( X \) relative to \( \eta \). This means that, for every \( A \in A \), one has
\[
0 \leq \eta^+(A) = \eta(A \cap X^+) = \mu(A \cap X^+) - \nu(A \cap X^+);
\]
\[
0 \leq \eta^-(A) = -\eta(A \cap X^-) = \nu(A \cap X^-) - \mu(A \cap X^-).
\]
In particular we get

\[(19) \quad \mu(A \cap X^+) \geq \nu(A \cap X^+) \text{ and } \mu(A \cap X^-) \leq \nu(A \cap X^-), \quad \forall A \in \mathcal{A}.\]

(i). Define the measure \( \mu \lor \nu = \mu + \eta^- \). Using (18) we have

\[(20) \quad (\mu \lor \nu)(A) = \mu(A \cap X^+) + \nu(A \cap X^-), \quad \forall A \in \mathcal{A}.\]

Notice that, using (19), it follows that, for every \( A \in \mathcal{A} \), one has the inequalities

\[\begin{align*}
(\mu \lor \nu)(A \cap X^+) &= \mu(A \cap X^+) \geq \nu(A \cap X^+), \\
(\mu \lor \nu)(A \cap X^-) &= \nu(A \cap X^-) \geq \mu(A \cap X^-),
\end{align*}\]

In particular, this gives

\[(\mu \lor \nu)(A) = (\mu \lor \nu)(A \cap X^+) + (\mu \lor \nu)(A \cap X^-) \geq \mu(A \cap X^+) + \mu(A \cap X^-) = \mu(A),\]

\[(\mu \lor \nu)(A) = (\mu \lor \nu)(A \cap X^+) + (\mu \lor \nu)(A \cap X^-) \geq \nu(A \cap X^+) + \nu(A \cap X^-) = \nu(A),\]

for every \( A \in \mathcal{A} \), so \( \mu \lor \nu \) indeed has property (a).

To prove property (b), start with some “honest” measure \( \omega \) on \( \mathcal{A} \), with \( \mu, \nu \leq \omega \), and let us show that \( \mu \lor \nu \leq \omega \). This is quite clear, since for any \( A \in \mathcal{A} \), using (20) we have

\[\omega(A) = \omega(A \cap X^+) + \omega(A \cap X^-) \geq (\mu \lor \nu)(A).\]

The uniqueness of \( \mu \lor \nu \) is now clear from (a) and (b).

(ii). Remark that, using the Minimality Theorem 8.3, for the measure \( \eta = \mu - \nu \), it follows that \( \eta^+ \leq \mu \). In particular, \( \eta^+ \) is a finite “honest” measure, and so is the difference \( \mu - \eta^+ \). Put \( \mu \land \nu = \mu - \eta^+ \). Using (17) we have

\[(21) \quad (\mu \land \nu)(A) = \mu(A \cap X^-) + \nu(A \cap X^+), \quad \forall A \in \mathcal{A}.\]

Notice that, using (19), it follows that, for every \( A \in \mathcal{A} \), one has the inequalities

\[\begin{align*}
(\mu \land \nu)(A \cap X^+) &= \nu(A \cap X^+) \leq \mu(A \cap X^+), \\
(\mu \land \nu)(A \cap X^-) &= \mu(A \cap X^-) \geq \nu(A \cap X^-),
\end{align*}\]

In particular, this gives

\[(\mu \land \nu)(A) = (\mu \land \nu)(A \cap X^+) + (\mu \land \nu)(A \cap X^-) \leq \mu(A \cap X^+) + \mu(A \cap X^-) = \mu(A),\]

\[(\mu \land \nu)(A) = (\mu \land \nu)(A \cap X^+) + (\mu \land \nu)(A \cap X^-) \leq \nu(A \cap X^+) + \nu(A \cap X^-) = \nu(A),\]

for every \( A \in \mathcal{A} \), so \( \mu \land \nu \) indeed has property (a).

To prove property (b), start with some “honest” measure \( \lambda \) on \( \mathcal{A} \), with \( \mu, \nu \leq \omega \), and let us show that \( \mu \land \nu \geq \lambda \). This is quite clear, since for any \( A \in \mathcal{A} \), using (21) we have

\[\lambda(A) = \lambda(A \cap X^+) + \omega(A \cap X^-) \leq \mu(A \cap X^+) + \nu(A \cap X^-) = (\mu \land \nu)(A).\]

The uniqueness of \( \mu \land \nu \) is now clear from (a) and (b).

