5. Banach algebras

In this section we discuss an important concept in Functional Analysis. The result presented here are needed in Section 7 as well as in Chapter V.

Definition. Let \( \mathbb{K} \) be either \( \mathbb{R} \) or \( \mathbb{C} \). A normed algebra over \( \mathbb{K} \) is an algebra \( \mathcal{A} \) over \( \mathbb{K} \), which also carries a norm \( \| \cdot \| \), which is sub-multiplicative, in the sense that:

- \( \| xy \| \leq \| x \| \cdot \| y \|, \forall x, y \in \mathcal{A} \)

(When there is no danger of confusion, the norm will be omitted from the notation.)

If \( \mathcal{A} \) is also a Banach space, then it will be called a Banach algebra.

Remarks 5.1. Suppose \( \mathcal{A} \) is a normed algebra.

A. For any element \( a \in \mathcal{A} \), one defines the left multiplication map \( L_a \) and the right multiplication map \( R_a \) by

\[
L_a : \mathcal{A} \ni x \mapsto ax \in \mathcal{A},
\]

\[
R_a : \mathcal{A} \ni x \mapsto xa \in \mathcal{A}.
\]

It is a consequence of sub-multiplicativity that both \( L_a, R_a : \mathcal{A} \to \mathcal{A} \) are linear continuous, and one has the inequalities \( \| L_a \| \leq \| a \| \) and \( \| R_a \| \leq \| a \| \), for all \( a \in \mathcal{A} \).

B. As a consequence of sub-multiplicativity, one has the inequality

\[
\| xy - ab \| \leq \| x \| \cdot \| y - b \| + \| x - a \| \cdot \| b \|, \forall a, b, x, y \in \mathcal{A}.
\]

In particular, when we equip the product space \( \mathcal{A} \times \mathcal{A} \) with the product topology, the multiplication map \( \mathcal{M} : \mathcal{A} \times \mathcal{A} \ni (x, y) \mapsto xy \in \mathcal{A} \) is continuous. Indeed, if \( (x_n)_{n=1}^{\infty} \subset \mathcal{A} \) is convergent to \( a \in \mathcal{A} \), and \( (y_n)_{n=1}^{\infty} \subset \mathcal{A} \) is convergent to \( b \in \mathcal{A} \), then

\[
\lim_{n \to \infty} \| x_n y_n - ab \| = 0,
\]

i.e. \( \lim_{n \to \infty} x_n y_n = ab \).

C. If \( \mathcal{A} \) is a normed algebra, then its completion \( \overline{\mathcal{A}} \) is a Banach algebra. The point here is of course the fact that \( \overline{\mathcal{A}} \) can be equipped with a multiplication operation, which turns it into a normed algebra. This multiplication is defined as follows. One starts with two elements \( x, y \in \overline{\mathcal{A}} \), represented by two Cauchy sequences \( (x_n)_{n=1}^{\infty} \subset \mathcal{A} \) and \( (y_n)_{n=1}^{\infty} \subset \mathcal{A} \) respectively, and using (1), one has

\[
\| x_m y_n - x_n y_n \| \leq \| x_m \| \cdot \| y_m - y_n \| + \| x_m - x_n \| \cdot \| y_n \|, \forall m, n \geq 1,
\]

which clearly gives the fact that \( (x_n y_n)_{n=1}^{\infty} \) is a Cauchy sequence, hence it defines some element \( xy \in \overline{\mathcal{A}} \). Of course, one has to show that this definition is independent of the choice of the two Cauchy sequences. This is pretty clear, for if one takes two other Cauchy sequences \( (x'_n)_{n=1}^{\infty} \subset \mathcal{A} \) and \( (y'_n)_{n=1}^{\infty} \subset \mathcal{A} \) with

\[
\lim_{n \to \infty} (x'_n - x_n) = \lim_{n \to \infty} (y'_n - y_n) = 0,
\]

then again by (1) we have

\[
\| x'_n y'_n - x_n y_n \| \leq \| x'_n \| \cdot \| y'_n - y_n \| + \| x'_n - x_n \| \cdot \| y_n \|, \forall n \geq 1,
\]

so \( \lim_{n \to \infty} (x'_n y'_n - x_n y_n) = 0 \). It is clear that the norm on \( \overline{\mathcal{A}} \) is sub-multiplicative.

Definition. If \( \mathcal{B} \) is a Banach algebra, then for a subset \( \mathcal{M} \subset \mathcal{B} \) we denote by \( \text{alg}(\mathcal{M}) \) the subalgebra of \( \mathcal{B} \), generated by \( \mathcal{M} \), which consists all linear combinations of (finite) products of elements in \( \mathcal{M} \). (This is the smallest subalgebra of \( \mathcal{B} \) which
contains $M$. Its closure $\text{alg}(M)$, in $B$ (which can be treated as the completion), will be referred to as the *Banach subalgebra of $B$, generated by $M$.*

**Exercise 1**: Let $B$ be a Banach algebra, and let $M \subset B$, be a *commuting subset*, i.e.

$$xy = yx, \ \forall x, y \in M.$$ 

Let $A$ be the Banach subalgebra of $B$ generated by $M$. Prove that $A$ is a *commutative algebra*, i.e.

$$ab = ba, \ \forall a, b \in A.$$

**Examples 5.1.** Let $K$ be either $\mathbb{R}$ or $\mathbb{C}$.

A. Let $X$ be a compact space. Then the Banach space (see Section 5)

$$C^K(X) = \{ f : X \to K : f \text{ continuous } \},$$

is a Banach algebra over $K$, when equipped with the pointwise multiplication, and the norm

$$\|f\| = \sup_{x \in X} |f(x)|, \ f \in C(X).$$

B. More generally, if $\Omega$ is a topological space, then the Banach space

$$C^K_c(\Omega) = \{ f : \Omega \to K : f \text{ continuous, bounded } \}$$

is a Banach algebra.

C. If $\Omega$ is a locally compact space, then $C^K_c(\Omega)$ is a normed algebra. Its completion $C^K_0(\Omega)$ is a Banach algebra.

D. Take $X$ to be a Banach space over $K$, and define

$$\mathcal{L}(X) = \{ T : X \to X : T \text{ linear and continuous } \},$$

equipped with the pointwise vector space structure. The multiplication is the composition, and the norm is:

$$\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}, \ T \in \mathcal{L}(X).$$

Then $\mathcal{L}(X)$ is a Banach algebra.

**Definition.** A normed algebra $A$ is said to be *unital*, if it has a unit element 1, i.e.

$$1 \cdot x = x \cdot 1 = x, \ \forall x \in A.$$ 

**Remark 5.2.** Suppose $A$ is a unital normed algebra with unit $1 \neq 0$. Then there exists a new sub-multiplicative norm $\| \cdot \|'$, which is equivalent to $\| \cdot \|$, and such that $\|1\|' = 1$. Indeed, if $\|1\| \neq 1$, then by sub-multiplicativity we have $\|1\| \leq \|1\|^2$, which forces $\|1\| > 1$. We then define

$$\|a\|' = \|L_a\|, \ \forall a \in A.$$ 

On the one hand, by Remark 5.1.A, we know already that we have the inequality

$$\|a\|' \leq \|a\|, \ \forall a \in A.$$ 

On the other hand, we also have

$$\|a\| = \|L_a1\| \leq \|L_a\| \cdot \|1\| = \|a\|' \cdot \|1\|, \ \forall a \in A.$$ 

The new norm $\| \cdot \|'$ will be called the *unital correction of $\| \cdot \|$.*

The Banach algebras described in Examples 5.1.A, 5.1.B and 5.1.D are all unital. For A and B the unit is the constant function 1. For D the unit is the identity map $I : X \to X$. In these examples the norm is the correct one, in the sense that the unit has norm 1.
The following is a familiar construction in algebra, which is often used when dealing with non-unital algebras.

**Definition.** Suppose $A$ is a $K$-algebra. Define the vector space $\tilde{A} = A \times K$, equipped with the operation
\[
(a, \alpha) \cdot (b, \beta) = (ab + \alpha b + \beta a, \alpha \beta), \quad a, b \in A, \ \alpha, \beta \in K.
\]
Then $\tilde{A}$ is a unital $K$-algebra, with unit $1 = (0, 1)$. We identify $A$ as a subalgebra in $\tilde{A}$ via the map
\[
\iota: A \ni a \mapsto (a, 0) \in \tilde{A}.
\]
The pair $(\tilde{A}, \iota)$ is referred to as the **unitization** of $A$. Its key feature is the following:

- whenever $B$ is a unital $K$-algebra, and $\Phi: A \to B$ is an algebra homomorphism, there exists a unique algebra homomorphism $\tilde{\Phi}: \tilde{A} \to B$, such that $\tilde{\Phi}|_A = \Phi$ and $\tilde{\Phi}(1) = 1$.

Explicitly, $\tilde{\Phi}(a, \alpha) = \Phi(a) + \alpha 1$, $\forall (a, \alpha) \in \tilde{A}$.

Remark that this construction makes sense even in the case when $A$ already has a unit. The unit $1$ in $\tilde{A}$ is then regarded as a “new” unit.

**Remarks 5.3.** In the normed algebra setting, it is natural to ask the following question. **Given a normed algebra $A$, does there exist a sub-multiplicative norm on the unitized algebra $\tilde{A}$, which extends the given norm on $A$?**

A. The answer to the above question is affirmative. One example is the norm $\| \cdot \|_1$ on $\tilde{A}$, defined by
\[
\|(a, \alpha)\|_1 = \|a\| + |\alpha|, \quad \forall (a, \alpha) \in \tilde{A}.
\]
Notice that this norm has the properties

(i) $\|\iota(a)\|_1 = \|a\|$, $\forall a \in A$;

(ii) $\|1\|_1 = 1$.

A sub-multiplicative norm on $\tilde{A}$, with the properties (i) and (ii) above, will be called a **unitization norm**.

B. In principle there exist many unitization norms on $\tilde{A}$. One way to construct such a norm is as follows (see Example 5.2 below for an illustration). Assume there exists an isometric algebra homomorphism $\Phi: A \to B$, where $B$ is a unital normed algebra (whose unit is denoted by $1_B$) with $\|1_B\| = 1$, and such that $1_B \not\in \Phi(A)$. The second condition ensures that the homomorphism $\tilde{\Phi}: \tilde{A} \to B$, defined by $\tilde{\Phi}(a, \alpha) = \Phi(a) + \alpha 1_B$, $\forall a \in A, \ \alpha \in K$, is injective. With the help of the homomorphism $\tilde{\Phi}$ one can then define a unitization norm $\| \cdot \|'$ on $\tilde{A}$ by
\[
\|x\|' = \|\tilde{\Phi}(x)\|, \quad \forall x \in \tilde{A}.
\]

C. If $\| \cdot \|'$ is any unitization norm on $\tilde{A}$, then one has the inequality
\[
\|x\|' \leq \|\iota(x)\|, \quad \forall x \in \tilde{A}.
\]
This is pretty obvious, for if $x = (a, \alpha)$, then we can also write $x = \iota(a) + \alpha 1$, which by the properties (i) and (ii) gives
\[
\|x\|' \leq \|\iota(a)\|' + |\alpha| \cdot \|1\|' = \|a\| + |\alpha| = \|x\|_1.
\]
Of course, (3) gives the fact that the identity map
\[
\text{Id}: (\tilde{A}, \| \cdot \|_1) \to (\tilde{A}, \| \cdot \|')
\]
is continuous.

D. If \( \mathcal{A} \) is a Banach algebra, then any unitization norm on \( \tilde{\mathcal{A}} \) is equivalent to the norm \( \| \cdot \|_1 \). This is a consequence of the Inverse Mapping Theorem, which gives the continuity of the identity map

\[
\text{Id} : (\tilde{\mathcal{A}}, \| \cdot \|') \to (\tilde{\mathcal{A}}, \| \cdot \|_1).
\]

Since \( (\tilde{\mathcal{A}}, \| \cdot \|_1) \) is complete, it follows that any unitization norm turns \( \tilde{\mathcal{A}} \) into a Banach algebra.

