3. Measurable spaces and measurable maps

In this section we discuss a certain type of maps related to \( \sigma \)-algebras.

**Definitions.** A **measurable space** is a pair \((X, A)\) consisting of a (non-empty) set \(X\) and a \(\sigma\)-algebra \(A\) on \(X\).

Given two measurable spaces \((X, A)\) and \((Y, B)\), a **measurable map** \(T : (X, A) \to (Y, B)\) is simply a map \(T : X \to Y\), with the property

\[
T^{-1}(B) \in A, \quad \forall B \in B.
\]

**Remark 3.1.** In terms of the constructions outlined in Section 2, measurability for maps can be characterized as follows. Given measurable spaces \((X, A)\) and \((Y, B)\), and a map \(T : (X, A) \to (Y, B)\), the following are equivalent:

(i) \(T : (X, A) \to (Y, B)\) is measurable;
(ii) \(T^*B \subset A\);
(iii) \(T_A \supset B\).

Recall \(T^*B = \{T^{-1}(B) : B \in B\}\);
\(T_A = \{B \subset Y : T^{-1}(B) \in A\}\).

With these equalities, everything is immediate.

The following summarizes some useful properties of measurable maps.

**Proposition 3.1.** Let \((X, A)\) be a measurable space.

(i) If \(A'\) is any \(\sigma\)-algebra, with \(A' \subset A\), then the identity map \(\text{Id}_X : (X, A) \to (X, A')\) is measurable.
(ii) For any subset \(M \subset X\), the inclusion map \(\iota : (M, A|_M) \hookrightarrow (X, A)\) is measurable.
(iii) If \((Y, B)\) and \((Z, \mathcal{E})\) are measurable spaces, and if \((X, A) \xrightarrow{T} (Y, B) \xrightarrow{S} (Z, \mathcal{E})\) are measurable maps, then the composition \(S \circ T : (X, A) \to (Z, \mathcal{E})\) is again a measurable map.

**Proof.** (i). This is trivial, since \((\text{Id}_X)^*A' = A' \subset A\).
(ii). This is again trivial, since \(\iota^*A = A|_M\).
(iii). Start with some set \(C \in \mathcal{E}\), and let us prove that \((S \circ T)^{-1}(C) \in A\). We know that \((S \circ T)^{-1} = T^{-1}(S^{-1}(C))\). Since \(S\) is measurable, we have \(S^{-1}(C) \in B\), and since \(T\) is measurable, we have \(T^{-1}(S^{-1}(C)) \in A\). \(\square\)

Often, one would like to check the measurability condition (1) on a small collection of \(B\)'s. Such a criterion is the following.

**Lemma 3.1.** Let \((X, A)\) and \((Y, B)\) be measurable spaces. Assume \(B = \Sigma(\mathcal{E})\), for some collection of sets \(\mathcal{E} \subset \mathcal{P}(Y)\). For a map \(T : X \to Y\), the following are equivalent:

(i) \(T : (X, A) \to (Y, B)\) is measurable;
(ii) \(T^{-1}(E) \in A, \forall E \in \mathcal{E}\).
PROOF. The implication (i) \(\Rightarrow\) (ii) is trivial.

To prove the implication (ii) \(\Rightarrow\) (i), assume (ii) holds. We first observe that condition (ii) reads \(f^* \mathcal{E} \subseteq \mathcal{A}\). Since \(\mathcal{A}\) is a \(\sigma\)-algebra, we get the inclusion
\[
\Sigma(f^* \mathcal{E}) \subseteq \mathcal{A}.
\]
Using the Generating Theorem 2.2, we have
\[
f^* \mathcal{B} = f^* \Sigma(\mathcal{E}) = \Sigma(f^* \mathcal{E}) \subseteq \mathcal{A},
\]
and, by the preceding remark, we are done. \(\square\)

COROLLARY 3.1. Let \((X, \mathcal{A})\) be a measurable space, let \(Y\) be a topological Hausdorff space which is second countable, and let \(\mathcal{S}\) be a sub-base for the topology of \(Y\). For a map \(T : X \to Y\), the following are equivalent:

(i) \(T : (X, \mathcal{A}) \to (Y, \mathcal{S}(Y))\) is a measurable map;

(ii) \(T^{-1}(\mathcal{S}) \in \mathcal{A}, \forall \mathcal{S} \in \mathcal{S}\).

PROOF. Immediate from the above Lemma, and Proposition 2.2, which states that \(\mathcal{S}(Y) = \Sigma(\mathcal{S})\). \(\square\)

We know (see Section 19) that the type \(\Sigma\) is consistent and natural. In particular, measurability behaves nicely with respect to products and disjoint unions. More explicitly one has the following.

PROPOSITION 3.2. Let \((X_i, \mathcal{A}_i)_{i \in I}\) be a collection of measurable spaces. Consider the sets \(X = \prod_{i \in I} X_i\) and \(Y = \bigsqcup_{i \in I} X_i\), and the \(\sigma\)-algebras
\[
\mathcal{A} = \Sigma(X_i \mathcal{A}_i) \text{ and } \mathcal{B} = \bigvee_{i \in I} \mathcal{A}_i.
\]
Let \((Z, \mathcal{G})\) be a measurable space.

(i) If we denote by \(\pi_i : X \to X_i\), \(i \in I\), the projection maps, then a map \(f : (Z, \mathcal{G}) \to (X, \mathcal{A})\) is measurable, if and only if, all the maps \(\pi_i \circ f : (Z, \mathcal{G}) \to (X_i, \mathcal{A}_i), i \in I,\) are measurable.

(ii) If we denote by \(\epsilon_i : X_i \to Y\), \(i \in I\), the inclusion maps, then a map \(g : (Y, \mathcal{B}) \to (Z, \mathcal{G})\) is measurable, if and only if, all the maps \(g \circ \epsilon_i \circ f : (X_i, \mathcal{A}_i) \to (Z, \mathcal{G}), i \in I,\) are measurable.

PROOF. (i). By the definition of the product \(\sigma\)-algebra, we know that
\[
\mathcal{A} = \Sigma(\bigcup_{i \in I} \pi_i^* \mathcal{A}_i).
\]
If we fix some index \(i \in I\), then the obvious inclusion \(\pi_i^* \mathcal{A}_i \subseteq \mathcal{A}\) immediately shows that \(\pi_i : (X, \mathcal{A}) \to (X_i, \mathcal{A}_i)\) is measurable. Therefore, if \(f : (Z, \mathcal{G}) \to (X, \mathcal{A})\) is measurable, then by Proposition 3.1 it follows that all compositions \(\pi_i \circ f : (Z, \mathcal{G}) \to (X_i, \mathcal{A}_i), i \in I,\) are measurable.

Conversely, assume all the compositions \(\pi_i \circ f\) are measurable, and let us show that \(f : (Z, \mathcal{G}) \to (X, \mathcal{A})\) is measurable. By Lemma 3.1 and (2), all we need to prove is the fact that
\[
f^* \bigcup_{i \in I} \pi_i^* \mathcal{A}_i \subseteq \mathcal{G},
\]
which is equivalent to
\[
f^* (\pi_i^* \mathcal{A}_i) \subseteq \mathcal{G}, \forall i \in I.
\]
But this is obvious, because \( f^* (\pi_i^* A_i) = (\pi_i \circ f)^* A_i \), and \( \pi_i \circ f \) is measurable, for all \( i \in I \).

(ii). By the definition of the \( \sigma \)-algebra sum, we know that

\[
\mathcal{B} = \bigcap_{i \in I} \epsilon_i^* A_i.
\]

If we fix some index \( i \in I \), then the obvious inclusion \( \epsilon_i^* A_i \supseteq \mathcal{B} \) immediately shows that \( \epsilon_i : (X_i, A_i) \to (Y, \mathcal{B}) \) is measurable. Therefore, if \( g : (Y, \mathcal{B}) \to (Z, \mathcal{G}) \) is measurable, then by Proposition 3.1 it follows that all compositions \( g \circ \epsilon_i : (X_i, A_i) \to (Z, \mathcal{G}), i \in I \), are measurable.