The notion of a finite signed measure can be generalized to the complex case.

**Definition.** Suppose \( \mathcal{A} \) is a \( \sigma \)-algebra on a non-empty set \( X \). A function \( \mu : \mathcal{A} \to \mathbb{C} \) is called a **complex measure on** \( \mathcal{A} \), if it is \( \sigma \)-additive in the sense that

\[(\text{ADD}_\sigma) \quad \text{for any pairwise disjoint sequence } (A_n)_{n=1}^\infty \subset \mathcal{A}, \text{ one has the equality}\]

\[(22) \quad \mu\left( \bigcup_{n=1}^\infty A_n \right) = \sum_{n=1}^\infty \mu(A_n).\]
Remark that the condition $\mu(\emptyset) = 0$ is automatic in this case. Note also that a map $\mu : A \to \mathbb{C}$ is a complex measure, if and only if the maps $\text{Re}\mu$ and $\text{Im}\mu$ are finite signed measures.

The following result describes an important construction.

**Theorem 7.4.** Let $A$ be a $\sigma$-algebra, and let $\mu$ be either a signed measure, or a complex measure on $A$. For every $A \in A$, we define

$$
\nu(A) = \sup \left\{ \sum_{k=1}^{\infty} |\mu(A_k)| : \{A_k\}_{k=1}^{\infty} \subset A, \text{pairwise disjoint}, \bigcup_{k=1}^{\infty} A_k = A \right\}.
$$

The map $\nu : A \to [0, \infty]$ is an “honest” measure on $A$.

**Proof.** The first step in the proof is contained in the following.

**Claim 1:** For any pairwise disjoint sequence $(A_n)_{n=1}^{\infty} \subset A$, one has the inequality

$$
\nu\left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \nu(A_n).
$$

Denote the right hand side of (24) by $S$, and denote the union $\bigcup_{n=1}^{\infty} A_n$ simply by $A$. Start now with some pairwise disjoint sequence $(D_k)_{k=1}^{\infty} \subset A$, with $\bigcup_{k=1}^{\infty} D_k = A$. For every $k \geq 1$, we have $D_k = \bigcup_{n=1}^{\infty} (D_k \cap A_n)$, with $(D_k \cap A_n)_{n=1}^{\infty} \subset A$ pairwise disjoint, so we have

$$
|\mu(D_k)| = \left| \sum_{n=1}^{\infty} \mu(D_k \cap A_n) \right| \leq \sum_{n=1}^{\infty} |\mu(D_k \cap A_n)|, \ \forall k \geq 1.
$$

Summing up then yields

$$
\sum_{k=1}^{\infty} |\mu(D_k)| \leq \sum_{k=1}^{\infty} \left[ \sum_{n=1}^{\infty} |\mu(D_k \cap A_n)| \right] = \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{\infty} |\mu(D_k \cap A_n)| \right].
$$

Since for each $n \geq 1$, the sequence $(D_k \cap A_n)_{k=1}^{\infty} \subset A$ is pairwise disjoint, and satisfies $\bigcup_{k=1}^{\infty} (D_k \cap A_n) = A_n$, by the definition of $\nu$, we get

$$
\sum_{k=1}^{\infty} |\mu(D_k \cap A_n)| \leq \nu(A_n), \ \forall n \geq 1.
$$

Using these estimates in (25), we then get

$$
\sum_{k=1}^{\infty} |\mu(D_k)| \leq \sum_{n=1}^{\infty} \nu(A_n).
$$

Since the inequality $\sum_{k=1}^{\infty} |\mu(D_k)| \leq S$ holds for all pairwise disjoint sequences $(D_k)_{k=1}^{\infty} \subset A$, with $\bigcup_{k=1}^{\infty} D_k = A$, by the definition of $\nu$ we get $\nu(A) \leq S$, and the Claim is proven.