**Exercise 2**

Consider the algebra \( \mathbb{C}[t] \) of polynomials in one variable, and put

\[
\mathcal{A} = \{ P \in \mathbb{C}[t] : P(0) = 0 \}.
\]

Equip \( \mathcal{A} \) with the norm

\[
\| P \| = \sup_{t \in [1,2]} |P(t)|, \quad P \in \mathcal{A},
\]

so that if one defines for each \( P \in \mathcal{A} \) the function \( f_P : [1,2] \to \mathbb{C} \) by \( f_P(t) = P(t), \ t \in [1,2] \), one gets an isometric algebra homomorphism

\[
\Phi : \mathcal{A} \ni P \mapsto f_P \in \mathbb{C}([1,2]).
\]

Prove that the unitization norm \( \| \cdot \|' \) on \( \tilde{\mathcal{A}} \), defined by \( \Phi \) as in Remark 5.3.B, is not equivalent to the norm \( \| \cdot \|_1 \).

**Example 5.2.** Let \( \Omega \) be a locally compact space \( \Omega \), and consider Banach algebra \( \mathbb{C}^K_0(\Omega) \), identified as \( \mathbb{C}^K_0(\Omega) = \{ f \in \mathbb{C}^K(\Omega^\alpha) : f(\infty) = 0 \} \), where \( \Omega^\alpha \) is the Alexandrov compactification of \( \Omega \). If we apply the construction outlined in Remark 5.3.B, for the inclusion \( \Phi : \mathbb{C}^K_0(\Omega) \hookrightarrow \mathbb{C}^K(\Omega^\alpha) \), then this gives rise to a unitization norm, denoted \( \| \cdot \|_{st} \), defined as follows. If one starts with a pair \( x = (f, \lambda) \in \tilde{\mathbb{C}^K_0}(\Omega) \), then the element \( \phi_x = \tilde{\Phi}(f, \lambda) = \Phi(f) + \lambda I \in \mathbb{C}^K(\Omega^\alpha) \) is precisely the function defined by

\[
\phi_x(p) = \begin{cases} f(p) + \lambda & \text{if } p \in \Omega \\ \lambda & \text{if } p = \infty \end{cases}
\]

so that

\[
\|(f, \lambda)\|_{st} = \max_{p \in \Omega^\alpha} |\phi_x(p)| = \max_{p \in \Omega} \{ \max_{p \in \Omega} |f(p) + \lambda|, |\lambda| \}.
\]

The norm \( \| \cdot \|_{st} \) is referred to as the standard norm. Of course, by construction

\[
\tilde{\Phi} : (\tilde{\mathbb{C}^K_0}(\Omega), \| \cdot \|_{st}) \to \mathbb{C}^K(\Omega^\alpha)
\]

is an isometric algebra homomorphism. It turns out that \( \tilde{\Phi} \) is in fact an isomorphism. All we need is the surjectivity, which is pretty clear. Indeed, if one starts with some continuous function \( g : \Omega^\alpha \to \mathbb{K} \), then if we define \( f : \Omega \to \mathbb{K} \) by \( f(p) = g(p) - g(\infty), \ \forall p \in \Omega \), then clearly \( f \in \mathbb{C}^K_0(\Omega) \), and moreover if we put \( \lambda = g(\infty) \), then we clearly have \( g = \tilde{\Phi}(f, \lambda) \).

**Remarks 5.4.**

A. As we have seen in I.5, the construction of the compact space \( \Omega^\alpha \) makes sense also when \( \Omega \) is already compact. At the Banach algebra level this corresponds to the unitization of an already unital Banach algebra.

B. In practice one would like to realize the unitization of algebras of the form \( \mathbb{C}^K_0(\Omega) \) in a more concrete way. This can be seen in the following situation. Assume
$T$ is some compact Hausdorff space, and $p$ is some point in $T$. Consider the open set $\Omega = T \setminus \{ p \}$, which is a locally compact space, when equipped with the induced topology. Consider the space $\mathfrak{A}_p = \{ f \in C^0_K(T) : f(p) = 0 \}$. It is clear that $\mathfrak{A}_p$ is a closed subalgebra of $C^0_K(T)$. Moreover, the restriction map gives rise to an isometric algebra isomorphism

\[ R : \mathfrak{A}_p \ni f \mapsto f|_\Omega \in C^0_K(\Omega). \]

Of course, if we denote the inverse of $R$ by $\Phi$, then we have an isometric algebra homomorphism

\[ \Phi : C^0_K(\Omega) \to C^0_K(T), \]

with $\text{Ran} \ \Phi = \mathfrak{A}_p$. Finally, if we consider the extension of $\Phi$ to an algebra homomorphism

\[ \tilde{\Phi} : \tilde{C}^0_K(\Omega) \to C^0_K(T), \]

defined by

\[ \tilde{\Phi}(f, \lambda) = \Phi(f) + \lambda 1, \ \forall f \in C^0_K(\Omega), \ \lambda \in \mathbb{K}, \]

then $\tilde{\Phi}$ is an isometric algebra isomorphism, when $\tilde{C}^0_K(\Omega)$ is equipped with the standard norm $\| \cdot \|_{\text{st}}$.

An important role in Banach algebra theory is played by the invertible elements. The following technical result is very useful (note that the fact that we work in a Banach algebra is essential in the proof).

**Lemma 5.1.** Suppose $A$ is a unital Banach algebra, and $a \in A$ is an invertible element. Suppose $x \in A$ is an element with

\[ \| x - a \| < \frac{1}{\| a^{-1} \|}. \]

Then $x$ is also invertible, and

\[ \| x^{-1} - a^{-1} \| \leq \frac{\| a^{-1} \|^2 \cdot \| x - a \|}{1 - \| a^{-1} \| \cdot \| x - a \|}. \]

**Proof.** Define the element $y = 1 - a^{-1}x$. We have

\[ \| y \| = \| a^{-1}(a - x) \| \leq \| a^{-1} \| \cdot \| a - x \| < 1. \]

Since we have

\[ \| y^n \| \leq \| y \|^n, \ \forall n \geq 1, \]

it follows that the series

\[ 1 + y + y^2 + y^3 + \ldots \]

is convergent to an element $b \in A$, and we have

\[ \| 1 - b \| \leq \| y \| + \| y \|^2 + \cdots \leq \sum_{n=1}^{\infty} (\| a^{-1} \| \cdot \| x - a \|)^n = \frac{\| a^{-1} \| \cdot \| x - a \|}{1 - \| a^{-1} \| \cdot \| x - a \|}. \]

By the way $b$ is defined, it is obvious that $b(1 - y) = (1 - y)b = 1$, so we get the fact that $a^{-1}x = 1 - y$ is invertible, and $(a^{-1}x)^{-1} = b$. Because $a$ is invertible, it follows that $x$ is invertible, with inverse $x^{-1} = ba^{-1}$. We then have

\[ \| x^{-1} - a^{-1} \| = \| (1 - b)a^{-1} \| \leq \| 1 - b \| \cdot \| a^{-1} \| \leq \frac{\| a^{-1} \|^2 \cdot \| x - a \|}{1 - \| a^{-1} \| \cdot \| x - a \|}. \]

\[ \Box \]
Corollary 5.1. If \( \mathcal{A} \) is a unital Banach algebra, then the set
\[
\text{GL}(\mathcal{A}) = \{ a \in \mathcal{A} : a \text{ invertible} \}
\]
is open in \( \mathcal{A} \), and the map
\[
\text{GL}(\mathcal{A}) \ni a \mapsto a^{-1} \in \text{GL}(\mathcal{A})
\]
is continuous.

**Proof.** Immediate from the above result. \( \square \)

**Exercise 3.** Suppose \( \mathcal{A} \) is a Banach algebra and \( (a_n)_{n=1}^{\infty} \) is a sequence of invertible elements which converges to some element \( a \in \mathcal{A} \). Prove that the following are equivalent:

(i) \( \lim_{n \to \infty} \|a_n^{-1}\| = \infty; \)

(ii) the sequence \( (\|a_n^{-1}\|)_{n \geq 0} \) is unbounded;

(iii) \( a \) is non-invertible.

**Definition.** Let \( \mathcal{A} \) be a Banach algebra over \( \mathbb{K} \). A *character* of \( \mathcal{A} \) is a non-trivial (i.e. not identically zero) algebra homomorphism \( \gamma : \mathcal{A} \to \mathbb{K} \), which means that:

- \( \gamma \) is linear;
- \( \gamma \) is multiplicative, i.e. \( \phi(ab) = \phi(a)\phi(b), \forall a, b \in \mathcal{A} \).

Remark that, if \( \mathcal{A} \) has unit 1, then the fact that \( \gamma \) is non-trivial forces the equality \( \gamma(1) = 1 \).

We then define the space
\[
\text{Char}(\mathcal{A}) = \{ \gamma : \mathcal{A} \to \mathbb{K} : \gamma \text{ character} \}.
\]

**Proposition 5.1 (Automatic continuity of characters).** Let \( \mathcal{A} \) be a Banach algebra over \( \mathbb{K} \), and let \( \gamma : \mathcal{A} \to \mathbb{K} \) be a character. Then \( \gamma \) is continuous, and has norm \( \|\gamma\| \leq 1 \).

**Proof.** We can assume that \( \mathcal{A} \) is a unital Banach algebra, and \( \|1\| = 1 \). If this is not the case, we extend \( \gamma \) to a character \( \tilde{\gamma} \) of \( \tilde{\mathcal{A}} = \mathcal{A} \times \mathbb{K} \) by
\[
\tilde{\gamma}(a, \alpha) = \gamma(a) + \alpha, \forall a \in \mathcal{A}, \alpha \in \mathbb{K}.
\]
In this case we have \( \gamma = \tilde{\gamma} \circ \iota \), where \( \iota : \mathcal{A} \ni a \mapsto (a, 0) \in \tilde{\mathcal{A}} \) is the inclusion map, so it will suffice to show that \( \tilde{\gamma} \) is continuous and has norm \( \|\tilde{\gamma}\| \leq 1 \). (By Remark 5.3.D, we know that \( \tilde{\mathcal{A}} \) is a Banach algebra, when equipped with any unitization norm.)

In order to prove the desired statement, we need to prove that
\[
(\forall a \in \mathcal{A}) \quad |\gamma(a)| \leq \|a\|.
\]
We argue by contradiction. Assume there exists \( a_0 \in \mathcal{A} \) with \( |\gamma(a_0)| > \|a_0\| \). Put \( \lambda = \gamma(a_0) \), so that the element \( a_1 = \lambda^{-1}a_0 \) has norm \( \|a_1\| < 1 \). By Lemma 5.1, applied to \( a = 1 \) and \( x = 1 - a_1 \), it follows that the element \( 1 - a_1 = 1 - \lambda^{-1}a_0 \) is invertible. Since \( \lambda \neq 0 \), it follows that the element \( z = \lambda(1 - a_1) = \lambda 1 - a_0 \) is invertible. Notice however that by construction we have
\[
\gamma(z) = \lambda\gamma(1) - \gamma(a_0) = \lambda - \gamma(a_0) = 0.
\]
Of course this forces \( 1 = \gamma(1) = \gamma(z^{-1}z) = \gamma(z^{-1})\gamma(z) = 0 \), which is impossible. \( \square \)
**Corollary 5.2.** Let $\mathcal{A}$ be a unital Banach algebra. When equipped with the $w^*$-topology, the space $\text{Char}(\mathcal{A})$ is compact.