Conversely, assume all the compositions \( g \circ \epsilon_i \) are measurable, and let us show that \( g : (Y, \mathcal{B}) \to (Z, \mathcal{G}) \) is measurable. This is equivalent to the inclusion \( g_* \mathcal{B} \supseteq \mathcal{G} \). By (3) we immediately have

\[
g_* \mathcal{B} = g_* \left( \bigcap_{i \in I} \epsilon_i^* A_i \right) = \bigcap_{i \in I} g_* (\epsilon_i^* A_i).
\]

We know however that, since \( g \circ \epsilon_i \) are all measurable, we have

\[
g_* (\epsilon_i^* A_i) = (g \circ \epsilon_i)_* A_i \supseteq \mathcal{G}, \ \forall \ i \in I,
\]

so the desired inclusion is an immediate consequence of (4). \( \square \)

**Conventions.** Let \((X, A)\) be a measurable space. An extended real-valued function \( f : (X, A) \to [-\infty, \infty] \) is said to be a **measurable function**, if it is measurable in the above sense as a map \( f : (X, A) \to ([-\infty, \infty], Bor([\mathbb{R}])) \). If \( f \) has values in \( \mathbb{R} \), this is equivalent to the fact that \( f \) is a measurable map \( f : (X, A) \to (\mathbb{R}, Bor(\mathbb{R})) \) is measurable. Likewise, a complex valued function \( f : (X, A) \to \mathbb{C} \) is measurable, if it is measurable as a map \( f : (X, A) \to (\mathbb{C}, Bor(\mathbb{C})) \). If \( \mathbb{K} \) is one of the fields \( \mathbb{R} \) or \( \mathbb{C} \), we define the set

\[
\mathcal{B}_\mathbb{K}(X, A) = \{ f : (X, A) \to \mathbb{K} : f \text{ measurable function } \}.
\]

**Remark 3.2.** Let \((X, A)\) be a measurable space. If \( A \subseteq \mathbb{R} \) is a dense subset, then the results from Section 2, combined with Lemma 2.1, show that the measurability of a function \( f : (X, A) \to [-\infty, \infty] \) is equivalent to any of the following conditions:

- \( f^{-1}((a, \infty]) \in A, \ \forall a \in A; \)
- \( f^{-1}([a, \infty]) \in A, \ \forall a \in A; \)
- \( f^{-1}([\infty, a)) \in A, \ \forall a \in A; \)
- \( f^{-1}((\infty, a]) \in A, \ \forall a \in A. \)

**Definition.** If \( X \) and \( Y \) are topological Hausdorff spaces, a map \( T : X \to Y \) is said to be **Borel measurable**, if \( T \) is measurable as a map

\[
T : (X, Bor(X)) \to (Y, Bor(Y)).
\]

In the cases when \( Y = \mathbb{R}, \mathbb{C}, [-\infty, \infty] \), a Borel measurable map will be simply called a **Borel measurable function**.

For \( \mathbb{K} = \mathbb{R}, \mathbb{C} \), we define

\[
\mathcal{B}_\mathbb{K}(X) = \{ f : X \to \mathbb{K} : f \text{ Borel measurable function } \}.
\]
Remark 3.3. If $X$ and $Y$ are topological Hausdorff spaces, then any continuous map $T : X \to Y$ is Borel measurable. This follows from Lemma 3.1, from the fact that

$$\text{Bor}(Y) = \Sigma \{ D \subset Y : D \text{ open} \},$$

and the fact that $T^{-1}(D)$ is open, hence in $\text{Bor}(X)$, for every open set $D \subset Y$.

Measurable maps behave nicely with respect to “measurable countable operations,” as suggested by the following result.

**Proposition 3.3.** Let $(X, \mathcal{A})$ and $(Z, \mathcal{B})$ be a measurable spaces, let $I$ be a set which is at most countable, and let $(Y_i)_{i \in I}$ be a family of topological Hausdorff spaces, each of which is second countable. Suppose a measurable map $T_i : (X, \mathcal{A}) \to (Y_i, \text{Bor}(Y_i))$ is given, for each $i \in I$. Define the map $T : X \to \prod_{i \in I} Y_i$ by

$$T(x) = (T_i(x))_{i \in I}, \ \forall x \in X.$$

Equip the product space $Y = \prod_{i \in I} Y_i$ with the product topology.

For any measurable map $g : (Y, \text{Bor}(Y)) \to (Z, \mathcal{B})$, the composition $g \circ T : (X, \mathcal{A}) \to (Z, \mathcal{B})$ is measurable.

**Proof.** We know (see Corollary 2.3) that we have the equality

$$\text{Bor}(Y) = \Sigma \bigwedge_{i \in I} \text{Bor}(Y_i).$$

By Proposition 3.2, the map $T : (X, \mathcal{A}) \to (Y, \text{Bor}(Y))$ is measurable, so by Proposition 3.1, the composition $g \circ T : (X, \mathcal{A}) \to (Z, \mathcal{B})$ is also measurable. \qed

The above result has many useful applications.

**Corollary 3.2.** Suppose $(X, \mathcal{A})$ is a measurable space, and $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then, when equipped with point-wise addition and multiplication, the set $\mathcal{B}_\mathbb{K}(X, \mathcal{A})$ is a unital $\mathbb{K}$-algebra.

**Proof.** Clearly the constant function $1$ is measurable.

Also, if $f \in \mathcal{B}_\mathbb{K}(X, \mathcal{A})$ and $\lambda \in \mathbb{K}$, then the function $\lambda f$ is again measurable, since it can be written as the composition $M_\lambda \circ f$, where $M_\lambda : \mathbb{K} \ni \alpha \mapsto \lambda \alpha \in \mathbb{K}$ is obviously continuous.

Finally, let us show that if $f_1, f_2 \in \mathcal{B}_\mathbb{K}(X, \mathcal{A})$, then $f_1 + f_2$ and $f_1 \cdot f_2$ again belong to $\mathcal{B}_\mathbb{K}(X, \mathcal{A})$. This is however immediate from Proposition 3.3, applied to the index set $I = \{1, 2\}$, the spaces $Y_1 = Y_2 = \mathbb{K}$, and the continuous maps

$$g_1 : \mathbb{K}^2 \ni (\lambda_1, \lambda_2) \mapsto \lambda_1 + \lambda_2 \in \mathbb{K},$$

$$g_2 : \mathbb{K}^2 \ni (\lambda, \lambda_2) \mapsto \lambda_1 \cdot \lambda_2 \in \mathbb{K}. \quad \square$$

**Corollary 3.3.** If $(X, \mathcal{A})$ is a measurable space, then a complex valued function $f : X \to \mathbb{C}$ is measurable, if and only if the real valued functions $\text{Re} f, \text{Im} f : X \to \mathbb{R}$ are measurable.

**Proof.** If $f$ is measurable, the composing $f$ with the continuous maps

$$\rho : \mathbb{C} \ni z \mapsto \text{Re} z \in \mathbb{R} \quad \text{and} \quad \gamma : \mathbb{C} \ni z \mapsto \text{Im} z \in \mathbb{R},$$

immediately gives the measurability of $\text{Re} f = \rho \circ f$ and $\text{Im} f = \gamma \circ f$. \qed
Conversely, if both $\text{Re} \ f, \ \text{Im} \ f : X \to \mathbb{R}$ then the measurability of $f$ follows from Proposition 3.3, applied to $Y_1 = Y_2 = \mathbb{R},$ the functions $f_1 = \text{Re} \ f$ and $f_2 = \text{Im} \ f,$ and to the continuous function

$$g : \mathbb{R}^2 \ni (a, b) \mapsto a + bi \in \mathbb{C}. \square$$

**Corollary 3.4.** Let $(X, A)$ be a measurable space, let $I$ be a set which is at most countable, and let $f_i : (X, A) \to [\infty, \infty], \ i \in I$ be collection of measurable functions. Then the functions $g, h : X \to [-\infty, \infty], \ defined \ by$

$$g(x) = \inf \{f_i(x) : i \in I\} \ and \ h(x) = \sup \{f_i(x) : i \in I\}, \ \forall \ x \in X,$$

are both measurable.

**Proof.** Define the maps $m, M : \prod_{i \in I} [-\infty, \infty] \to [-\infty, \infty]$ by

$$m(x) = \inf \{x_i : i \in I\} \ and \ M(x) = \sup \{x_i : i \in I\}, \ \forall x = (x_i)_{i \in I} \in \prod_{i \in I} [-\infty, \infty].$$

By Proposition 3.3, it suffices to prove the (Borel) measurability of the maps $m$ and $M.$

To prove the measurability of $m,$ we are going to show that

$$m^{-1}((-\infty, a)) \in \text{Bor} \left(\prod_{i \in I} [-\infty, \infty]\right), \ \forall a \in \mathbb{R}.$$ But this is quite obvious, since a point $x = (x_i)_{i \in I}$ belongs to $m^{-1}((-\infty, a)),$ if and only if there exists some $j \in I$ with $x_i < a.$ In other words, if we define the projections $\pi_j : \prod_{i \in I} [-\infty, \infty] \to [-\infty, \infty],$ then we have

$$m^{-1}((-\infty, a)) = \bigcup_{j \in I} \pi_j((-\infty, a)).$$

This shows that in fact $m^{-1}((-\infty, a))$ is open, hence clearly Borel.

To prove the measurability of $M,$ we are going to show that

$$M^{-1}((a, \infty)) \in \text{Bor} \left(\prod_{i \in I} [-\infty, \infty]\right), \ \forall a \in \mathbb{R}.$$ But this is again clear, since, as before, we have the equality

$$M^{-1}((a, \infty)) = \bigcup_{j \in I} \pi_j((a, \infty)),$$

which shows that in fact $M^{-1}((a, \infty))$ is open, hence Borel. \square

**Corollary 3.5.** Let $(X, A)$ be a measurable space, and let $f_n : (X, A) \to [\infty, \infty], \ n \in \mathbb{N}$ be sequence of measurable functions. Then the functions $g, h : X \to [-\infty, \infty], \ defined \ by$

$$g(x) = \lim \inf \ f_n(x) \ and \ h(x) = \lim \sup \ f_n(x), \ \forall x \in X,$$

are both measurable.