**Claim 2:** For any finite pairwise disjoint collection $(A_n)_{n=1}^{N} \subset A$, one has the inequality

$$
\nu(A_1 \cup \cdots \cup A_N) \geq \nu(A_1) + \cdots + \nu(A_N).
$$
We use induction on $N$, and we see immediately that it suffices only to prove the case $N = 2$. Fix for the moment a pairwise disjoint sequence $(D_k)_{k=1}^\infty \subset \mathcal{A}$, with $\bigcup_{k=1}^\infty D_k = A_1$, and denote the sum $\sum_{k=1}^\infty |\mu(D_k)|$ by $R$. Suppose we have a pairwise disjoint sequence $(E_j)_{j=1}^\infty \subset \mathcal{A}$, with $\bigcup_{j=1}^\infty E_j = A_2$. If we combine it with the $D_k$’s, i.e. we define

$$F_p = \begin{cases} D_{p/2} & \text{if } p \text{ is even} \\ E_{(p+1)/2} & \text{if } p \text{ is odd} \end{cases}$$

then we get a new pairwise disjoint sequence $(F_p)_{p=1}^\infty \subset \mathcal{A}$, with $\bigcup_{p=1}^\infty F_p = A_1 \cup A_2$.

By the definition of $\nu$ we will then get

$$\nu(A_1 \cup A_2) \geq \sum_{p=1}^\infty |\mu(F_p)| = \sum_{k=1}^\infty |\mu(D_k)| + \sum_{j=1}^\infty |\mu(E_j)| = R + \sum_{j=1}^\infty |\mu(E_j)|.$$

Taking supremum over all pairwise disjoint sequences $(E_j)_{j=1}^\infty \subset \mathcal{A}$, with $\bigcup_{j=1}^\infty E_j = A_2$, the above inequality yields $\mu(A_1 \cup A_2) \geq R + \nu(A_2)$, so now we have

$$\nu(A_1 \cup A_2) \geq \nu(A_2) + \sum_{k=1}^\infty |\mu(D_k)|.$$

Taking supremum over all pairwise disjoint sequences $(D_k)_{k=1}^\infty \subset \mathcal{A}$, with $\bigcup_{k=1}^\infty D_k = A_1$, the above inequality finally gives $\nu(A_1 \cup A_2) \geq \nu(A_2) + \nu(A_1)$, and the Claim is proven.

We are now in position to prove that $\nu$ is a measure on $\mathcal{A}$. The equality $\nu(\emptyset) = 0$ is trivial. To prove $\sigma$-additivity, we start with some pairwise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$, and we must prove the equality

$$\nu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \nu(A_n).$$

On the one hand, using Claim 1, we know that we have the inequality $\nu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \nu(A_n)$. On the other hand, if we denote the union $\bigcup_{n=1}^\infty A_n$ simply by $A$, then using Claim 2, we see that

$$\nu(A) \geq \nu(A \setminus [A_1 \cup \cdots \cup A_N]) + \nu(A_1) + \cdots + \nu(A_N) \geq \nu(A_1) + \cdots + \nu(A_N), \quad \forall N \geq 1,$$

which immediately gives the other inequality $\nu(A) \geq \sum_{n=1}^\infty \nu(A_n)$. □

**Definition.** With the notations above, and under the hypothesis of Theorem 8.6, the “honest” measure $\nu$, defined by (23), is called the *variation measure of $\mu$*, and will be denoted by $|\mu|$. By construction, we have the inequality

$$|\mu(A)| \leq |\mu|(A), \quad \forall A \in \mathcal{A}.$$

**Remark 7.2.** Let $\mu$ be either a signed measure, or a complex measure on the $\sigma$-algebra $\mathcal{A}$. Exactly as with numbers (or functions), the measure $|\mu|$ has a minimality property, which can be stated as follows. *Whenever $\nu$ is an “honest” measure on $\mathcal{A}$ with

$$|\mu(A)| \leq \nu(A), \quad \forall A \in \mathcal{A},$$

it follows that we have

$$|\mu|(A) \leq \nu(A), \quad \forall A \in \mathcal{A}.$$*
This is quite clear, because for any pairwise disjoint sequence \( (A_n)_{n=1}^{\infty} \subset \mathcal{A} \), with \( \bigcup_{n=1}^{\infty} A_n = A \), one has the inequality
\[
\sum_{n=1}^{\infty} |\mu(A_n)| \leq \sum_{n=1}^{\infty} \nu(A_n) = \nu(A),
\]
and then the desired inequality follows by taking the supremum in the left hand side.