**Proof.** Since the unit ball $(l^1)^*_{\mathcal{A}}$ is compact, when equipped with the $w^*$-topology (see Section 3), all we need to show is the fact that $\text{Char}(\mathcal{A})$ is $w^*$-closed in $(l^1)_{\mathcal{A}}$. Take a net $(\gamma_{\lambda})_{\lambda \in \Lambda} \subseteq \text{Char}(\mathcal{A})$, and let $\phi \in (l^1)_{\mathcal{A}}$, with $\phi = w^*\lim_{\lambda \in \Lambda} \gamma_{\lambda}$, and let us show that $\phi \in \text{Char}(\mathcal{A})$. By the definition of the $w^*$-topology, we know that

$$\phi(a) = \lim_{\lambda \in \Lambda} \gamma_{\lambda}(a), \quad \forall a \in \mathcal{A}.$$  

This clearly implies the fact that $\phi$ is linear, multiplicative, as well as the fact that $\phi$ is not identically zero (since $\phi(1) = \lim_{\lambda \in \Lambda} \gamma_{\lambda}(1) = 1$).

The above result can be generalized to include the non-unital case.

**Proposition 5.2.** Suppose $\mathcal{A}$ is a Banach algebra. Let $\tilde{\mathcal{A}}$ be the unitization of $\mathcal{A}$, and let $\iota: \mathcal{A} \to \tilde{\mathcal{A}}$ denote the inclusion map. The correspondence

$$\iota^* : \text{Char}(\tilde{\mathcal{A}}) \ni \phi \longmapsto \phi \circ \iota \in (l^1)^*_{\mathcal{A}}$$

has the following properties:

(i) $\iota^*$ is injective;

(ii) $\text{Ran} \iota^* = \text{Char}(\mathcal{A}) \cup \{0\}$;

(iii) when we equip $\tilde{\mathcal{A}}$ with any unitization norm, and the spaces $\text{Char}(\tilde{\mathcal{A}})$ and $\text{Char}(\mathcal{A}) \cup \{0\}$ with the $w^*$-topology, the map

$$\iota^* : \text{Char}(\tilde{\mathcal{A}}) \to \text{Char}(\mathcal{A}) \cup \{0\}$$

is a homeomorphism;

(iv) in particular, $\text{Char}(\mathcal{A}) \cup \{0\}$ is compact, in the $w^*$-topology.

**Proof.** (i). Let $\phi_1, \phi_2 \in \text{Char}(\tilde{\mathcal{A}})$ be such that $\phi_1 \circ \iota = \phi_2 \circ \iota$, and let us show that $\phi_1 = \phi_2$. Recall that $\tilde{\mathcal{A}} = \mathcal{A} \times \mathbb{K}$ has unit $1 = (0, 1)$. Since any element $x = (a, \lambda) \in \tilde{\mathcal{A}}$ can be written as

$$x = (a, 0) + \lambda(0, 1) = \iota(a) + 1,$$

we get

$$\phi_1(x) = (\phi_1 \circ \iota)(a) + \lambda \phi_1(1) = (\phi_2 \circ \iota)(a) + \lambda = (\phi_2 \circ \iota)(a) + \lambda \phi_2(1) = \phi_2(x),$$

so we indeed have $\phi_1 = \phi_2$.

(ii). It is obvious that, if $\phi \in \text{Char}(\tilde{\mathcal{A}})$, then $\iota^*(\phi) = \phi \circ \iota : \mathcal{A} \to \mathbb{K}$ is an algebra homomorphism. Then either $\iota^*(\phi) = 0$, or $\iota^*(\phi)$ is a character on $\mathcal{A}$. This proves the inclusion

$$\text{Ran} \iota^* \subseteq \text{Char}(\mathcal{A}) \cup \{0\}.$$

Conversely, if we start with some $\gamma \in \text{Char}(\mathcal{A}) \cup \{0\}$, then $\gamma : \mathcal{A} \to \mathbb{K}$ is an algebra homomorphism (possibly zero), and using the key feature of $\tilde{\mathcal{A}}$ (see the definition preceding Remarks 5.3), there exists a (unique) algebra homomorphism $\tilde{\gamma} : \tilde{\mathcal{A}} \to \mathbb{K}$, with $\tilde{\gamma}(1) = 1$, and $\tilde{\gamma} \circ \iota = \gamma$. Obviously the first property says that $\tilde{\gamma}$ is a character on $\mathcal{A}$, while the second property reads $\iota^*(\tilde{\gamma}) = \gamma$.

(iii)-(iv). Using the fact that the character space $\text{Char}(\tilde{\mathcal{A}})$ is compact, combined with (i) and (ii), we see that it suffices to show that $\iota^*$ is continuous. But this fact is clear. \[\square\]
Corollary 5.3. Let $\mathcal{A}$ be a Banach algebra. If the character space $\text{Char}(\mathcal{A})$ is non-empty, then it is locally compact, when equipped with the $w^*$-topology.

Proof. Put $K = \text{Char}(\mathcal{A}) \cup \{0\} \subset (\mathcal{A}^*)_1$, which is compact in the $w^*$-topology. Then $\text{Char}(\mathcal{A}) = K \setminus \{0\}$ is open in $K$, so it is locally compact, when equipped with the induced topology from $K$, which is precisely the $w^*$-topology. \hfill \Box

Remark 5.5. Suppose $\mathcal{A}$ is a non-unital Banach algebra. It is possible for the character space $\text{Char}(\mathcal{A})$ to be empty (see Exercise ??). It is also possible for the character space $\text{Char}(\mathcal{A})$ to be non-empty and compact (see Exercise ?? below). In the event when $\text{Char}(\mathcal{A})$ is non-compact, it is locally compact, and using the results from Section I.5, it follows that the compact space $\text{Char}(\mathcal{A}) \cup \{0\}$ is identified with $\text{Char}(\mathcal{A})^\alpha$ - the Alexandrov compactification of $\text{Char}(\mathcal{A})$ - with 0 identified with the point at $\infty$.

In fact the topological (i.e. homeomorphic) identifications

\[(5) \quad \text{Char}(\hat{\mathcal{A}}) \simeq \text{Char}(\mathcal{A}) \cup \{0\} \simeq \text{Char}(\mathcal{A})^\alpha\]

work also in the case when $\text{Char}(\mathcal{A})$ is already compact, then only difference being that in this case the point at $\infty$ is isolated.

Using (5) we see that $\text{Char}(\mathcal{A})$ always gets topologically identified with an open subset of the compact space $\text{Char}(\hat{\mathcal{A}})$. This identification is implemented by the injective map

\[J : \text{Char}(\mathcal{A}) \ni \gamma \mapsto \hat{\gamma} \in \text{Char}(\hat{\mathcal{A}}),\]

where for a character $\gamma \in \text{Char}(\mathcal{A})$, one defines the character $\hat{\gamma} \in \text{Char}(\hat{\mathcal{A}})$ by

\[\hat{\gamma}(a, \alpha) = \gamma(a) + \alpha, \quad \forall a \in \mathcal{A}, \alpha \in \mathbb{K}.

Exercise 4. Any Banach space $X$ can be turned into a “trivial” Banach algebra by defining the product as

\[xy = 0, \quad \forall x, y \in X.\]

Prove that, when equipped with this trivial Banach algebra structure, the character space is empty.

Exercise 5. Consider the vector space $\mathbb{C}^2$, equipped with the norm

\[\|(\lambda_1, \lambda_2)\|_\infty = \max\{|\lambda_1|, |\lambda_2|\}.\]

Equipp $\mathbb{C}^2$ with the product

\[(\lambda_1, \lambda_2) \cdot (\mu_1, \mu_2) = (\lambda_1 \mu_1, 0).\]

Prove that $\mathbb{C}^2$ is a commutative non-unital Banach algebra, whose character space is a singleton, hence compact.

Theorem 5.1. Let $\Omega$ be a locally compact space, and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. For each point $p \in \Omega$, define the evaluation map

\[\epsilon_p : C_0^K(\Omega) \ni f \mapsto f(p) \in \mathbb{K}.\]

(i) For each $p \in \Omega$, the map $\epsilon_p$ defines a character of the Banach algebra $C_0^K(\Omega)$.

(ii) When we equip the character space $\text{Char}(C_0^K(\Omega))$ with the $w^*$-topology, the correspondence

\[E : \Omega \ni p \mapsto \epsilon_p \in \text{Char}(C_0^K(\Omega))\]

defines a homeomorphism.
Pick a point $p$ by compactness it follows that

and the claim is proven

Next we show that $E$ is continuous. This amounts to showing that, for a net $(p_\lambda)_{\lambda \in \Lambda} \subset \Omega$ which converges to a point $p \in \Omega$, we also have

$$w^* \lim_{\lambda \in \Lambda} \epsilon_{p_\lambda} = \epsilon_p.$$  

This is however trivial, since the above condition reads

$$\lim_{\lambda \in \Lambda} f(p_\lambda) = f(p), \quad \forall f \in C^K(\Omega).$$

To conclude the proof in our particular case, it suffices to show that $E$ is surjective. (By the compactness assumption, this will force $E$ to be a homeomorphism). Start with some character $\gamma : C^K(\Omega) \to \mathbb{K}$, and consider the closed subspace $\mathcal{F} = \text{Ker } \gamma$.

**Claim 1:** For every finite subset $\mathcal{F} \subset \mathcal{J}$, the closed set

$$A(\mathcal{F}) = \{ p \in \Omega : f(p) = 0, \quad \forall f \in \mathcal{F} \}$$

is non-empty.

Indeed, if we list the set $\mathcal{F} = \{ f_1, \ldots, f_n \}$, then the function $g = |f_1|^2 + \cdots + |f_n|^2 = f_1 f_1 + \cdots + f_n f_n$ will satisfy

$$\gamma(g) = \gamma(f_1) \gamma(f_1) + \cdots + \gamma(f_n) \gamma(f_n) = 0,$$

hence $g \in \mathcal{G}$. Obviously, $g$ cannot be invertible in $C^K(\Omega)$, because otherwise we will have $1 = \gamma(1) = \gamma(g^{-1}) \gamma(g) = 0$. In particular, this forces the set $G = \{ p \in \Omega : g(p) = 0 \}$ to be non-empty. Then we are done since obviously $G \subset A(\mathcal{F})$.

Using the above claim, combined with the fact that for any sequence $\mathcal{F}_1, \ldots, \mathcal{F}_k \in \mathcal{P} \text{fin}(\mathcal{J})$, we have

$$A(\mathcal{F}_1) \cap \cdots \cap A(\mathcal{F}_k) = A(\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k) \neq \emptyset,$$

by compactness it follows that

$$\bigcap_{\mathcal{F} \in \mathcal{P} \text{fin}(\mathcal{J})} A(\mathcal{F}) \neq \emptyset.$$  

Pick a point $p$ in the above intersection, so that we have

$$f(p) = 0, \quad \forall f \in \mathcal{J} = \text{Ker } \gamma.$$  

**Claim 2:** One has the equality $\gamma = \epsilon_p$.

Indeed, if we start with some arbitrary function $f \in C^K(\Omega)$, and we set $\mu = \gamma(f)$, then the function $g = \mu(1-f)$ will satisfy $\gamma(g) = \mu \gamma(1) - \gamma(f) = \mu - \gamma(f) = 0$, so using (6) we will have $g(p) = 0$, i.e. $\mu = f(p)$. In other words, we get $\gamma(f) = f(p) = \epsilon_p(f)$, and the claim is proven
Having proven property (ii) in the particular case when $\Omega$ is compact, we now proceed with the proof in the case when $\Omega$ is non-compact. Consider the Alexandrov compactification $\Omega^\alpha$ of $\Omega$, and denote by

$$E^\alpha : \Omega^\alpha \to \text{Char}(C^K(\Omega^\alpha))$$

the corresponding $E$-map, which we know to be a homeomorphism, by the particular case above.