**Proof.** For every $n \in \mathbb{N},$ define the functions $g_n, h_n : X \to [-\infty, \infty]$ by

$$g_n(x) = \inf \{f_k(x) : k \geq n\} \ and \ h_n(x) = \sup \{f_k(x) : k \geq n\}, \ \forall x \in X.$$ By Corollary 3.5, we know that $g_n$ and $h_n$ are measurable for all $n \in \mathbb{N}.$ Since

$$g(x) = \sup \{g_n(x) : n \in \mathbb{N}\} \ and \ h(x) = \inf \{h_n(x) : n \in \mathbb{N}\}, \ \forall x \in X,$$
the fact that both $g$ and $h$ are measurable follows again from Corollary 3.5. □

**Corollary 3.6.** Let $(X, \mathcal{A})$ be a measurable space, and let

$$f_n : (X, \mathcal{A}) \to [-\infty, \infty], \quad n \in \mathbb{N}$$

be sequence of measurable functions, with the property that, for each $x \in X$, the sequence $(f_n(x))_{n=1}^{\infty} \subset [-\infty, \infty]$ has a limit. Then the function $f : X \to [-\infty, \infty]$, defined by

$$f(x) = \lim_{n \to \infty} f_n(x), \quad \forall x \in X,$$

is again measurable.

**Proof.** Immediate from the above result. □

**Exercise 1.** If $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, are continuous functions, and if $f(x) = \lim_{n \to \infty} f_n(x)$ exists, for every $x \in \mathbb{R}$, then by the above Corollary we know that $f : \mathbb{R} \to [-\infty, \infty]$ is Borel measurable. Prove that the converse is not true. More explicitly, prove that there is no sequence $(f_n)_{n=1}^{\infty}$ of continuous functions, with

$$\lim_{n \to \infty} f_n(x) = \kappa Q(x), \quad \forall x \in \mathbb{R}.$$

**Hint:** Use Baire’s Theorem.

**Exercise 2.** Prove that a function $f : \mathbb{R} \to \mathbb{R}$, which is continuous everywhere, except for a countable set of points, is Borel measurable. As an application, prove that any monotone function is Borel measurable.

Corollary 3.6 can be generalized, as follows.

**Theorem 3.1.** Let $(X, \mathcal{A})$ be a measurable space, let $Y$ be a separable metric space, and let

$$T_n : (X, \mathcal{A}) \to (Y, \text{Bor}(Y)), \quad n \in \mathbb{N}$$

be a sequence of measurable maps. Assume that, for every $x \in X$, the sequence $(T_n(x))_{n=1}^{\infty} \subset Y$ is convergent. Define the map $T : X \to Y$ by

$$T(x) = \lim_{n \to \infty} T_n(x), \quad \forall x \in X.$$  

Then $T : (X, \mathcal{A}) \to (Y, \text{Bor}(Y))$ is a measurable map.

**Proof.** Denote by $d$ the metric on $Y$. The collection

$$\mathcal{V} = \{B_r(y) : y \in Y, \ r > 0\}$$

is a base for the topology of $Y$. Since $Y$ is second countable, it suffices then to show that

$$T^{-1}(B_r(y)) \in \mathcal{A}, \quad \forall y \in Y, \ r > 0 .$$

**Claim:** For every $y \in Y$ and $r > 0$ one has the equality

$$T^{-1}(B_r(y)) = \bigcup_{m, n=1}^{\infty} T_k^{-1}(B_{r-\frac{1}{n}}(y)).$$

(5)

(6)
Denote the set in the right hand side simply by \( A \). Start first with some \( x \in A \). There exist some \( m, n \in \mathbb{N} \) such that
\[
x \in \bigcap_{k=m}^{\infty} T_k^{-1}(B_{r-\frac{1}{n}}(y)),
\]
which means that
\[
T_k(x) \in B_{r-\frac{1}{n}}(y), \quad \forall k \geq m,
\]
that is,
\[
d(T_k(x), y) < r - \frac{1}{n}, \quad \forall k \geq m.
\]
Pasing to the limit \((k \to \infty)\) then yields
\[
d(T(x), y) \leq r - \frac{1}{n} < r,
\]
which means that \( T(x) \in B_r(y) \), i.e. \( x = T^{-1}(B_r(y)) \), thus proving the inclusion \( A \subset T^{-1}(B_r(y)) \).

Conversely, if \( x \in T^{-1}(B_r(y)) \), we get \( T(x) \in (B_r(y), \text{i.e.} \quad d(T(x), y) < r \).
Choose an integer \( n \) such that
\[
d(T(x), y) < r - \frac{2}{n}.
\]
Since \( \lim_{k \to \infty} T_k(x) = T(x) \), there exists some \( m \in \mathbb{N} \) such that
\[
d(T_k(x), T(x)) < \frac{2}{n}, \quad \forall k \geq m.
\]
Combining this with (7) then gives
\[
d(T_k(x), y) \leq d(T(x), y) + d(T_k(x), T(x)) < r - \frac{2}{n} + \frac{1}{n} = r - \frac{1}{n}, \quad \forall k \geq m,
\]
which means that
\[
x \in \bigcap_{k=m}^{\infty} T_k^{-1}(B_{r-\frac{1}{n}}(y)),
\]
hence \( x \) indeed belongs to \( A \).

Having proven (6) we now observe that, since the \( T_k \)'s are measurable, it follows that
\[
T_k^{-1}(B_{r-\frac{1}{n}}(y)) \in \mathcal{A}, \quad \forall k, n \in \mathbb{N}, r > 0.
\]
Using the fact that \( \mathcal{A} \) is closed under countable intersections, it follows that
\[
\bigcap_{k=m}^{\infty} T_k^{-1}(B_{r-\frac{1}{n}}(y)) \in \mathcal{A}, \quad \forall m, n \in \mathbb{N}, r > 0.
\]
Finally, using the fact that \( \mathcal{A} \) is closed under countable unions, the desired property (5) follows. \( \square \)

**Exercise 3.** Let \((X, \mathcal{A})\) be a measurable space, and let \((X_n)_{n=1}^{\infty}\) be a sequence of sets in \( \mathcal{A} \), with \( X = \bigcup_{n=1}^{\infty} X_n \). Suppose \((Y, \mathcal{B})\) is a measurable space, and \( F : X \to Y \) is a map, such that
\[
F|_{X_n} : (X_n, \mathcal{A}|_{X_n}) \to (Y, \mathcal{B})
\]
is measurable, for all \( n \in \mathbb{N} \). Prove that \( f : (X, \mathcal{A}) \to (Y, \mathcal{B}) \) is measurable.
Exercise 4*. Let $\Omega_1 \subset \mathbb{R}^n$ be an open set, and let $f_1, \ldots, f_n : \Omega_1 \to \mathbb{R}$ be $C^1$ functions, with the property that the matrix

$$A(p) = \left[ \frac{\partial f_j}{\partial x_k}(p) \right]_{j,k=1}^n$$

is invertible, for every point $p \in \Omega_1$. Define the map

$$F : \Omega_1 \ni p \mapsto (f_1(p), \ldots, f_n(p)) \in \mathbb{R}^n.$$  

(i) Prove that the set $\Omega_2 = F(\Omega_1)$ is open in $\mathbb{R}^n$.  

(ii) Although $F : \Omega_1 \to \Omega_2$ may fail to be injective, prove that there exists a Borel measurable map $\phi : \Omega_2 \to \Omega_1$, with $F \circ \phi = \text{Id}_{\Omega_2}$.  

HINT: Use the Inverse Function Theorem, combined with Exercises 2 and 3. exercise.  

Exercise 5*. Let $P(z)$ be a non-constant polynomial with complex coefficients. Prove that there exists a Borel measurable function $f : \mathbb{C} \to \mathbb{C}$, such that

$$P(f(z)) = z, \quad \forall z \in \mathbb{C}.$$  

HINT: Use the preceding exercise, applied to the set $\Omega_1 = \{z \in \mathbb{C} : P'(z) \neq 0\}$.  

The preceding exercise can be generalized:  

Exercise 6*. Let $\Omega_1 \subset \mathbb{C}$ be a connected open set, and let $f : \Omega_1 \to \mathbb{C}$ be a non-constant holomorphic function. By the Open Mapping Theorem we know that the set $\Omega_2 = f(\Omega_1)$ is open. Prove that there exists a Borel measurable function $\phi : \Omega_2 \to \Omega_1$, such that $f \circ \phi = \text{Id}_{\Omega_2}$.  

HINT: Use Exercise 4, applied to the set $\Omega_0 = \{z \in \Omega_1 : f'(z) \neq 0\}$. Since $f$ is non-constant, the set $\Omega_1 \setminus \Omega_0$ is countable.  

We continue with a discussion on the role of elementary functions.  

Proposition 3.4. Let $(X, \mathcal{A})$ be a measurable space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For an elementary function $f \in \text{Elem}_\mathbb{K}(X)$, the following are equivalent:

(i) $f \in \mathcal{A}\text{-}\text{Elem}_\mathbb{K}(X)$;  

(ii) $f : (X, \mathcal{A}) \to \mathbb{K}$ is measurable.  

Proof. (i) $\Rightarrow$ (ii). We know that $\mathcal{A}\text{-}\text{Elem}_\mathbb{K} = \text{Span}_\mathbb{K}\{\mathcal{A}_A : A \in \mathcal{A}\}$. Since $\mathcal{B}_\mathbb{K}(X, \mathcal{A})$ is a vector space, it suffices to show only that $\mathcal{A}_A : (X, \mathcal{A}) \to \mathbb{K}$ is measurable, for all $A \in \mathcal{A}$. But this is trivial, since for every Borel set $B \subset \mathbb{R}$ one has either $\mathcal{A}_A^{-1}(B) = \varnothing$, or $\mathcal{A}_A^{-1}(B) = A$, or $\mathcal{A}_A^{-1}(B) = X$.  