In the case of signed measures, the variation measure is also given by the following.

**Proposition 7.2.** Let \( \mu \) be a signed measure on the \( \sigma \)-algebra \( \mathcal{A} \). Then one has the equality
\[
|\mu| = \mu^+ + \mu^-,
\]
where \( \mu = \mu^+ - \mu^- \) is the Hahn-Jordan decomposition of \( \mu \).

**Proof.** Denote the measure \( \mu^+ + \mu^- \) simply by \( \nu \). Remark that we obviously have
\[
-\nu(A) = -\mu^+(A) - \mu^-(A) \leq \mu^+(A) - \mu^-(A) \leq \mu^+(A) + \mu^-(A) = \nu(A), \quad \forall A \in \mathcal{A},
\]
which gives
\[
|\mu(A)| \leq \nu(A), \quad \forall A \in \mathcal{A}.
\]
By Remark 8.5, this forces the inequality \( |\mu| \leq \nu \).

To prove the other inequality, we start by fixing sets \( X^+, X^- \in \mathcal{A} \) as in Theorem 8.2. We decompose each set \( A \in \mathcal{A} \) as \( A = A^+ \cup A^- \), where \( A^\pm = A \cap X^\pm \), so that we have
\[
\nu(A) = \nu(A^+) + \nu(A^-) = \mu^+(A^+) + \mu^- (A^-) + \mu^+(A^-) + \mu^-(A^+) = \mu^+(A^+) + \mu^- (A^-).
\]
Notice now that \( \mu(A^+) = \mu^+(A^+) \geq 0 \), and \( -\mu(A^-) = \mu^-(A^-) \geq 0 \), which means that we have the equalities \( \mu^+(A^+) = |\mu(A^+)| \) and \( \mu^-(A^-) = |\mu(A^-)| \), so the above equality reads
\[
\nu(A) = |\mu(A^+)| + |\mu(A^-)|,
\]
and by the definition of \( |\mu| \) we then immediately get \( \nu(A) \leq |\mu|(A) \). \hfill \( \Box \)

An interesting consequence is the following.

**Corollary 7.2.** Let \( \mu \) be either a finite signed measure, or a complex measure on the \( \sigma \)-algebra \( \mathcal{A} \). Then the variation measure \( |\mu| \) is finite.

**Proof.** The signed measure case is clear from the above result.

In the complex case, we write \( \mu = \nu + i\eta \), with \( \nu \) and \( \eta \) finite signed measures on \( \mathcal{A} \). We apply the signed case, to get the fact that both \( |\nu| \) and \( |\eta| \) are finite. Notice that we have
\[
|\mu(A)| = |\nu(A) + i\eta(A)| \leq |\nu(A)| + |\eta(A)| \leq |\nu|(A) + |\eta|(A), \quad \forall A \in \mathcal{A},
\]
so by Remark 8.5 we get \( |\mu| \leq |\nu| + |\eta| \), and then the finiteness of \( |\mu| \) is a consequence of the finiteness of \( |\nu| \) and \( |\eta| \). \hfill \( \Box \)

**Exercise 2.** Let \( \mathcal{A} \) be a \( \sigma \)-algebra, and let \( \mathbb{K} \) be one of the fields \( \mathbb{R} \) or \( \mathbb{C} \). For the purpose of this exercise, let us agree to use the term \( \mathbb{K} \)-measure for designating either a finite signed measure (when \( \mathbb{K} = \mathbb{R} \)), or a complex measure (when \( \mathbb{K} = \mathbb{C} \)). Prove the following.
(i) The collection of all $K$-measures on $\mathcal{A}$ is a vector space.

(ii) For any two $K$-measures $\mu$ and $\nu$, one has the inequality

$$|\mu + \nu| \leq |\mu| + |\nu|.$$ 

(iii) For any $K$-measure $\mu$ and any $\alpha \in K$, one has the equality

$$|\alpha \mu| = |\alpha| \cdot |\mu|.$$