We know (see Example 5.2) that $C^K(\Omega^\alpha)$ is algebraically isomorphic to the unitization $\tilde{C}_K^0(\Omega)$. Using Proposition 5.2 (see also Remark 5.5) this isomorphism gives rise to a homeomorphism

$$F : \text{Char}(C^K(\Omega^\alpha)) \to \text{Char}(C_K^0(\Omega)) \cup \{0\}.$$ 

Explicitly, this homeomorphism is defined as follows. If we identify $C_K^0(\Omega) = \{f \in C^K(\Omega^\alpha) : f(\infty) = 0\}$ as a (closed) subalgebra of $C^K(\Omega^\alpha)$, then we have

$$F(\gamma) = \gamma|_{C_K^0(\Omega)}, \quad \forall \gamma \in \text{Char}(C^K(\Omega^\alpha)).$$

Let us consider now the composition

$$\tilde{E} = F \circ E^\alpha : \Omega^\alpha \to \text{Char}(C_K^0(\Omega)) \cup \{0\},$$

which is a homeomorphism. Remark that, since we obviously have $\tilde{E}(\infty) = 0$, it follows that

$$\tilde{E}|_{\Omega} : \Omega \to \text{Char}(C_K^0(\Omega))$$

is also a homeomorphism. (We identify $\Omega$ with the open set $\Omega^\alpha \setminus \{\infty\} \subset \Omega^\alpha.$) But now we are done, because one clearly has the equality $\tilde{E}|_{\Omega} = E.$ \qed

Having brought the $w^*$-topology into the picture, the following result is a welcome addition to the theory.

**PROPOSITION 5.3.** Let $A$ be a Banach algebra, with the property that its character space $\text{Char}(A)$ is non-empty. Equip $\text{Char}(A)$ with the $w^*$-topology, so that it becomes a locally compact space.

(i) For any element $a \in A$, the map

$$\hat{a} : \text{Char}(A) \ni \gamma \mapsto \gamma(a) \in \mathbb{C}$$

is continuous.

(ii) If $\text{Char}(A)$ is non-compact, then for any element $a \in A$, the continuous function $\hat{a} : \text{Char}(A) \to \mathbb{C}$ has limit $0$ at $\infty$.

(iii) The correspondence

$$\Gamma_A : A \ni a \mapsto \hat{a} \in C_0(\text{Char}(A))$$

is an algebra homomorphism. If $A$ is unital, with unit 1, then $\hat{1}$ is the constant function 1.

(iv) The map $\Gamma_A : A \to C_0(\text{Char}(A))$ is continuous, with $\|\Gamma_A\| \leq 1$.

**Proof.** (i). The continuity is trivial, by the way the $w^*$-topology is defined.

(ii). If $\text{Char}(A)$ is non-compact, then we know (see Remark 5.7, and Section I.5) that the Alexandrov compactification of $\text{Char}(A)$ is identified with $\text{Char}(A) \cup \{0\}$, with the zero map 0 playing the role of the point at $\infty$. The proof of the desired
statement amount to showing that whenever \((\gamma_{\lambda})_{\lambda \in \Lambda} \subset \text{Char}(\mathcal{A})\) is a net with \(w^*-\lim_{\lambda \in \Lambda} \gamma_{\lambda} = 0\), we have \(\lim_{\lambda \in \Lambda} \hat{a}(\gamma_{\lambda}) = 0\), and this is trivial.

(iii). In order to check that \(\Gamma_{\mathcal{A}}\) is an algebra homomorphism, we need to check that

- \(\hat{a} + \hat{b} = \hat{a} + \hat{b}, \forall a, b \in \mathcal{A}\);
- \(\alpha \hat{a} = \alpha \hat{a}, \forall a \in \mathcal{A}, \alpha \in \mathbb{C}\);
- \(\hat{a} \hat{b} = \hat{a} \hat{b}, \forall a, b \in \mathcal{A}\).

All these conditions are trivially verified. For example the first condition is checked pointwise:

\[
\hat{a} + \hat{b}(\gamma) = \gamma(a + b) = \gamma(a) + \gamma(b) = \hat{a}(\gamma) + \hat{b}(\gamma), \quad \forall \gamma \in \text{Char}(\mathcal{A}).
\]

Likewise, if \(\mathcal{A}\) is unital, then we have \(\hat{1}(\gamma) = \gamma(1) = 1, \forall \gamma \in \text{Char}(\mathcal{A})\).

(iv). By the definition of the norm on \(C_0(\text{Char}(\mathcal{A}))\), we have

\[
\|\hat{a}\| = \max \{ \|\hat{a}(\gamma)\| : \gamma \in \text{Char}(\mathcal{A})\} = \max \{ \|\gamma(a)\| : \gamma \in \text{Char}(\mathcal{A})\}.
\]

By Corollary 5.3 we know that \(\|\gamma(a)\| \leq \|a\|, \forall \gamma \in \text{Char}(\mathcal{A})\), and then the above equalities immediately prove the estimate \(\|\hat{a}\| \leq \|a\|\). □

**Definition.** Let \(\mathcal{A}\) be a Banach algebra, with non-empty character space \(\text{Char}(\mathcal{A})\). The continuous algebra homomorphism

\[
\Gamma_{\mathcal{A}} : \mathcal{A} \to C_0(\text{Char}(\mathcal{A})),
\]

defined above, is called the **Gelfand correspondence**. For an element \(a \in \mathcal{A}\), the function \(\Gamma_{\mathcal{A}}(a) = \hat{a} \in C_0(\text{Char}(\mathcal{A}))\) is called the **Gelfand transform of** \(a\).

As with ring theory, an important role in the study of algebras is played by **ideals**. Since the algebras we deal with are not necessarily commutative, there are three types of ideals one can consider, as explained below.

**Definitions.** Let \(\mathcal{A}\) be some algebra over a field \(\mathbb{K}\).

**A.** A subset \(J \subset \mathcal{A}\) is called a **left ideal in** \(\mathcal{A}\), if

- \(J\) is a linear subspace of \(\mathcal{A}\);
- \(ax \in J, \forall a \in \mathcal{A}, x \in J\).

**B.** A subset \(J \subset \mathcal{A}\) is called a **right ideal in** \(\mathcal{A}\), if

- \(J\) is a linear subspace of \(\mathcal{A}\);
- \(xa \in J, \forall a \in \mathcal{A}, x \in J\).

**C.** A subset \(J \subset \mathcal{A}\) is called a **two sided ideal in** \(\mathcal{A}\), if \(J\) is both a left and a right ideal in \(\mathcal{A}\).

Of course, if \(\mathcal{A}\) is commutative, these three notions coincide.

In the study of (Banach) algebras, one deals almost exclusively with two sided ideals. The reason is the following

**Fact:** If \(J\) is a two sided ideal in an algebra \(\mathcal{A}\), then the quotient space \(\mathcal{A}/J\) carries a unique algebra structure, which makes the quotient map

\[
Q : \mathcal{A} \ni a \mapsto [a] \in \mathcal{A}/J
\]

an algebra homomorphism. Moreover, if \(\mathcal{A}\) has a unit \(1\), and \(J \subseteq \mathcal{A}\), then the algebra \(\mathcal{A}/J\) has unit \(1 = Q(1)\).

In the Banach algebra setting, one has the following properties.
Theorem 5.2. Let $\mathcal{A}$ be a Banach algebra.

A. If $\mathcal{J}$ is a left (or right, or two sided) ideal in $\mathcal{A}$, then so is its closure $\overline{\mathcal{J}}$.

B. If $\mathcal{A}$ is a unital Banach algebra, and $\mathcal{J} \subseteq \mathcal{A}$ is a left (or right, or two sided) ideal, then $\overline{\mathcal{J}} \subseteq \mathcal{A}$.

C. If $\mathcal{J}$ is a closed two sided ideal in $\mathcal{A}$, then when equipped with the quotient norm, the algebra $\mathcal{A}/\mathcal{J}$ is a Banach algebra. Moreover, if $\mathcal{A}$ is unital, and $\mathcal{J} \subseteq \mathcal{A}$, then $\mathcal{A}/\mathcal{J}$ is a unital Banach algebra.

Proof. A. Assume $\mathcal{J}$ is a left ideal. Clearly $\overline{\mathcal{J}}$ is a linear subspace, so we only need to check the second condition. Fix $a \in \mathcal{A}$ and $x \in \mathcal{J}$, and let us prove that $ax \in \overline{\mathcal{J}}$. Since $x \in \mathcal{J}$, there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{J}$, with $x = \lim_{n \to \infty} x_n$. Then the continuity of the left multiplication map $L_a : \mathcal{A} \to \mathcal{A}$ gives

$$ax = L_a x = \lim_{n \to \infty} L_a x_n = \lim_{n \to \infty} ax_n.$$ 

Since $ax_n \in \mathcal{J}, \forall n \geq 1$, this forces $ax \in \overline{\mathcal{J}}$.

In the case when $\mathcal{J}$ is a right ideal, the proof is identical. In the case when $\mathcal{J}$ is two sided, we use the above cases.

B. Assume $\mathcal{J}$ is a left ideal. We argue by contradiction. Suppose $\mathcal{J} = \mathcal{A}$. In particular it follows that $\overline{\mathcal{J}}$ contains 1, the unit in $\mathcal{A}$. Since the set $GL(\mathcal{A})$ of all invertible elements is open, and contains 1, the fact that 1 belongs to $\mathcal{J}$ gives the fact that the intersection $GL(\mathcal{A}) \cap \overline{\mathcal{J}}$ is non-empty. In particular, this means that $\mathcal{J}$ contains some invertible element $x$. Of course, if we define $a = x^{-1} \in \mathcal{A}$, we have $\mathcal{J} \ni ax = 1$. Since $\mathcal{J}$ contains 1, it will contain all elements of $\mathcal{A}$, which contradicts the strict inclusion $\mathcal{J} \subsetneq \mathcal{A}$.

In the case when $\mathcal{J}$ is a right ideal, the proof is identical. In the case when $\mathcal{J}$ is two sided, we use the above cases.

C. Let us prove the first assertion. We know (see Section 2) that the quotient space $\mathcal{A}/\mathcal{J}$ is a Banach space, so the only thing we need to prove is the second condition in the definition. Start with two elements $v, w \in \mathcal{A}/\mathcal{J}$, and let us prove the inequality

$$\|vw\|_{\mathcal{A}/\mathcal{J}} \leq \|v\|_{\mathcal{A}/\mathcal{J}} \cdot \|w\|_{\mathcal{A}/\mathcal{J}}. \quad (7)$$

Use the notations from Section 1. For each $\varepsilon > 0$, we choose $a_{\varepsilon} \in v$ and $b_{\varepsilon} \in w$, such that

$$\|a_{\varepsilon}\| \leq \|v\|_{\mathcal{A}/\mathcal{J}} + \varepsilon \text{ and } \|b_{\varepsilon}\| \leq \|w\|_{\mathcal{A}/\mathcal{J}} + \varepsilon.$$ 

Since $a_{\varepsilon}b_{\varepsilon} \in vw$, we immediately get

$$\|vw\|_{\mathcal{A}/\mathcal{J}} \leq \|a_{\varepsilon}b_{\varepsilon}\| \leq \|a_{\varepsilon}\| \cdot \|b_{\varepsilon}\| \leq (\|v\|_{\mathcal{A}/\mathcal{J}} + \varepsilon) \cdot (\|w\|_{\mathcal{A}/\mathcal{J}} + \varepsilon).$$

Since the inequality $\|vw\|_{\mathcal{A}/\mathcal{J}} \leq (\|v\|_{\mathcal{A}/\mathcal{J}} + \varepsilon) \cdot (\|w\|_{\mathcal{A}/\mathcal{J}} + \varepsilon)$ holds for all $\varepsilon > 0$, it will clearly force (7).

□

Definition. Let $\mathcal{A}$ be an algebra over a field $\mathbb{K}$. A maximal left (or right, or two sided) ideal in $\mathcal{A}$ is a left (or right, or two sided) ideal $\mathcal{M} \subseteq \mathcal{A}$, with the property:

- there is no left (or right, or two sided) ideal $\mathcal{J}$ in $\mathcal{A}$, with $\mathcal{M} \subsetneq \mathcal{J} \subseteq \mathcal{A}$. 