(ii) $\Rightarrow$ (i). Assume now $f$ is measurable. List the range of $f$ as

$$f(X) = \{\lambda_1, \ldots, \lambda_n\},$$

with $\lambda_j \neq \lambda_k$, for all $j, k \in \{1, \ldots, n\}$ with $j \neq k$. Since $f$ is measurable, and the singleton sets $\{\lambda_1\}, \ldots, \{\lambda_n\}$ are in $\text{Bor}(\mathbb{K})$, it follows that the sets $A_j = f^{-1}(\{\lambda_j\})$, $j = 1, \ldots, n$ are all in $\mathcal{A}$. Since we clearly have

$$f = \lambda_1 \mathcal{A}_A + \cdots + \lambda_n \mathcal{A}_A,$$

it follows that $f$ indeed belongs to $\mathcal{A}\text{-}\text{Elem}_\mathbb{K}(X)$. $\square$
Remarks 3.4. A. If \((X,A)\) and \((Y,B)\) are measurable spaces, if \(T : (X,A) \to (Y,B)\) is a measurable map, and if \(f \in B\)-Elem\(_K(Y)\), then \(f \circ T \in A\)-Elem\(_K(X)\). This follows from the fact that the composition \(f \circ T : (X,A) \to \mathbb{K}\) is measurable, and elementary.

B. If \((X,A)\) is a measurable space, if \(f \in A\)-Elem\(_K(X)\), and if \(g : f(X) \to \mathbb{K}\) is an arbitrary function, then \(g \circ f \in A\)-Elem\(_K(X)\). This follows from the fact that, if one considers the finite set \(Y = f(X)\), and the \(\sigma\)-algebra \(\mathcal{P}(Y)\) on it, then

\[(X,A) \xrightarrow{f} (Y,\mathcal{P}(Y)) \xrightarrow{g} \mathbb{K}\]

are measurable. So \(g \circ f\) is also measurable, and obviously elementary.

The following is an interesting converse of Corollary 3.6.

Theorem 3.2. Let \((X,A)\) be a measurable space, and let \(f : (X,A) \to [-\infty, \infty]\) be a measurable function. Then there exists a sequence \(f_n^\infty_{n=1} \in A\)-Elem\(_R(X)\), such that

1. \(\inf \{ f(y) : y \in X \} \leq f_n(x) \leq \sup \{ f(z) : z \in X \}, \forall x \in X, n \geq 1;\)
2. \(\lim_{n \to \infty} f_n(x) = f(x), \forall x \in X.\)

Moreover,

1. if \(\inf \{ f(x) : x \in X \} > -\infty\), then the sequence \(f_n^\infty_{n=1}\) can be chosen to be non-decreasing, i.e. \(f_n \leq f_{n+1}, \forall n \in \mathbb{N}\);
2. if \(\sup \{ f(x) : x \in X \} < \infty\), then the sequence \(f_n^\infty_{n=1}\) can be chosen to be non-increasing, i.e. \(f_n \geq f_{n+1}, \forall n \in \mathbb{N}\);
3. if \(\inf \{ f(x) : x \in X \} > -\infty\) and \(\sup \{ f(x) : x \in X \} < \infty\), then the sequence \(f_n^\infty_{n=1}\) can be chosen either non-decreasing, or non-increasing, and such that it converges uniformly to \(f\), i.e.

\[\lim_{n \to \infty} \sup_{x \in X} \left| f_n(x) - f(x) \right| = 0.\]

Proof. We begin with a special case of (iii). Assume \(X = [0,1]\), \(A = Bor([0,1])\), and consider the inclusion \(F : [0,1] \hookrightarrow [-\infty, \infty]\). For each \(n \in \mathbb{N}\), define the intervals \(I_k^n, J_k^n, 0 \leq k \leq 2^n - 1\) by

\[I_k^n = \left[ \frac{k}{2^n}, \frac{(k+1)}{2^n} \right), \text{ if } 0 \leq k \leq 2^n - 2; \quad I_{2^n - 1}^n = \left[ \frac{(2^n - 1)}{2^n}, 1 \right],\]

\[J_k^n = \left( \frac{k}{2^n}, \frac{(k+1)}{2^n} \right], \text{ if } 1 \leq k \leq 2^n - 1; \quad J_0^n = [0,1/2^n].\]

We then define, for each \(n \in \mathbb{N}\), the functions \(g_n, h_n : [0,1] \to \mathbb{R}\) by

\[g_n = 2^{-n} \sum_{k=0}^{2^n-1} k \lambda_{I_k^n} \quad \text{and} \quad h_n = 2^{-n} \sum_{k=0}^{2^n-1} (k+1) \lambda_{J_k^n}.\]

Remark that

\[0 \leq g_n(s) < 1 \quad \text{and} \quad 0 < h_n(s) \leq 1, \quad \forall s \in [0,1].\]

Note that, for every \(n \in \mathbb{N}\), we have

\[g_n(0) = 0; \quad g_n(1) = (2^n - 1)/2^n; \quad (9) \]

\[h_n(0) = 1/2^n; \quad h_n(1) = 1. \quad (10)\]

Claim 1: The sequence \((g_n)_{n=1}^\infty\) is non-decreasing, and the sequence \((h_n)_{n=1}^\infty\) is non-increasing.
Using (9) and (10), we only need to examine the restrictions to the open interval $(0, 1)$. Fix some point $s \in (0, 1)$. For every integer $n \geq 1$, define
\[ p_n^s = \max \{ k \in \mathbb{Z} : 0 \leq \frac{k}{2^n} < s \}. \]
We clearly have $p_n^s < 2^n$ and
\[ \frac{p_n^s}{2^n} < s \leq \frac{p_n^s + 1}{2^n}. \]
We then have
\[ g_n(s) = \begin{cases} p_n^s/2^n & \text{if } s \neq (p_n^s + 1)/2^n, \\ (p_n^s + 1)/2^n & \text{if } s = (p_n^s + 1)/2^n. \end{cases} \]
We now estimate $g_{n+1}(s)$ and $h_{n+1}(s)$. First of all, using (11), we have
\[ \frac{2p_n^s}{2^n+1} < x \leq \frac{2p_n^s + 2}{2^n+1}, \]
which means that either $p_{n+1}^s = 2p_n^s$, or $p_{n+1}^s = 2p_n^s + 1$. This immediately gives
\[ h_{n+1}(s) = \frac{p_{n+1}^s + 1}{2^n+1} \leq \frac{2p_n^s + 2}{2^n+1} = \frac{p_n^s + 1}{2^n} = h_n(s). \]
Note that, if $s = (p_n^s + 1)/2^n$, we will have $p_{n+1}^s = 2p_n^s + 1$ and $s = (p_{n+1}^s + 1)/2^{n+1}$, so we get
\[ g_{n+1}(s) = (p_{n+1}^s + 1)/2^{n+1} = (2p_n^s + 2)/2^{n+1} = (p_n^s + 1)/2^n = g_n(s). \]
If $s \neq (p_n^s + 1)/2^n$, then
\[ g_n(s) = \frac{p_n^s}{2^n} = \frac{2p_n^s}{2^n} \leq \frac{p_{n+1}^s}{2^{n+1}} \leq g_{n+1}(s). \]

Claim 2: For every $s \in [0, 1]$ one has
\[ \lim_{n \to \infty} \left[ \sup_{s \in [0, 1]} |g_n(s) - s| \right] = \lim_{n \to \infty} \left[ \sup_{s \in [0, 1]} |h_n(s) - s| \right] = 0. \]
To prove this fact we are going to estimate the differences $|g_n(s) - s|$ and $|h_n(s) - s|$. If $s = 0$ or $s = 1$, then the equalities (9) and (10) immediately show that
\[ |g_n(s) - s| \leq \frac{1}{2^n} \text{ and } |h_n(s) - s| \leq \frac{1}{2^n}, \forall n \in \mathbb{N}. \]
If $s \in (0, 1)$, then the definitions of $g_n(s)$ and $h_n(s)$ clearly show that
\[ s, g_n(s), h_n(s) \in [p_n^s/2^n, (p_n^s + 1)/2^n], \]
and then we see that we again have the inequalities (13). Since (13) now holds for all $s \in [0, 1]$, the Claim immediately follows.

We proceed now with the proof of the theorem. Define
\[ \alpha = \inf \{ f(x) : x \in X \} \text{ and } \beta = \sup \{ f(x) : x \in X \}. \]
If $\alpha = \beta$, there is nothing to prove. Assume $\alpha < \beta$. Depending on the finitude of $\alpha$ and $\beta$, we define a homeomorphism $\Phi : [\alpha, \beta] \to [0, 1]$, as follows.

(a) If $\alpha > -\infty$ and $\beta < \infty$, we define
\[ \Phi(s) = \frac{s - \alpha}{\beta - \alpha}, \forall s \in [\alpha, \beta]. \]
(b) If \( \alpha > -\infty \) and \( \beta = \infty \), we define
\[
\Phi(s) = \begin{cases} 
\frac{\pi}{2} \arctan(s - \alpha) & \text{if } s \neq \beta \\
1 & \text{if } s = \beta
\end{cases}
\]
(c) If \( \alpha = -\infty \) and \( \beta < \infty \), we define
\[
\Phi(s) = \begin{cases} 
1 + \frac{\pi}{2} \arctan(s - \beta) & \text{if } s \neq \alpha \\
0 & \text{if } s = \alpha
\end{cases}
\]
(d) If \( \alpha = -\infty \) and \( \beta = \infty \), we define
\[
\Phi(s) = \begin{cases} 
0 & \text{if } s = \alpha \\
\frac{1}{2} + \frac{1}{\pi} \arctan(s - \beta) & \text{if } \alpha < s \beta \\
1 & \text{if } s = \beta
\end{cases}
\]
Notice that \( \Phi(\alpha) = 0, \Phi(\beta) = 1 \), and
\[
\alpha \leq s < t \leq \beta \Rightarrow \Phi(s) < \Phi(t).
\]

After these preparations, we proceed with the proof. We begin with the special cases (i), (ii) and (iii).

If \( \alpha > -\infty \), we define the functions \( f_n = \Phi^{-1} \circ g_n \circ \Phi \circ f \). Since \( \Phi \) and \( \Phi^{-1} \) are increasing, and \( (g_n)_{n=1}^{\infty} \) is non-decreasing, it follows that \( (f_n)_{n=1}^{\infty} \) is non-decreasing. Since \( 0 \leq g_n(s) < 1, \forall s \in [0, 1] \), we see that \( \alpha \leq f_n(x) < \beta, \forall x \in X \). In particular, we have \(-\infty < f_n(x) < \infty \), for all \( n \) and \( x \). It is obvious that \( f_n \) is elementary, measurable, and since \( \lim_{n \to \infty} g_n(s) = s, \forall s \in [0, 1] \) (by Claim 2), we immediately get \( \lim_{n \to \infty} f_n(x) = f(x), \forall x \in X \).

If \( \beta < \infty \), we define the functions \( f_n = \Phi^{-1} \circ h_n \circ \Phi \circ f \). Since \( \Phi \) and \( \Phi^{-1} \) are increasing, and \( (h_n)_{n=1}^{\infty} \) is non-increasing, it follows that \( (f_n)_{n=1}^{\infty} \) is non-increasing. Since \( 0 < h_n(s) \leq 1, \forall s \in [0, 1] \), we see that \( \alpha < f_n(x) \leq \beta, \forall x \in X \). In particular, we have \(-\infty < f_n(x) < \infty \), for all \( n \) and \( x \). It is obvious that \( f_n \) is elementary, measurable, and since \( \lim_{n \to \infty} h_n(s) = s, \forall s \in [0, 1] \) (by Claim 2), we immediately get \( \lim_{n \to \infty} f_n(x) = f(x), \forall x \in X \).

If \( \alpha > -\infty \) and \( \beta < \infty \), then we can take \( f_n = \Phi^{-1} \circ g_n \circ \Phi \circ f, \forall n \), or we can take \( f_n = \Phi^{-1} \circ h_n \circ \Phi \circ f, \forall n \). The inequalities (13), combined with the definition (c) of \( \Phi \), show that
\[
|f_n(x) - f| \leq \frac{\beta - \alpha}{2n}, \forall x \in X, n \in \mathbb{N},
\]
with any of the above choices for \((f_n)_{n=1}^{\infty}\).

Having proven the cases (i), (ii) and (iii), we now examine the general situation, when \( \alpha = -\infty \) and \( \beta = \infty \). Consider the functions \( f', f'' : X \to [-\infty, \infty] \) defined by
\[
f'(x) = \max\{f(x), 0\} \text{ and } f''(x) = \min\{f(x), 0\}, \forall x \in X.
\]
By Corollary 3.4, both \( f' \) and \( f'' \) are measurable. Since \( \inf_{x \in X} f'(x) \geq 0 \), by part (i), there exists a sequence \((f'_n)_{n=1}^{\infty} \in A\text{-Elem}_g(X)\), such that \( \lim_{n \to \infty} f'_n(x) = f'(x), \forall x \in X \). Since \( \sup_{x \in X} f''(x) \leq 0 \), by part (ii), there exists a sequence \((f''_n)_{n=1}^{\infty} \in A\text{-Elem}_g(X)\), such that \( \lim_{n \to \infty} f''_n(x) = f''(x), \forall x \in X \). Define the elementary functions \( f_n = f'_n + f''_n, n \in \mathbb{N} \). Clearly the \( f_n \)'s are all in \( A\text{-Elem}_g(X) \).

We now check that
\[
\lim_{n \to \infty} f_n(x) = f(x), \forall x \in X.
\]
There are two cases to examine: (a) \( f(x) \geq 0 \); (b) \( f(x) \leq 0 \).
In case (a), we have \( f'(x) = f(x) \) and \( f''(x) = 0 \), so \( \lim_{n \to \infty} f'_n(x) = f(x) \) and \( \lim_{n \to \infty} f''_n(x) = 0 \).
In case (b), we have \( f'(x) = 0 \) and \( f''(x) = f(x) \), so \( \lim_{n \to \infty} f'_n(x) = 0 \) and \( \lim_{n \to \infty} f''_n(x) = f(x) \).
In either case, the equality (14) follows. \( \square \)

We conclude this section with a discussion on an interesting measurable space, that appears often in connection with probability theory.

**Example 3.1.** Consider the space \( T = \{0,1\}^\infty \), i.e.
\[
T = \{a = (\alpha_n)_{n=1}^\infty : \alpha_n \in \{0,1\}, \forall n \in \mathbb{N}\}.
\]
We call \( T \) the *space of infinite coin flippings*, having in mind that an element of \( T \) is the same as the outcome of an infinite sequence of coin flips (think 0 as corresponding to tails, and 1 as corresponding to heads). Equip \( T \) with the product topology. By Tikhonov’s Theorem, \( T \) is compact. The product topology on \( T \) is in fact given by a metric \( d \) defined by
\[
d(a, b) = \sum_{n=1}^{\infty} \left| \frac{\alpha_n - \beta_n}{2^n} \right|, \quad \forall a = (\alpha_n)_{n=1}^\infty, b = (\beta_n)_{n=1}^\infty \in T.
\]
For every number \( r \geq 2 \) we define a map \( \phi_r : T \to [0,1] \) by
\[
\phi_r(a) = (r-1) \sum_{n=1}^{\infty} \frac{\alpha_n}{r^n}, \quad \forall a = (\alpha_n)_{n=1}^\infty \in T.
\]
It is pretty clear that
\[
|\phi_r(a) - \phi_r(b)| \leq (r-1)d(a, b), \quad \forall a, b \in T,
\]
so the maps \( \phi_r : T \to [0,1], r \geq 2 \) are continuous. In particular, the set \( K_r = \phi_r(T) \) is a compact subset of \([0,1]\).

Define
\[
T_0 = \{a = (\alpha_n)_{n \in \mathbb{N}} \in T : \text{the set } \{n \in \mathbb{N} : \alpha_n = 0\} \text{ is infinite}\}.
\]
The set \( T \smallsetminus T_0 \) can be described as:
\[
T \smallsetminus T_0 = \{(\alpha_n)_{n \in \mathbb{N}} \in T : \text{there exists } N \in \mathbb{N}, \text{ such that } \alpha_n = 1, \forall n \geq N\}.
\]
The following are well known (see Appendix B, the proof of Proposition B.2).

**Facts:**
1. The set \( T \smallsetminus T_0 \) is countable
2. For any \( r \geq 2 \), and elements \( a = (\alpha_n)_{n=1}^\infty, b = (\beta_n)_{n=1}^\infty \in T_0 \), the following are equivalent:
   - there exists \( N \in \mathbb{N} \) such that \( \alpha_N = 1, \beta_N = 0 \), and \( \alpha_n = \beta_n \), for all \( n \in \mathbb{N} \) with \( n < N \);
   - \( \phi_r(a) > \phi(b) \).
   In particular, the map \( \phi_r|_{T_0} : T_0 \to [0,1] \) is injective.

The above constructions have a remarkable feature.

**Theorem 3.3.** Use the notations above. For a number \( r \geq 2 \) and subset \( A \subset T \), the following are equivalent:
1. \( A \in \text{Bor}(T) \);
2. \( \phi_r(A) \in \text{Bor}(K_r) \).
PROOF. Throughout the proof the number \( r \) will be fixed. The map \( \phi_r \) will be denoted by \( \phi \), and the compact set \( K_r \) will be denoted by \( K \).

Since \( \phi : T \to K \) is continuous, it is measurable, i.e. we have the implication
\[
B \in \text{Bor}(K) \Rightarrow \phi^{-1}(B) \in \text{Bor}(T).
\]

Before we proceed with the actual proof, we need some preparations. Remark that, since \( \phi : T \to K \) is surjective, we have the equality
\[
\phi(\phi^{-1}(C)) = C, \ \forall C \subset K.
\]

Claim 1: If a subset \( C \subset K \) is at most countable, if and only if the set \( \phi^{-1}(C) \subset T \) is at most countable.