Remarks 5.6. A. Let $\mathcal{A}$ be an algebra over a field $K$, and let $\mathcal{M}$ be a two sided ideal in $\mathcal{A}$ with the property that the quotient algebra $\mathcal{A}/\mathcal{M}$ is one dimensional. Then $\mathcal{M}$ is a maximal ideal. Indeed, if $\mathcal{M}$ were not maximal, we could find another two sided ideal $\mathcal{J}$ with $\mathcal{M} \subseteq \mathcal{J} \subseteq \mathcal{A}$.

As in the proof of Proposition 5.1, this would give rise to an injective, but not surjective linear map $\mathcal{J}/\mathcal{M} \rightarrow \mathcal{A}/\mathcal{M}$, which is impossible, since the target space is one dimensional.

B. If $\mathcal{A}$ is an algebra over a field $K$, and if $\mathcal{A}$ has a unit, then for every left (or right, or two sided) ideal $\mathcal{J} \subseteq \mathcal{A}$, there exists at least one maximal (or right, or two sided) ideal $\mathcal{M}$ with $\mathcal{J} \subset \mathcal{M}$. This follows from Zorn’s Lemma, applied to the collection

$$\mathcal{C} = \{ \mathcal{N} \subset \mathcal{A} : \mathcal{N} \text{ left (or right, or two sided) ideal}, \mathcal{J} \subset \mathcal{N} \text{ and } 1 \not\in \mathcal{N} \},$$

ordered with inclusion.

C. Suppose $\mathcal{A}$ is a Banach algebra, which has at least one character. By the remark A above, it follows that the collection

$$\text{Max}(\mathcal{A}) = \{ \mathcal{M} \subset \mathcal{A} : \mathcal{M} \text{ maximal two sided ideal in } \mathcal{A} \}$$

is non-empty, and one has a correspondence

$$(8) \quad \text{Char}(\mathcal{A}) \ni \gamma \mapsto \text{Ker} \gamma \in \text{Max}(\mathcal{A}).$$

This will be referred to as the character correspondence. Remark that this correspondence is injective. Indeed, if $\gamma_1$ and $\gamma_2$ are characters on $\mathcal{A}$, with $\text{Ker} \gamma_1 = \text{Ker} \gamma_2 = \mathcal{M}$, and if we take $a \in \mathcal{A}$ with $\gamma_1(a) = 1$ (which is possible, since $\gamma_1$ is non-zero), then on the one hand, we also have $\gamma_2(a) \neq 0$ (since $a \not\in \mathcal{M}$). On the other hand, since $\gamma_1(a^2 - a) = \gamma_1(a)^2 - \gamma_1(a) = 0$, it follows that $a \in \mathcal{M}$, which in turn forces $\gamma_2(a)^2 - \gamma_2(a) = \gamma_2(a^2 - a) = 0$, and since $\gamma_2(a) \neq 0$, it follows that $\gamma_2(a) = 1 = \gamma_1(a)$. Finally, if we start with an arbitrary element $x \in \mathcal{A}$, then

$$\gamma_1(\gamma_1(x)a - x) = \gamma_1(x)\gamma_1(a) - \gamma_1(x) = 0,$$

which gives the fact that $\gamma_1(x)a - x$ belongs to $\mathcal{M}$, and in particular we get the fact that

$$0 = \gamma_2(\gamma_1(x)a - x) = \gamma_1(x)\gamma_2(a) - \gamma_2(x) = \gamma_1(x) - \gamma_2(x).$$

D. If $\mathcal{A}$ is a unital Banach algebra, then every maximal left (or right, or two sided) ideal is closed. Indeed, if $\mathcal{M}$ is a maximal left (or right, or two sided) ideal in $\mathcal{M}$, then by Theorem 5.1 we have the inclusions

$$\mathcal{M} \subset \overline{\mathcal{M}} \subseteq \mathcal{A},$$

with the closure $\overline{\mathcal{M}}$ again a left (or right, or two sided) ideal. The maximality of $\mathcal{M}$ then forces the equality $\overline{\mathcal{M}} = \mathcal{M}$.

We will return to the study of the relationship between characters and maximal ideals later in this section.

Exercise 6. Let $n \geq 2$ be an integer. Equip the space $\mathbb{C}^n$ with any norm, and take $\mathcal{A} = \mathcal{L}(\mathbb{C}^n)$. Prove that the only two sided ideals of $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$. Conclude that the Banach algebra $\mathcal{A}$ has no characters.
We now introduce an important tool in the analysis in Banach algebras.

**Definition.** Let $A$ be a complex unital Banach algebra (the term “complex” means that the algebra is over $\mathbb{C}$). For an element $a \in A$, the set

$$\text{Spec}_A(a) = \{ \lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } A \}$$

is called the **spectrum of $a$ relative to $A$**.

The following result collects some of the properties of the spectrum.

**Theorem 5.3.** Let $A$ be a complex unital Banach algebra.

(i) For any $a \in A$, the set $\text{Spec}_A(a)$ is compact and non-empty.

(ii) If $A$ and $B$ are unital Banach algebras, and $\Phi : A \rightarrow B$ is a unital homomorphism of algebras, i.e. $\Phi(1) = 1$, then for every element $a \in A$ one has the inclusion

$$\text{Spec}_B(\Phi(a)) \subset \text{Spec}_A(a).$$

**Proof.** (i) Let us show first that $\text{Spec}_A(a)$ is closed. We prove this indirectly, by showing that its complement is open. By definition, we have

$$\mathbb{C} \setminus \text{Spec}_A(a) = \{ \lambda \in \mathbb{C} : \lambda 1 - a \text{ invertible in } A \}.$$

If we define the map

$$\varphi : \mathbb{C} \ni \lambda \mapsto \lambda 1 - a \in A,$$

then we have the equality

$$\mathbb{C} \setminus \text{Spec}_A(a) = \varphi^{-1}(GL(A)),$$

and then the fact that $\mathbb{C} \setminus \text{Spec}_A(a)$ is open is a consequence of the (obvious) continuity of $\varphi$, combined with the fact that $GL(A)$ is open.

Next we prove that $\text{Spec}_A(a)$ is bounded. For the remainder of the proof, we are going to assume that the norm satisfies $\|1\| = 1$. (If not, we replace it with its unital correction, which does not change anything, as far as the statements are concerned).

Observe that if $\lambda \in \mathbb{C}$ satisfies $|\lambda| > \|a\|$, then the element $x(\lambda) = \lambda 1 - a$ will satisfy the inequality

$$\|x(\lambda) - 1\| = \|a\| < |\lambda| = \|\lambda 1\|^{-1},$$

so by Lemma 5.1, it follows that $x(\lambda)$ is invertible. This means that $\lambda \notin \text{Spec}_A(a)$.

The above argument proves that

$$\text{Spec}_A(a) \subset \{ \lambda \in \mathbb{C} : |\lambda| \leq \|a\| \},$$

so indeed $\text{Spec}_A(a)$ is bounded.

We now prove that the set $\text{Spec}_A(a)$ is non-empty. We argue by contradiction. Assume $\text{Spec}_A(a) = \emptyset$. Define the function $F : \mathbb{C} \rightarrow A$ by $F(\zeta) = (\zeta 1 - a)^{-1}$. On the one hand, $F$ is continuous.

On the other hand, if $|\zeta| > 2\|a\|$, then $\|\zeta^{-1}a\| < \frac{1}{2}$, so by Lemma 5.1 we have

$$\|(1 - \zeta^{-1}a)^{-1} - 1\| \leq \frac{\|\zeta^{-1}a\|}{1 - \|\zeta^{-1}a\|} = \frac{1}{1 - \|\zeta^{-1}a\|} - 1 \leq \frac{1}{1 - \frac{1}{2}} - 1 = 1,$$

which gives $\|(1 - \zeta 1 - a)^{-1}\| \leq 2$, thus proving the inequality

$$\|(\zeta 1 - a)^{-1}\| = \|\zeta^{-1}(1 - \zeta^{-1}a)^{-1}\| \leq \frac{2}{|\zeta|}.$$
Among other things, this proves
\[ \lim_{\zeta \to \infty} \| F(\zeta) \| = 0. \]
Finally, it is clear that
\[ F(\lambda) - F(\zeta) = (\lambda 1 - a)^{-1} - (\zeta 1 - a)^{-1} = (\lambda 1 - a)^{-1} \left[ 1 - (\lambda 1 - a)(\zeta 1 - a)^{-1} \right] = (\lambda 1 - a)^{-1} \left[ (\zeta 1 - a) - (\lambda 1 - a) \right] (\zeta 1 - a)^{-1} = -(\lambda - \zeta) F(\lambda) F(\zeta), \]
so by the continuity of $F$, we get
\[ \lim_{\lambda \to \zeta} \frac{1}{\lambda - \zeta} [F(\lambda) - F(\zeta)] = -F(\zeta) F(\zeta) = -(\zeta 1 - a)^{-2}, \quad \forall \zeta. \]
For each $\phi \in \mathcal{A}^*$ (the topological dual of $\mathcal{A}$) we consider the map $F_{\phi} = \phi \circ F : \mathbb{C} \to \mathbb{C}$. By (10) it is clear that $F_{\phi}$ is holomorphic (being complex differentiable). Since by (9) we also have $\lim_{\zeta \to \infty} F_{\phi}(\zeta) = 0$, using Liouville’s Theorem, it follows that $F_{\phi}$ is constant zero. Fix now $\zeta \in \mathbb{C}$. The fact that
\[ \phi(F(\zeta)) = F_{\phi}(\zeta) = 0, \quad \forall \phi \in \mathcal{A}^*, \]
combined with the Hahn-Banach Theorem (see Appendix D), forces $F(\zeta) = 0$, which is clearly impossible.

(ii) This inclusion is an obvious consequence of the implication
\[ \lambda 1 - a \text{ invertible in } \mathcal{A} \implies \lambda 1 - \Phi(a) \text{ invertible in } \mathcal{B}. \]

**Definition.** Suppose $\mathcal{A}$ is a complex unital Banach algebra. For an element $a \in \mathcal{A}$, the number
\[ \text{rad}_\mathcal{A}(a) = \max \{ |\lambda| : \lambda \in \text{Spec}_\mathcal{A}(a) \} \]
is called the spectral radius of $a$ in $\mathcal{A}$. Recall that, during the proof of the Theorem, when we showed that $\text{Spec}_\mathcal{A}(a)$ is bounded, we actually proved the inequality:
\[ \text{rad}_\mathcal{A}(a) \leq \| a \|, \quad \forall a \in \mathcal{A} \]
(In fact we showed the inequality $\text{rad}_\mathcal{A}(a) \leq \| a \|'$, where $\| . \|'$ is the unital correction of $\| . \|$. The inequality (11) follows then from the inequality $\| a \|' \leq \| a \|$.)

Remarks 5.7. Suppose $\mathcal{B}$ is a complex unital Banach algebra, and $\mathcal{A}$ is a closed subalgebra of $\mathcal{A}$, which contains the unit of $\mathcal{B}$

A. As a particular case of property (ii) above, we see that for every $a \in \mathcal{A}$ one has the inclusion
\[ \text{Spec}_\mathcal{A}(a) \supset \text{Spec}_\mathcal{B}(a). \]
In other words, "the spectrum increases as the algebra gets smaller." There are examples when the inclusion is strict, and such examples will be discussed in the next section.