Suppose \( C \) is at most countable countable. If we take \( A_0 = \phi^{-1}(C) \cap T_0 \), and \( A_1 = \phi^{-1}(C) \setminus T_0 \), then obviously \( \phi^{-1}(C) = A_0 \cup A_1 \). Since \( A_1 \subset T \setminus T_0 \), and \( T \setminus T_0 \) is countable, it follows that \( A_1 \) is at most countable, so we only need to prove that \( A_0 \) is at most countable. But since \( \phi|_{T_0} \) is injective, and \( A_0 \subset T_0 \), it follows that \( \phi|_{A_0} : A_0 \to C \) is injective, and then the fact that \( C \) is at most countable, forces \( A_0 \) to be at most countable.

Conversely, if \( \phi^{-1}(C) \) is at most countable, then so is \( \phi(\phi^{-1}(C)) \). By (16) we are done.

For each subset \( A \subset T \), we define
\[
\langle A \rangle = \phi^{-1}(\phi(A)).
\]

Remark that \( A \subset \langle A \rangle, \ \forall A \subset T \). Note also that, for any family \( (A_i)_{i \in I} \) of subsets of \( T \), one has the equality
\[
\langle \bigcup_{i \in I} A_i \rangle = \phi^{-1}\left( \phi\left( \bigcup_{i \in I} A_i \right) \right) = \phi^{-1}\left( \bigcup_{i \in I} \phi(A_i) \right) = \bigcup_{i \in I} \phi^{-1}(\phi(A_i)) = \bigcup_{i \in I} \langle A_i \rangle.
\]

As an application of Claim 1, to the set \( C = \phi(T \setminus T_0) \), we see that

(\ast) the set \( \langle T \setminus T_0 \rangle \) is at most countable.

Claim 2: For any subset \( A \subset T_0 \), one has the inclusion
\[
\langle A \rangle \setminus A \subset \langle T \setminus T_0 \rangle.
\]

In particular, the difference \( \langle A \rangle \setminus A \) is at most countable.

Start with an arbitrary element \( x \in \langle A \rangle \setminus A \). This means that \( x \notin A \), but \( \phi(x) \in \phi(A) \), which means that there exists some \( a \in A \), with \( \phi(x) = \phi(a) \). Assume now \( x \notin T \setminus T_0 \), which means that \( x \in T_0 \). But then, the fact that \( x, a \in T_0 \), combined with the injectivity of \( \phi|_{T_0} \) will force \( x = a \), which is impossible since \( a \in A \).

Claim 3: For any set \( A \subset T \), the difference \( \langle A \rangle \setminus A \) is at most countable.

Take \( A_0 = A \cap T_0 \) and \( A_1 = A \setminus A_0 \). Notice that, since \( A_1 \subset T \setminus T_0 \), we have
\[
\langle A_1 \rangle = \phi^{-1}(\phi(A_1)) \subset \phi^{-1}(\phi(T \setminus T_0)) = \langle T \setminus T_0 \rangle,
\]
so it follows that \( \langle A_1 \rangle \) is at most countable. We obviously have \( A = A_0 \cup A_1 \), so by (17)
\[
\langle A \rangle = \langle A_0 \rangle \cup \langle A_1 \rangle.
\]

But now we are done, since
\[
\langle A \rangle \setminus A = (\langle A_0 \rangle \cup \langle A_1 \rangle) \setminus (A_0 \cup A_1) \subset (\langle A_0 \rangle \setminus A_0) \cup \langle A_1 \rangle,
\]
and both \( \langle A_0 \rangle \setminus A_0 \) (by Claim 2) and \( \langle A_1 \rangle \) are at most countable.
Claim 4: For any subset \( A \subset T \), one has the inclusion
\[
\phi(T \setminus A) \supset K \setminus \phi(A),
\]
and the difference \( \phi(T \setminus A) \setminus (K \setminus \phi(A)) \) is at most countable.

The inclusion (18) is pretty obvious, from the surjectivity of \( \phi \). In order to prove that the difference

\[
C = \phi(T \setminus A) \setminus (K \setminus \phi(A)) = \phi(T \setminus A) \cap \phi(A)
\]
is countable, by Claim 1, it suffices to prove that \( \phi^{-1}(C) \) is countable. We have

\[
\phi^{-1}(C) = \phi^{-1}(\phi(T \setminus A) \cap \phi(A)) = \phi^{-1}(\phi(T \setminus A)) \cap \phi^{-1}(\phi(A)) = \langle T \setminus A \rangle \cap \langle A \rangle.
\]

We can write \( \phi^{-1}(C) = A_1 \cup A_2 \), where

\[
A_1 = (T \setminus A) \cap \langle A \rangle \quad \text{and} \quad A_2 = [(T \setminus A) \setminus (T \setminus A)] \cap \langle A \rangle,
\]
so it suffices to prove that both \( A_1 \) and \( A_2 \) are at most countable. But these facts are immediate from Claim 3, since \( A_1 = \langle A \rangle \setminus A \), and \( A_2 \subset (T \setminus A) \setminus (T \setminus A) \).

We can now proceed with the proof of the theorem. Define

\[
\mathcal{A} = \{ A \subset T : \phi(A) \in \text{Bor}(K) \},
\]
so that what we need to prove is the equality \( \mathcal{A} = \text{Bor}(T) \).

First, remark that, if \( A \in \mathcal{A} \), then \( \phi(A) \in \text{Bor}(K) \), and the fact that \( \phi \) is Borel measurable will force \( \langle A \rangle = \phi^{-1}(\phi(A)) \) to be a Borel set in \( T \). But since \( \langle A \rangle \setminus A \) is countable, hence Borel, it follows that

\[
A = \langle A \rangle \setminus (\langle A \rangle \setminus A)
\]
is again Borel. Therefore, we have the inclusion \( \mathcal{A} \subset \text{Bor}(T) \).

Second, remark that if \( F \subset T \) is a compact subset, then the continuity of \( \phi \) gives the fact that \( \phi(F) \) is compact, hence Borel. This then forces \( F \in \mathcal{A} \). Therefore \( \mathcal{A} \) contains the collection \( \mathcal{C}_T \) of all compact subsets of \( T \).

Now we have

\[
\mathcal{C}_T \subset \mathcal{A} \subset \text{Bor}(T) = \Sigma(\mathcal{C}_T),
\]
so all we need to prove is the fact that \( \mathcal{A} \) is a \( \sigma \)-algebra, i.e. we have the properties

(\( a \)) \( A \in \mathcal{A} \Rightarrow T \setminus A \in \mathcal{A} \);

(\( b \)) for any sequence \( (A_n)_{n=1}^\infty \subset \mathcal{A} \), the union \( \bigcup_{n=1}^\infty A_n \) also belongs to \( \mathcal{A} \).

To check (\( a \)) start with some set \( A \in \mathcal{A} \). We know that \( \phi(A) \in \text{Bor}(K) \), and we want to show that \( \phi(T \setminus A) \) is again Borel. By Claim 4, we know we can write

\[
\phi(T \setminus A) = [K \setminus \phi(A)] \cup C,
\]
for some set \( C \subset K \) which is at most countable. Since \( C \) and \( K \setminus \phi(A) \) are Borel, this shows that \( \phi(T \setminus A) \) is also Borel.

Property (\( b \)) is obvious, since \( \phi(A_n) \), \( n \geq 1 \) are all Borel, and

\[
\phi\left( \bigcup_{n=1}^\infty A_n \right) = \bigcup_{n=1}^\infty \phi(A_n). \quad \Box
\]

Corollary 3.7. Use the above notations. For a number \( r \geq 2 \) and a subset \( B \subset K_r \), the following are equivalent:

(i) \( B \in \text{Bor}(K_r) \);

(ii) \( \phi^{-1}_r(B) \in \text{Bor}(T) \).
Proof. The implication (i) \(\Rightarrow\) (ii) is trivial, since \(\phi_r\) is continuous, hence measurable.

Conversely, if the set \(A = \phi_r^{-1}(B)\) is Borel, then by the Theorem, \(\phi_r(A)\) is Borel. But since \(\phi_r\) is surjective, we have \(B = \phi_r(A)\).

Comments. From the above results, we see that \(\phi_r : T \to K_r\) "almost preserves Borel structures." More explicitly, if one considers the maps

\[
\Phi_r : \mathcal{P}(T) \ni A \mapsto \phi_r(A) \in \mathcal{P}(K_r),
\]

\[
\Psi_r : \mathcal{P}(K_r) \ni B \mapsto \phi_r^{-1}(B) \in \mathcal{P}(T),
\]

then

- \((\Phi_r \circ \Psi_r)(B) = B\), for all \(B \subset K_r\);
- \((\Psi_r \circ \Phi_r)(A) \supset A\), and \((\Phi_r \circ \Psi_r)(A) \setminus A\) is at most countable, for all \(A \subset T\);
- \(B \in \text{Bor}(K_r) \iff \Psi_r(B) \in \text{Bor}(T)\);
- \(A \in \text{Bor}(T) \iff \Phi_r(A) \in \text{Bor}(K_r)\).

In the particular case \(r = 2\), we know that \(K_2 = [0,1]\), so we can think the measurable space \([0,1], \text{Bor}([0,1])\) as "approximatively the same" as the measurable space \((T, \text{Bor}(T))\).

The case \(r = 3\) will be an interesting one, especially for constructing various counter-examples. The compact set \(K_3 \subset [0,1]\) is called the ternary Cantor set.

It turns out that there exists another useful description of the ternary Cantor set \(K_3\), which yields some interesting properties.

Notations. We keep the notations above. An element \(a = (\alpha_n)_{n=1}^{\infty} \in T\) will be called finite, if there exists some \(N \in \mathbb{N}\), such that \(\alpha_n = 0\), \(\forall n \geq 0\). We define

\[
T_{\text{fin}} = \{a \in T : a \text{ finite}\}.
\]

Remark that \(T_{\text{fin}} \subset T_0\). In particular the map \(\phi_3|_{T_{\text{fin}}} : T_{\text{fin}} \to K_3\) is injective.

For \(a \in T_{\text{fin}}\) we define its length as

\[
\ell(a) = \min\{N \in \mathbb{N} : \alpha_n = 0, \ \forall n \geq N\} - 1.
\]

With this definition, for every \(a = (\alpha_n)_{n=1}^{\infty} \in T_{\text{fin}}\), we have

\[
\alpha_{\ell(a)} = 1 \text{ and } \alpha_n = 0, \ \forall n > \ell(a).
\]

We define

\[
\Lambda = \{(k,a) \in \mathbb{Z} \times T_{\text{fin}} : k \geq \ell(a)\}.
\]

Finally, for every pair \(\lambda = (k,a) \in \Lambda\), we define the open interval

\[
I_\lambda = (\phi_3(a) + \frac{1}{3^{k+1}}, \phi_3(a) + \frac{2}{3^{k+1}}).
\]

Remark that, using (19) we have

\[
\phi_3(a) \leq 2 \sum_{n=1}^{\ell(a)} \frac{2}{3^n} = 1 - \frac{1}{3^{\ell(a)}},
\]

with the convention that the sum is 0, if \(\ell(a) = 0\). We then get

\[
\phi_3(a) + \frac{2}{3^{k+1}} \leq 1 - \frac{1}{3^{\ell(a)}} + \frac{2}{3^{k+1}} < 1 - \frac{1}{3^{\ell(a)}} + \frac{1}{3^k} \leq 1,
\]
which gives the inclusion $I_\lambda \subset (0, 1)$.

The following result describes an alternative construction of $K_3$.

**Theorem 3.4.** Use the notations above.

(i) The set $T_{fin}$ is dense in $T$;
(ii) The system $(I_\lambda)_{\lambda \in \Lambda}$ is pair-wise disjoint.
(iii) $\bigcup_{\lambda \in \Lambda} = [0, 1] \setminus K_3$.

**Proof.** The map $\phi_3$ will be simply denoted by $\phi$, and the Cantor set $K_3$ will be denoted simply by $K$.

(i) Fix some element $a = (\alpha_n)_{n=1}^{\infty} \in T$. For every integer $k \geq 1$ define the element $a_k = (\alpha_n^k)_{n=1}^{\infty} \in T$, by

$$\alpha_n^k = \begin{cases} 
\alpha_n & \text{if } n \leq k \\
0 & \text{if } n > k
\end{cases}$$

It is obvious that $a_k \in T_{fin}, \forall k \in \mathbb{N}$. The inequality

$$d(a, a_k) = \sum_{n=k+1}^{\infty} \frac{\alpha_n}{2^{n-k}} \leq \sum_{n=k+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^k}, \forall k \in \mathbb{N}$$

then immediately shows that $\lim_{k \to \infty} a_k = a$.

(ii) Assume $\lambda, \mu \in \Lambda$ are such that $\lambda \neq \mu$, and let us prove that $I_\lambda \cap I_\mu = \emptyset$. Let $\lambda = (j, a)$ and $\mu = (k, b)$, where $a = (\alpha_n)_{n=1}^{\infty}$ and $b = (\beta_n)_{n=1}^{\infty}$ are elements in $T_{fin}$ with $\ell(a) \leq j$ and $\ell(b) \leq k$. Since $\lambda \neq \mu$, we have one (or both) of the following cases: (A) $\alpha \neq \beta$, or (B) $j \neq k$.

In case (A) we take

$$m = \min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\}.$$ 

Without any loss of generality, we can assume that $\alpha_m = 0$ and $\beta_m = 1$. Note that $k \geq \ell(b) \geq m \geq 1$. We are going to prove that $I_\lambda \cap I_\mu = \emptyset$, by showing that the right end-point of $I_\lambda$ is not greater than the left end-point of $I_\mu$, that is,

$$\phi(a) + \frac{2}{3^{k+1}} \leq \phi(b) + \frac{1}{3^{k+1}}.$$ 

Define the number

$$M = \sum_{n=1}^{m-1} \frac{\alpha_n}{3^n} = \sum_{n=1}^{m-1} \frac{\beta_n}{3^n},$$

with the convention that $M = 0$, if $m = 1$. We have:

$$\phi(a) = 2M + 2 \sum_{n=m+1}^{\ell(a)} \frac{\alpha_n}{3^n} \leq 2M + 2 \sum_{n=m+1}^{\ell(a)} \frac{1}{3^n} = 2M + \frac{1}{3^{m-1}} - \frac{1}{3^{\ell(a)}};$$

$$\phi(b) = 2M + \frac{2}{3^m} + 2 \sum_{n=m+1}^{\ell(b)} \frac{\beta_n}{3^n} \geq 2M + \frac{2}{3^m}.$$ 

The inequality (20) then follows immediately from:

$$\phi(a) + \frac{2}{3^{k+1}} \leq 2M + \frac{1}{3^{m-1}} - \frac{1}{3^{\ell(a)}} + \frac{2}{3^{k+1}} < 2M + \frac{1}{3^{m-1}} - \frac{1}{3^{\ell(a)}} + \frac{1}{3} \leq 2M + \frac{1}{3^m} < 2M + \frac{2}{3^m} \leq \phi(b) < \phi(b) + \frac{1}{3^{k+1}}.$$
In case (b), based on the fact that we have proven case (a), we can assume, without any loss of generality, that \(a = b\) and \(j < k\). In this case we have
\[
\phi(b) + \frac{2}{3^{k+1}} = \phi(a) + \frac{2}{3^{k+1}} < \phi(a) + \frac{1}{3^k} \leq \phi(a) + \frac{1}{3^j+1},
\]
which means that the right end-point of \(I_\mu\) is not greater than the left end-point of \(I_\lambda\), so again we get \(I_\lambda \cap I_\mu = \emptyset\).

For the proof of (iii) we are going to use the space
\[
P = \{0, 1, 2\}^{\mathbb{R}_0} = \{(\alpha_n)_{n=1}^\infty : \alpha_n \in \{0, 1, 2\}, \ \forall n \in \mathbb{N}\}.
\]
Exactly as is the case with \(T\), the product space \(P\) is compact with respect to the product topology, which is given by the metric
\[
d(a, b) = \sum_{n=1}^{\infty} \frac{|\alpha_n - \beta_n|}{2^n}, \ \forall a = (\alpha_n)_{n=1}^\infty, \ b = (\beta_n)_{n=1}^\infty \in P.
\]
Then map \(\psi : P \to [0, 1]\), defined by
\[
\psi(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}, \ \forall a = (\alpha_n)_{n=1}^\infty \in P,
\]
satisfies
\[
|\psi(a) - \psi(b)| \leq d(a, b), \ \forall a, b \in P,
\]
hence it is continuous. Note also that \(\psi\) is surjective. We can write \(\phi = \psi \circ \rho\), where
\[
\rho : \{0, 1\}^{\mathbb{R}_0} \ni (\alpha_n)_{n=1}^\infty \longmapsto (2\alpha_n)_{n=1}^\infty \in \{0, 1, 2\}^{\mathbb{R}_0}.
\]
Note also that \(\rho : T \to P\) is continuous, since we clearly have
\[
d(\rho(a), \rho(b)) \leq 2d(a, b), \ \forall a, b \in T.
\]

We now proceed with the proof of (iii). Denote the open set \(\bigcup_{I_\lambda \in \Lambda} I_\lambda\) simply by \(D\). Since \(T_{\text{fin}}\) is dense in \(T\), it follows that \(\phi(T_{\text{fin}})\) is dense in \(K = \phi(T)\). Therefore, in order to prove the inclusion \(K \subset [0, 1] \setminus D\), using the surjectivity of \(\psi\), it suffices to prove the inclusion
\[
\phi(T_{\text{fin}}) \subset [0, 1] \setminus D.
\]
Using the map \(\psi : P \to [0, 1]\), the above inclusion is equivalent to
\[
(21) \quad P \setminus \rho(T_{\text{fin}}) \supset \psi^{-1}(D).
\]
In order to prove the inclusion \([0, 1] \setminus D \subset K\), again using the surjectivity of \(\psi\), it suffices to prove the inclusion
\[
(22) \quad \psi^{-1}(D) \supset P \setminus \psi^{-1}(K).
\]
To prove (21) start with some element \(a = (\alpha_n)_{n=1}^\infty \in \psi^{-1}(D)\), which means that there exists some \(b \in T_{\text{fin}}\), and an integer \(k \geq \ell(b)\), such that \(\psi(a) \in I_{(k, b)}\), i.e.
\[
(23) \quad \frac{2\beta_1}{3} + \cdots + \frac{2\beta_k}{3^k} + \frac{1}{3^{k+1}} \leq \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} < \frac{2\beta_1}{3} + \cdots + \frac{2\beta_k}{3^k} + \frac{2}{3^{k+1}}.
\]
We prove that $a \notin \rho(T_{\tilde{f}})$ by contradiction. Assume $a \in \rho(T_{\tilde{f}})$, which means that there exists $b = (\gamma_n)_{n=1}^\infty \in T_{\tilde{f}}$, such that $\alpha_n = 2\gamma_n, \forall n \in \mathbb{N}$. Define the element $\tilde{\beta} = (\tilde{\beta}_n)_{n=1}^\infty \in T_{\tilde{f}}$ by

$$\tilde{\beta}_n = \begin{cases} \beta_n & \text{if } n \leq k \\ 1 & \text{if } n = k + 1 \\ 0 & \text{if } n > k + 1 \end{cases}$$