B. For an element $a \in \mathcal{A}$, the following are equivalent:

(i) $\text{Spec}_\mathcal{A}(a) = \text{Spec}_\mathcal{B}(a)$;
(ii) $\mathcal{A} \supset \{ (\lambda 1 - a)^{-1} : \lambda \in \mathbb{C} \setminus \text{Spec}_\mathcal{B}(a) \}$. 

\[ \square \]
Indeed, if the equality (i) holds, then one has the equality
\[ C \setminus \text{Spec}_B(a) = C \setminus \text{Spec}_A(a), \]
which gives in particular the fact that for every \( \lambda \in C \setminus \text{Spec}_B(a) \), the element \( \lambda 1 - a \) is also invertible in \( A \), so its inverse \( (\lambda 1 - a)^{-1} \) must belong to \( A \). Conversely, if one has the inclusion (ii), then for every \( \lambda \in C \setminus \text{Spec}_B(a) \) it follows that \( \lambda 1 - a \) is in fact invertible in \( A \), which means that we have the inclusion \( C \setminus \text{Spec}_B(a) \subset C \setminus \text{Spec}_A(a) \). In other words, we get \( \text{Spec}_A(a) \subset \text{Spec}_B(a) \), which by part A above forces the equality (i).

The following exercise clarifies how much the spectrum can increase.

**Exercise 7.** (Shilov’s boundary property). Let \( B \) be a complex unital Banach algebra, and let \( A \) be a closed subalgebra of \( B \), which contains the unit of \( B \). For then for every element \( a \in B \) one has the inclusion
\[ \partial \text{Spec}_B(a) \subset \text{Spec}_A(a). \]
(Note that the boundary \( \partial \) is taken in \( C \). For example \( \partial [0, 1] = [0, 1] \).)

**HINT:** Use Exercise 5.

**Exercise 8.** Let \( B \) be a complex unital Banach algebra, and let \( A \) be a closed subalgebra of \( B \), which contains the unit of \( B \). Prove that if \( a \in A \) has its \( A \)-spectrum a **perfect** set, i.e. \( \partial \text{Spec}_A(a) = \text{Spec}_A(a) \), then one has the equality \( \text{Spec}_A(a) = \text{Spec}_B(a) \).

**Exercise 9.** Prove the following continuity result for the spectrum. Let \( A \) be a complex unital Banach algebra, and let \( (a_n)_{n=1}^{\infty} \subset A \) be a sequence which converges to some \( a \in A \). If \( D \subset \mathbb{C} \) is some open set, with \( D \supset \text{Spec}_A(a) \), then there exists some integer \( d \geq 1 \), such that
\[ \text{Spec}_A(a_n) \subset D, \quad \forall n \geq d. \]

**Examples 5.3.**

A. Let \( n \geq 1 \) be an integer. Equip \( \mathbb{C}^n \) with any norm (it turns out that any two norms on \( \mathbb{C}^n \) are equivalent) so that \( \mathbb{C}^n \) becomes a Banach space, by Exercise 1 in Section 2. If we work in the unital Banach algebra \( \mathcal{L}(\mathbb{C}^n) \), then for an element \( T \in \mathcal{L}(\mathbb{C}^n) \), its spectrum is
\[ \text{Spec}_{\mathcal{L}(\mathbb{C}^n)}(T) = \{ \lambda \in \mathbb{C} : \lambda \text{ eigenvalue for } T \}. \]
In other words, for a complex number \( \lambda \), the condition \( \lambda \in \text{Spec}_{\mathcal{L}(\mathbb{C}^n)}(T) \) is equivalent to the existence of a vector \( \xi \in \mathbb{C}^n \) with \( \|\xi\| = 1 \), such that \( T\xi = \lambda \xi \).

B. Let \( X \) be some compact Hausdorff space. Work in the Banach algebra \( C(X) \). Then for an element \( f \in C(X) \), one has the equality
\[ \text{Spec}_{C(X)}(f) = \text{Ran } f. \]
This equality is based on the observation that a function \( g \in C(X) \) is invertible in \( C(X) \), if and only if \( g(p) \neq 0 \), \( \forall p \in X \).

**Remark 5.8.** Suppose \( A \) is a unital Banach algebra. For each element \( a \in A \), the following are equivalent
\[ (i) \ a \text{ is invertible in } A; \]
\[ (ii) \ \text{the left multiplication } L_a \text{ is invertible in } \mathcal{L}(A). \]
Indeed, the implication \((i) \Rightarrow (ii)\) is trivial, since the correspondence 

\[ \Lambda : A \ni a \longmapsto L_a \in \mathcal{L}(A) \]

is a unital algebra homomorphism. Conversely, if \(L_a\) is invertible in \(\mathcal{L}(A)\), and if we define \(b = (L_a)^{-1}(1) \in A\), then on the one hand, we have

\[(12) \quad ab = a \cdot (L_a)^{-1}(1) = L_a[(L_a)^{-1}(1)] = 1.\]

On the other hand, using this equality, we also have

\[ L_a(ba) = aba = a = L_a(1),\]

and then the fact that \(L_a\) is invertible will force \(ba = 1\), which combined with \((12)\) gives the invertibility of \(a\) (and \(a^{-1} = b\)).

**COMMENT.** The above remark shows, among other things, that when \(A\) is a complex unital Banach algebra, one has the equality

\[ \text{Spec}_A(a) = \text{Spec}_{\mathcal{L}(A)}(L_a), \quad \forall a \in A. \]

In other words, the problem of computing the spectrum of an element in a complex unital Banach algebra is equivalent to the computation of the spectrum of a linear continuous map on a complex Banach space. In connection with example A, one may think that this is equivalent with an eigenvalue problem. As example B suggests, this is not the correct point of view. For example if we work with the Banach algebra \(A = C([0, 1])\) and with the function \(f(t) = t, \ t \in [0, 1]\), then the linear continuous map \(L_f \in \mathcal{L}(A)\) has no eigenvalues, although we have the equality

\[ \text{Spec}_{\mathcal{L}(A)}(L_f) = \text{Spec}_A(f) = \text{Ran} f = [0, 1]. \]

The following exercise describes a weaker notion of eigenvalues.

**Exercise 10*. Let \(X\) be a complex Banach space, and let \(T \in \mathcal{L}(X)\). We call a complex number \(\lambda\) an *approximate eigenvalue for* \(T\), if there exists a sequence \((x_n)_{n=1}^{\infty} \subset X\) with

- \(\|x_n\| = 1, \ \forall n \geq 1;\)
- \(\lim_{n \to \infty} \|Tx_n - \lambda x_n\| = 0.\)

Prove the following.

A. If \(\lambda\) is an approximate eigenvalue for \(T\), then \(\lambda \in \partial \text{Spec}_{\mathcal{L}(X)}(T)\).

B. If \(\lambda \in \partial \text{Spec}_{\mathcal{L}(X)}(T)\), then \(\lambda\) is an approximate eigenvalue for \(T\).

C. Analyze the example given in the comment above, and for each \(\lambda \in [0, 1]\) indicate how a sequence of functions \((g_n)_{n=1}^{\infty} \subset C([0, 1])\) can be constructed, such that \(\|g_n\| = 1, \ \forall n \geq 1\), and \(\lim_{n \to \infty} \|f g_n - \lambda g_n\| = 0.\)

**HINT:** For part B use Exercise 5.

The following discussion deals with the notion of *Functional Calculus*. The general theme of functional calculus is to make sense of expressions like \(f(a)\), where \(a\) is an element in a Banach algebra \(A\), and \(f\) is some kind of function \(f : \mathbb{C} \to \mathbb{C}\).

**Example 5.4.** Fix some element \(a\) in a complex unital Banach algebra \(A\). Suppose \(f : \mathbb{C} \to \mathbb{C}\) is a polynomial function, that is

\[ f(\zeta) = \alpha_0 + \alpha_1 \zeta + \cdots + \alpha_n \zeta^n, \quad \forall \zeta \in \mathbb{C},\]

where \(\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{C}\) are some constants. We define the element

\[ f(a) = \alpha_0 + \alpha_1 a + \cdots + \alpha_n a^n.\]
If we denote by $\text{Pol}(\mathbb{C})$ the algebra of polynomial functions, then the correspondence

$$\text{Pol}_a : \text{Pol}(\mathbb{C}) \ni f \mapsto f(a) \in \mathcal{A}$$

is called the **polynomial functional calculus on** $a$.

The exercise below summarizes the properties of polynomial functional calculus.

**Exercise 11**. With the notations above, prove the following.

A. The polynomial functional calculus defines a unital algebra homomorphism $\text{Pol}_a : \text{Pol}(\mathbb{C}) \rightarrow \mathcal{A}$.

B. For every polynomial function $f$, one has the equality

$$\text{Spec}_\mathcal{A}(f(a)) = f(\text{Spec}_\mathcal{A}(a)).$$

The above equality is referred to as the **Spectral Mapping Formula**.

**Hint:** For part B, prove the two inclusions separately. To get the inclusion $\subseteq$ start with some $\lambda \in \text{Spec}_\mathcal{A}(f(a))$, consider the polynomial $g(\zeta) = \lambda - f(\zeta)$, and use the Fundamental Theorem of algebra to find $\mu_1, \ldots, \mu_n \in \mathbb{C}$ such that

$$g(\zeta) = (-1)^n \alpha_n (\mu_1 - \zeta) \cdots (\mu_n - \zeta), \ \forall \zeta \in \mathbb{C}$$

Use part A to conclude that

$$\lambda 1 - f(a) = (-1)^n \alpha_n (\mu_1 - 1) \cdots (\mu_n - 1),$$

and conclude that for one of the $\mu_i$'s, say $\mu_k$, the element $\mu_k 1 - 1$ is not invertible, so $\mu_k \in \text{Spec}_\mathcal{A}(a)$.

To prove the inclusion $\supseteq$ start with some $\mu \in \text{Spec}_\mathcal{A}(a)$, put $\lambda = f(\mu)$ and consider the polynomial function $g(\zeta) = \lambda - f(\zeta)$, which factors as

$$g(\zeta) = (\mu - \zeta) h(\zeta),$$

for some other polynomial $h(\zeta)$. Use part A again, to conclude that

$$\lambda 1 - f(a) = (\mu 1 - 1) h(a),$$

and conclude that $\lambda 1 - f(a)$ is not invertible, hence $\lambda \in \text{Spec}_\mathcal{A}(f(a))$.

**Example 5.5.** The following is an extension of polynomial functional calculus. Fix some element $a$ in a complex unital Banach algebra $\mathcal{A}$. Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is an **entire** function, that is $f$ is holomorphic, which means that it it defined by a power series

$$f(\zeta) = \sum_{n=1}^{\infty} \alpha_n \zeta^n, \ \forall \zeta \in \mathbb{C},$$

where $(\alpha_n)_{n=0}^{\infty} \subset \mathbb{C}$ are some constants. Remark that, since $f$ is entire, the power series has infinite radius of convergence, i.e.

$$\sum_{n=0}^{\infty} |\alpha_n| \rho^n < \infty, \ \forall \rho \geq 0.$$

In particular (use the convention $a^0 = 1$), we have

$$\sum_{n=0}^{\infty} ||\alpha_n a^n|| \leq \sum_{n=0}^{\infty} |\alpha_n|^n \cdot ||a||^n < \infty,$$

which in particular gives the fact that the sequence of partial sums $(b_k)_{k=1}^{\infty} \subset \mathcal{A}$, defined by $b_k = \sum_{n=0}^{k} \alpha_n a^n$, has a limit which we denote by $f(a)$. If we denote by $\text{Ent}(\mathbb{C})$ the algebra of polynomial functions, then the correspondence

$$\text{Ent}_a : \text{Ent}(\mathbb{C}) \ni f \mapsto f(a) \in \mathcal{A}$$

is called the **entire functional calculus on** $a$. 
The exercise below summarizes the properties of entire functional calculus.

Exercise 12*. With the notations above, prove the following.
A. The entire functional calculus defines a unital algebra homomorphism
\( \text{Ent}_a : \text{Ent}(\mathbb{C}) \to \mathcal{A} \).
B. For every entire function \( f \), one has Spectral Mapping Formula:
\[ \text{Spec}_\mathcal{A}(f(a)) = f(\text{Spec}_\mathcal{A}(a)). \]

HINT: For part B. follow the hints in the preceding exercise, with suitable modification. Instead of the Fundamental Theorem of Algebra, use the fact that for an entire function \( f \), its range \( f(\mathbb{C}) \) is either a point, or \( \mathbb{C} \). The factoring property used in the proof of the inclusion “\( \subset \)” still holds with entire functions.