With this definition, the inequalities (23) give

$$\phi(b) < \phi(b) + \frac{1}{3^{k+1}} < \phi(c) < \phi(\tilde{b}).$$

By Fact 2 above, there exist $N, N' \in \mathbb{N}$ such that

- $\gamma_N = 1$, $\beta_N = 0$, and $\gamma_n = \beta_n$, for all $n \in \mathbb{N}$ with $n < N$;
- $\gamma_{N'} = 0$, $\beta_{N'} = 1$, and $\gamma_n = \tilde{\beta}_n$, for all $n \in \mathbb{N}$ with $n < N'$.

We will examine three cases: (A) $N < N'$, (B) $N = N'$, or (C) $N > N'$.

Case (B) is clearly impossible. In case (A), the inequality $N < N'$ forces $\beta_N = 0$, $\gamma_N = 1$ and $\beta_N = \gamma_N$, which means that $\beta_N = 1 \neq \beta_N = 0$. This clearly forces $N = k + 1 > \ell(b)$, which in particular gives $\beta_n = \beta_n = 0, \forall n > N$, so we clearly have $\gamma_n \geq \beta_n, \forall n \in \mathbb{N}$, so we get $\phi(c) \geq \phi(\tilde{b})$, thus contradicting (24). In case (C), we have $\gamma_{N'} = 0$, $\beta_{N'} = 1$, and since $N' < N$, we also have $\beta_{N'} = \gamma_{N'} = 0$.

As before this would force $N' = k + 1$. We then have

$$\phi(c) = 2 \sum_{n=1}^\infty \frac{\gamma_n}{3^n} = 2 \sum_{n=1}^{N-1} \frac{\gamma_n}{3^n} + 2 \gamma_N 3^{N'} + 2 \sum_{n=N+1}^\infty \frac{\gamma_n}{3^n} = 2 \sum_{n=1}^k \frac{\beta_n}{3^n} + 0 + 2 \sum_{n=k+2}^\infty \frac{\gamma_n}{3^n} =$$

$$= \phi(b) + 2 \sum_{n=k+2}^\infty \frac{\gamma_n}{3^n} \leq \phi(b) + 2 \sum_{n=k+1}^\infty \frac{1}{3^n} = \phi(b) + \frac{1}{3^{k+1}},$$

again contradicting (24).

To prove (22), we start with some element $a \in P \bowtie \psi^{-1}(K)$, and we show that $\psi(a) \in D$. The fact that $a \notin \psi^{-1}(K)$ forces the fact that $a \notin \rho(T)$. In particular, this gives the fact that $a = (\alpha_n)_{n=1}^\infty \in \{0, 1, 2\}^\mathbb{N}$ and there exists some $n \in \mathbb{N}$ such that $\alpha_n = 1$. Put

$$N = \min\{n \in \mathbb{N} : \alpha_n = 1\}.$$

Define the elements $b = (\beta_n)_{n=1}^\infty \in \{0, 1\}^\mathbb{N}$, by

$$\beta_n = \begin{cases} \alpha_n/2 & \text{if } n < N \\ 0 & \text{if } n \geq N \end{cases}$$

Notice that $b \in T_{\tilde{f}}$, and $\ell(b) \leq N - 1$. Notice also that $2\beta_n = \alpha_n$, for all $n \in \mathbb{N}$ with $n < N - 1$. In particular, using the equality $\alpha_N = 1$, this gives

$$\phi(b) + \frac{1}{3^N} = 2 \sum_{n=1}^{N-1} \frac{\beta_n}{3^n} + \frac{\alpha_N}{3^N} = \sum_{n=1}^N \frac{\alpha_n}{3^n} \leq \sum_{n=1}^\infty \frac{\alpha_n}{3^n} = \psi(a);$$

and

$$\phi(b) + \frac{2}{3^N} = 2 \sum_{n=1}^{N-1} \frac{\gamma_n}{3^n} + \frac{\alpha_N}{3^N} + \sum_{n=N+1}^\infty \frac{2}{3^n} = \sum_{n=1}^N \frac{\alpha_n}{3^n} + \sum_{n=N+1}^\infty \frac{2}{3^n} \geq \sum_{n=1}^\infty \frac{\alpha_n}{3^n} = \psi(a).$$

$\square$
Consider the pair \( \lambda = (N - 1, \beta) \in \Lambda \). We are going to show that \( \psi(a) \in I_\lambda \), i.e. we have the inequalities

\[
\phi(b) + \frac{1}{3^N} < \psi(a) < \phi(b) + \frac{2}{3^N}.
\]

By (25) and (26) it suffices to prove only that

\[
\psi(a) \neq \phi(b) + \frac{1}{3^N} \text{ and } \psi(a) \neq \phi(b) + \frac{2}{3^N}.
\]

If \( \psi(a) = \phi(b) + \frac{1}{3^N} \), then by the inequalities (25), we are forced to have

\[
(28) \quad \alpha_n = 0, \quad \forall \ n > N.
\]

If \( \psi(a) = \phi(b) + \frac{2}{3^N} \), then by the inequalities (26), we are forced to have

\[
(29) \quad \alpha_n = 2, \quad \forall \ n > N.
\]

If (28) holds, we define \( c = (\gamma_n)_{n=1}^\infty \in T \), by

\[
\gamma_n = \begin{cases} \alpha_n / 2 & \text{if } n < N \\ 0 & \text{if } n = N \\ 1 & \text{if } n > N \end{cases}
\]

and we will have

\[
\phi(c) = 2 \sum_{n=1}^\infty \gamma_n \frac{1}{3^n} = \frac{2}{3} + \sum_{n=N+1}^\infty \frac{1}{3^n} = \sum_{n=1}^{N-1} \frac{\alpha_n}{3^n} + \frac{1}{3^N} = \psi(a),
\]

thus forcing \( \psi(a) \in K \), which is impossible.

If (29) holds, we define \( c = (\gamma_n)_{n=1}^\infty \in T \), by

\[
\gamma_n = \begin{cases} \alpha_n / 2 & \text{if } n \neq N \\ 1 & \text{if } n = N \\ 0 & \text{if } n > N \end{cases}
\]

and we will have

\[
\phi(c) = 2 \sum_{n=1}^{N-1} \frac{\gamma_n}{3^n} + \frac{2}{3^N} = \frac{2}{3} + \sum_{n=1}^{N-1} \frac{\gamma_n}{3^n} + \frac{1}{3^N} + \sum_{n=N+1}^\infty \frac{2}{3^n} = \sum_{n=1}^\infty \frac{\alpha_n}{3^n} = \psi(a),
\]

thus forcing again \( \psi(a) \in K \), which is impossible. \( \Box \)

**Exercise 7.** Using the notations above, prove that the set

\[
[0, 1] \setminus K_3 = \bigcup_{\lambda \in \Lambda} I_\lambda
\]

is dense in \([0, 1]\).

**Hints:** Define the set

\[
P_0 = \{ (\alpha_n)_{n=1}^\infty \in \{0, 1, 2\}^{\mathbb{N}} : \text{the set } \{ n \in \mathbb{N} : \alpha_n = 1 \} \text{ is infinite} \}.
\]

Prove that \( P_0 \) is dense in \( P \), and prove that \( \psi(P) \subset [0, 1] \setminus K \). (Use the arguments employed in the proof of part (iii).)

**Remarks 3.5.** If we set \( \Lambda_n = \Lambda \cap \{ \{n\} \times P \} \), then we can write the complement of the ternary Cantor set as

\[
[0, 1] \setminus K_3 = \bigcup_{n=0}^\infty D_n,
\]
where

\[ D_n = \bigcup_{\lambda \in \Lambda_n} I_\lambda. \]

Then the system of open sets \((D_n)_{n \geq 0}\) is pair-wise disjoint. Moreover, each \(D_n\) is a union of \(2^n\) disjoint intervals of length \(1/3^{n+1}\).

Since \(\text{card}\ T_0 = \mathfrak{c}\), and the map \(\phi_3|_{T_0} : T_0 \to K_3\) is injective, we get \(\text{card}\ K_3 \geq \mathfrak{c}\). Since we also have \(\text{card}\ K_3 \leq \text{card}\ \mathbb{R} = \mathfrak{c}\), we get in fact the equality

\[ \text{card}\ K_3 = \mathfrak{c}.\]