Exercise 13*. Let \( \mathcal{A} \) be a complex unital Banach algebra. Prove the Spectral Radius Formulas:
\[ \text{rad}_\mathcal{A}(a) = \lim_{n \to \infty} \|a^n\|^{1/n} = \inf \{ \|a^n\|^{1/n} : n \geq 1 \}. \]

HINTS: This is a very difficult problem! It requires several results from Complex Analysis. Follow the steps below to carry on the proof. We can assume that \( a \neq 0 \) (otherwise everything is trivial). Define the numbers
\[ L = \limsup_{n \to \infty} \|a^n\|^{1/n} \text{ and } M = \inf \{ \|a^n\|^{1/n} : n \geq 1 \}. \]

It is obvious that \( L \geq M \). Denote the spectral radius \( \text{rad}_\mathcal{A}(a) \) simply by \( R \). The proof will be divided in two parts, by showing the inequalities: (a) \( R \geq L \); (b) \( R \leq M \).

To prove (a) we define \( D \) to be the open disk in \( \mathbb{C} \), of radius \( 1/R \), centered at 0. (In the case when \( R = 0 \), we put \( D = \mathbb{C} \).) Consider the map \( F : D \to \mathcal{A} \) defined by \( F(\zeta) = (1 - \zeta a)^{-1} \).

Remark first that if we consider \( D_0 \) to be the open disk of radius \( 1/\|a\| \), centered at 0, then we have the inclusion \( D_0 \subset D \). Moreover, since \( \|\zeta a\| < 1 \), \( \forall \zeta \in D_0 \), by (the proof of) Lemma 5.1 we have the equality
\[ F(\zeta) = \sum_{n=0}^{\infty} \zeta^n a^n, \ \forall \zeta \in D_0. \]

The key idea is to think \( F \) as a “holomorphic \( \mathcal{A} \)-valued function.” To avoid any problems with making sense of this notion, we use the following trick. For every linear continuous map \( \phi : \mathcal{A} \to \mathbb{C} \), we define \( F_\phi : D \to \mathbb{C} \) as the composition \( F_\phi = \phi \circ F \). It is easy to show that \( F_\phi \) is holomorphic. On the one hand, we have
\[ F_\phi(\zeta) = \sum_{n=0}^{\infty} \phi(a^n) \zeta^n, \ \forall \zeta \in D_0. \]

On the other hand, \( F_\phi \) is defined on the whole open disk \( D \). This means that the radius of \( D \) is no greater than the radius of convergence of the above power series. This gives the inequality
\[ \frac{1}{R} \leq \limsup_{n \to \infty} \frac{1}{\|\phi(a^n)\|^{1/n}}, \]

thus proving the inequality
\[ R \geq \limsup_{n \to \infty} \|\phi(a^n)\|^{1/n}, \ \forall \phi \in \mathcal{A}^*. \]

Analyze carefully this inequality, using for instance the equality
\[ \|a\| = \max_{\phi \in (\mathcal{A}^*)_1} |\phi(a)|, \]

to conclude that one actually has \( R \geq \limsup_{n \to \infty} \|a^n\|^{1/n} \).

To prove (b) fix some \( \lambda \in \text{Spec}_\mathcal{A}(a) \), with \( R = |\lambda| \). Argue that, for every \( n \geq 1 \), one has \( \lambda^n \in \text{Spec}_\mathcal{A}(a^n) \), so
\[ R^n = |\lambda^n| \leq \text{rad}_\mathcal{A}(a^n) \leq \|a^n\|, \]

which forces \( R \leq \|a^n\|^{1/n} \).
The following explains how the computation of the spectrum can be reduced to the commutative case.

**Remark 5.9.** Let $B$ be a complex unital Banach algebra, and let $a \in B$ be an arbitrary element. Then there exists a closed commutative subalgebra $A$ which contains the unit of $B$, such that

$$\text{Spec}_A(a) = \text{Spec}_B(a).$$

Indeed if we take

$$M = \{1, a\} \cup \{(\lambda 1 - a)^{-1} : \lambda \in \mathbb{C} \setminus \text{Spec}_B(a)\},$$

then on the one hand, it is clear that $M$ is commuting, hence by Remark 5.1, the closed subalgebra $A = \overline{\text{alg}(M)}$ is commutative. On the other hand, the equality $\text{Spec}_A(a) = \text{Spec}_B(a)$ is immediate by Remark 5.2.B.

**Comment.** The above observation indicates that, although most of the interesting Banach algebras are non-commutative, the study of commutative ones should be relevant. It turns out that a “commutative result” also has applications to the general case.

What makes the commutative theory particularly manageable is the following result from Commutative Algebra.

**Lemma 5.2.** Let $A$ be a commutative algebra with unit, over a field $K$. If $x \in A$ is a non-invertible element, then there exists at least one maximal ideal $M$ of $A$, which contains $x$.

**Proof.** Since we work in a commutative algebra, we use only the term “ideal.” Consider the space

$$J_x = \{ax : a \in A\}.$$

It is obvious that $J_x$ is an ideal, and moreover, since $x$ is non-invertible, it follows that $J_x \not\ni 1$. Use then Remark 5.2.B. \qed

In the Banach algebra setting, one has the following remarkable result.

**Theorem 5.4.** Let $A$ be a commutative complex unital Banach algebra.

A. The character correspondence

$$\text{Char}(A) \ni \gamma \mapsto \ker \gamma \in \text{Max}(A)$$

is bijective.

B. If $x \in A$ is a non-invertible element, then there exists at least one character $\gamma \in \text{Char}(A)$, such that $\gamma(x) = 0$.

**Proof.** The key step in proving the theorem is contained in the following

**Claim:** If $M$ is a maximal ideal in $A$, then the quotient algebra $A/M$ is one dimensional.

To prove this, we fix a maximal ideal $M$. On the one hand, using Remark 5.2.D it follows that $M$ is closed. Denote for simplicity the quotient algebra $A/M$ by $B$, and let $Q : A \rightarrow B$ denote the quotient map. By Theorem 5.1.C the quotient algebra $B$, when equipped with the quotient norm, becomes a unital Banach algebra, with unit $1 = Q(1)$. Of course, $B$ is commutative. Fix for the moment an arbitrary element $v \in B$. Choose some number $\lambda \in \text{Spec}_B(v)$. (Here we use in an essential way the fact that $B$ is a unital Banach algebra, so by Theorem 5.3 the spectrum is...
non-empty.) Let us examine the element \( w = \lambda 1 - v \in B \). Since \( w \) is non-invertible, the set
\[
\mathcal{J}_w = \{ bw : b \in B \}
\]
is an ideal in \( B \), which does not contain the unit 1. Then it is obvious that the set \( \mathcal{N} = Q^{-1}(\mathcal{J}_w) \) is an ideal in \( A \), which does not contain the unit 1. Since we clearly have
\[
\mathcal{N} \supset \text{Ker} Q = \mathcal{M},
\]
the maximality of \( \mathcal{M} \) will force the equality \( \mathcal{N} = \mathcal{M} \). In particular, by the surjectivity of \( Q \), this forces \( \mathcal{J}_w = \{ 0 \} \), i.e. \( w = 0 \), so we get the equality \( v = \lambda 1 \). This way we have shown that every element \( v \in B \) is a scalar multiple of the unit 1, so \( B \) is indeed one dimensional.

Having proven the Claim, we now proceed with the proof of the Theorem. We already know that the character correspondence is injective (Remark 5.2.C). To prove the surjectivity, we start with a maximal ideal \( \mathcal{M} \), so using the Claim (and the notations used in its proof) it follows that the quotient algebra \( B = A/\mathcal{M} \) is a one dimensional commutative unital Banach algebra. Obviously \( B \) is then isomorphic, as an algebra, to \( \mathbb{C} \), the isomorphism being
\[
\omega : \mathbb{C} \ni \lambda \mapsto \lambda 1 \in B,
\]
where 1 denotes the unit of \( B \). The composition \( \omega^{-1} \circ Q : A \to \mathbb{C} \) is then a character, which clearly has Ker \( \gamma = \text{Ker} Q = \mathcal{M} \).

Part B follows immediately from Lemma 5.3, combined with part A. \( \square \)

**Corollary 5.4.** If \( A \) is a complex commutative unital Banach algebra, then for every element \( a \in A \), one has the equality
\[
\text{Spec}_A(a) = \{ \gamma(a) : \gamma \in \text{Char}(A) \}.
\]

**Proof.** Fix \( a \in A \), and denote for simplicity the right hand side of the above equality by \( S \). On the one hand, if \( \lambda \in S \), there exists a character \( \gamma \in \text{Char}(A) \) with \( \gamma(a) = \lambda \). Since \( \gamma(1) = 1 \), this forces \( \gamma(1a - a) = 0 \), which means that the element \( \lambda 1 - a \) belongs to the maximal ideal \( \text{Ker} \gamma \). Since \( \text{Ker} \gamma \subseteq A \), it follows that \( \text{Ker} \gamma \) contains no invertible element, so in particular \( \lambda 1 - a \) is non invertible, hence \( \lambda \in \text{Spec}_A(a) \).

Conversely, if \( \lambda \in \text{Spec}_A(a) \), then \( \lambda 1 - a \) is non invertible, so by the above result, there exists some character \( \gamma \) with \( \gamma(1a - a) = 0 \), i.e. \( \gamma(a) = \lambda \). \( \square \)

The above result can be equivalently reformulated in terms of the Gelfand transform.

**Corollary 5.5.** If \( A \) is a complex commutative unital Banach algebra, then for every element \( a \in A \), one has the equality
\[
\text{Spec}_A(a) = \text{Spec}_{C(\text{Char}(A))}(\hat{a}) = \text{Ran} \hat{a}.
\]

In particular one has the equality
\[
\text{rad}_A(a) = ||\hat{a}||.
\]

**Proof.** Immediate from the definition of the Gelfand transform. The second equality is clear, since
\[
\text{rad}_A(a) = \max \{ |\lambda| : \lambda \in \text{Spec}_A(a) \} = \max \{ |\lambda| : \lambda \in \text{Ran} \hat{a} \} = \\
\max \{ |\hat{a}(\gamma)| : \gamma \in \text{Char}(A) \} = ||\hat{a}||. \quad \square
\]
Example 5.6. Let $B$ be a complex unital Banach algebra. For an element $a \in B$, we consider the commutative unital Banach subalgebra $A$ constructed in Remark 5.7, that is $A = \text{alg}(\mathcal{M})$, where

$$\mathcal{M} = \{1, a\} \cup \{(\lambda 1 - a)^{-1} : \lambda \in \mathbb{C} \setminus \text{Spec}_B(a)\}.$$  

We already know that we have the equality $\text{Spec}_A(a) = \text{Spec}_B(a)$. Let us consider the map

$$(14) \quad F_a : \text{Char}(A) \ni \gamma \longmapsto \gamma(a) \in \mathbb{C}.$$  

The map $F_a$ is obviously continuous.

Remark that $F_a$ is injective. Indeed, if $\gamma_1, \gamma_2 \in \text{Char}(A)$ satisfy $\gamma_1(a) = \gamma_2(a)$, then it is clear that we also have

$$\gamma_1((\lambda 1 - a)^{-1}) = \frac{1}{\lambda - \gamma_1(a)} = \frac{1}{\lambda - \gamma_2(a)} = \gamma_2((\lambda 1 - a)^{-1}), \quad \forall \lambda \in \mathbb{C} \setminus \text{Spec}_B(a),$$

which means that $\gamma_1|_{\mathcal{M}} = \gamma_2|_{\mathcal{M}}$. Then it follows that $\gamma_1|_{\text{alg}(\mathcal{M})} = \gamma_2|_{\text{alg}(\mathcal{M})}$, and by continuity, we must have $\gamma_1 = \gamma_2$.

Finally, by the properties of the Gelfand correspondence, we also know that $\text{Ran} F_a = \text{Spec}_A(a)$.

Combining all these facts, it follows that the map (14) establishes a homeomorphism

$$F_a : \text{Char}(A) \xrightarrow{\sim} \text{Spec}_A(a) = \text{Spec}_B(a).$$

Exercise 14. Let $A$ be a unital Banach algebra, and let $a, b \in A$ be commuting elements, i.e. satisfying $ab = ba$. Prove the following inclusions

(i) $\text{Spec}_A(a + b) \subset \text{Spec}_A(a) + \text{Spec}_A(b)$;
(ii) $\text{Spec}_A(ab) \subset \text{Spec}_A(a) \cdot \text{Spec}_A(b)$.

Hint: Construct a closed commutative subalgebra $U \subset A$, with $\{1, a, b\} \subset U$, and such that

$$\text{Spec}_A(x) = \text{Spec}_U(x), \quad \forall x \in \{a, b, a + b, ab\}.$$  

Then work in $U$ and use Gelfand transforms.

The exercise below discusses a very interesting example. We shall elaborate on it later in Chapter VII.

Exercise 15*. Consider the Banach space $\ell^1(\mathbb{Z})$.

A. Prove that, for functions $f, g \in \ell^1(\mathbb{Z})$, and an integer $k$, the function $h : \mathbb{Z} \to \mathbb{C}$ defined by $h(n) = f(n)g(k - n)$, $\forall n \in \mathbb{Z}$, is summable.

B. Use part A to define, for $f, g \in \ell^1(\mathbb{Z})$ the function $f \times g : \mathbb{Z} \to \mathbb{C}$ by

$$(f \times g)(k) = \sum_{n \in \mathbb{Z}} f(n)g(k - n), \quad \forall k \in \mathbb{Z}.$$  

Prove that $f \times g$ is summable, hence it defines an element in $\ell^1(\mathbb{Z})$. Moreover, prove that

$$\|f \times g\|_1 \leq \|f\|_1 \cdot \|g\|_1, \quad \forall f, g \in \ell^1(\mathbb{Z}).$$

C. Prove that the operation $\times$ is associative, and distributive with respect to addition. Using part B, conclude that, when equipped with $\times$ as the multiplication, $\ell^1(\mathbb{Z})$ becomes a Banach algebra. Moreover, prove that $\ell^1(\mathbb{Z})$ is a unital commutative algebra. The unit is the function $e : \mathbb{Z} \to \mathbb{C}$ defined by $e(n) = \delta_{n0}$. 
§5. Banach algebras

D. Consider the unit circle $T = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \}$. For each $\zeta \in T$, and each $f \in \ell^1(\mathbb{Z})$, consider the function $f_\zeta : \mathbb{Z} \to \mathbb{C}$ defined by

$$f_\zeta(n) = \zeta^n f(n), \quad \forall n \in \mathbb{Z}.$$ 

Prove that $f_\zeta$ is summable. Moreover, for a fixed $\zeta \in T$, the correspondence

$$\gamma_\zeta : \ell^1(\mathbb{Z}) \ni f \mapsto \sum_{n \in \mathbb{Z}} f_\zeta(n) \in \mathbb{C}$$

defines a character on $\ell^1(\mathbb{Z})$.

E. Prove that the map

$$T \ni \zeta \mapsto \gamma_\zeta \in \text{Char}(\ell^1(\mathbb{Z}))$$

is a homeomorphism.

The above results state that after identifying the character space $\text{Char}(\ell^1(\mathbb{Z}))$ with $T$, the Gelfand correspondence gets identified with an algebra homomorphism

$$\ell^1(\mathbb{Z}) \ni f \mapsto \hat{f} \in C(T),$$

defined by

$$\hat{f}(\zeta) = \sum_{n \in \mathbb{Z}} f(n)\zeta^n, \quad \forall f \in \ell^1(\mathbb{Z}).$$

For $f \in \ell^1(\mathbb{Z})$ the function $\hat{f}$ is called the Fourier transform of $f$. The multiplication $\times$ is called the convolution product, and the Banach algebra $\ell^1(\mathbb{Z})$ defined above is called the Fourier algebra of $\mathbb{Z}$.

The machinery developed in this section will be employed in the study of some special types of Banach algebras, as we shall see later on in this chapter (see Section ??). In preparation for this, we conclude this section with a discussion on some special type of Banach algebras.

DEFINITIONS. Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. A map $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ is called an involution, if it satisfies the following conditions

- $(a + b)^* = a^* + b^*, \quad \forall a, b \in \mathcal{A};$
- $(\zeta a)^* = \overline{\zeta} a^*, \quad \forall a \in \mathcal{A}, \zeta \in \mathbb{C};$
- $(ab)^* = b^* a^*, \quad \forall a, b \in \mathcal{A};$
- $(a^*)^* = a, \quad \forall a \in \mathcal{A}.$

(Notice that if $\mathcal{A}$ has unit 1, then the above conditions force $1^* = 1.$) In this setting, the system $(\mathcal{A}, \bullet)$ is called an involutive algebra (or $\ast$-algebra).

Given involutive algebras $(\mathcal{A}, \bullet)$ and $(\mathcal{B}, \star)$, a map $\Phi : \mathcal{A} \to \mathcal{B}$ is called a $\ast$-homomorphism, if

- $\Phi$ is an algebra homomorphism;
- $\Phi(a^*) = \Phi(a)^*, \quad \forall a \in \mathcal{A}.$

A system $(\mathcal{A}, \bullet)$ is called an involutive normed algebra, if $\mathcal{A}$ is a normed algebra, and $\ast$ is an involution on $\mathcal{A}$, which is isometric, in the sense that

$$\|a^*\| = \|a\|, \quad \forall a \in \mathcal{A}.$$

If, in addition to these features, $\mathcal{A}$ is complete as a normed vector space, then $\mathcal{A}$ is called an involutive Banach algebra.
Remarks 5.10. A. If \((\mathcal{A}, \cdot)\) is an involutive normed algebra, then its completion \(\overline{\mathcal{A}}\) carries a natural involution, defined as the unique extension of the involution on \(\mathcal{A}\) (which is isometric) by continuity. When equipped with this involution, \(\overline{\mathcal{A}}\) becomes an involutive Banach algebra.

B. If \(\mathcal{A}\) is an involutive algebra, then the unitized algebra \(\hat{\mathcal{A}}\) carries a unique involution, defined as
\[
(a, \alpha)^* = (a^*, \bar{\alpha}), \quad \forall a \in \mathcal{A}, \alpha \in \mathbb{C}.
\]

C. Remark also that, if \(\mathcal{A}\) is an involutive normed algebra, then the unitization norm \(\| \cdot \|_1\) on \(\hat{\mathcal{A}}\), defined in Remark 5.3.A by
\[
\|(a, \alpha)\|_1 = \|a\| + |\alpha|, \quad \forall a \in \mathcal{A}, \alpha \in \mathbb{C},
\]
turns the unitized algebra \(\hat{\mathcal{A}}\) into an involutive normed algebra. A unitization norm on \(\hat{\mathcal{A}}\), which makes the involution isometric, will be called a unitization \(\cdot\)-norm.

D. In general, there may exist many unitization \(\cdot\)-norms on \(\hat{\mathcal{A}}\), but if \(\mathcal{A}\) is an involutive Banach algebra, then any such norm is equivalent to \(\| \cdot \|_1\).

Examples 5.7. A. If \(\Omega\) is a locally compact space, then \(C_0(\Omega)\) is an involutive Banach algebra, with the complex conjugation \(f \mapsto \overline{f}\) as its involution. The Banach algebra \(C_0(\Omega)\) is also an involutive Banach algebra, with the same involution. These involutions are called the standard involutions. Unless otherwise stated, these are the only involutions used on these algebras.

B. The Fourier algebra \(\ell^1(\mathbb{Z})\) becomes an involutive Banach algebra, with the involution \(\ell^1(\mathbb{Z}) \ni f \mapsto f^* \in \ell^1(\mathbb{Z})\) defined by
\[
f^*(n) = \overline{f(-n)}, \quad \forall n \in \mathbb{Z}.
\]

Definitions. If \((\mathcal{A}, \cdot)\) is a Banach algebra, then the dual space \(\mathcal{A}^*\) is equipped with a map
\[
\mathcal{A}^* \ni \phi \mapsto \phi^* \in \mathcal{A}^*,
\]
defined by
\[
\phi^*(a) = \overline{\phi(a^*),} \quad \forall a \in \mathcal{A}, \phi \in \mathcal{A}^*.
\]
The properties of this map are similar to the ones of the involution
- \((\phi + \psi)^* = \phi^* + \psi^*, \quad \forall \phi, \psi \in \mathcal{A}^*;\)
- \((\zeta \phi)^* = \overline{\zeta} \phi^*, \quad \forall \phi \in \mathcal{A}^*, \zeta \in \mathbb{C};\)
- \((\phi^*)^* = \phi, \quad \forall \phi \in \mathcal{A}^*.
\]
With this terminology, we say that a character \(\gamma \in \text{Char}(\mathcal{A})\) is involutive (or a \(*\)-character), if \(\gamma^* = \gamma\), which means that
\[
\gamma(a^*) = \overline{\gamma(a)}, \quad \forall a \in \mathcal{A}.
\]
(This means that \(\gamma : \mathcal{A} \to \mathbb{C}\) is a \(*\)-homomorphism, when \(\mathbb{C}\) is equipped with the complex conjugation as its involution.) The set of \(*\)-characters will be denoted by \(\text{Char}_*(\mathcal{A}, \cdot)\) (or simply \(\text{Char}_*(\mathcal{A})\), when there is no danger of confusion on the involution on \(\mathcal{A}\)).

Remark 5.11. If \((\mathcal{A}, \cdot)\) is an involutive Banach algebra, then \(\text{Char}_*(\mathcal{A})\) is \(w^*\)-closed in \(\text{Char}(\mathcal{A})\). If one considers the restriction map
\[
R : C_0(\text{Char}(\mathcal{A})) \ni f \mapsto f|_{\text{Char}_*(\mathcal{A})} \in C_0(\text{Char}_*(\mathcal{A})),
\]
then we can construct a new algebra homomorphism

$$\Gamma_{(A, \star)} : A \ni a \mapsto \hat{a} = R(\hat{a}) \in C_0(\text{Char}_\star(A)),$$

which is referred to as the Gelfand $\star$-correspondence. The key feature of this map is the fact that it defines a $\star$-homomorphism. For an element $a \in A$, we call the continuous function $\hat{a}$ the Gelfand $\star$-transform of $a$.

**Comment.** In general, the Gelfand $\star$-transform is not sensitive to the spectrum. This is due to the fact that, in general, the inclusion

$$\text{Char}_\star(A) \subset \text{Char}(A)$$

may be strict (see the exercise below). One can then talk about “nice” involutive Banach algebras, as being the ones for which one has the equality

$$\text{Char}_\star(A) = \text{Char}(A).$$

For “nice” involutive Banach algebras, the Gelfand correspondence and the Gelfand $\star$-correspondence coincide.

**Remark 5.12.** Let $(A, \star)$ be an involutive unital Banach algebra. For any element $a \in A$ one has the equality

$$\text{Spec}_A(a^*) = \{ \bar{\lambda} : \lambda \in \text{Spec}_A(a) \}.$$  

This is due to the fact that the involution $\star$ satisfies the equivalence

$$x \in GL(A) \iff x^* \in GL(A).$$

**Proposition 5.4.** Let $(A, \star)$ be an involutive unital commutative Banach algebra. The following conditions are equivalent.

(i) $(A, \star)$ is “nice”, in the sense that one has the equality

$$\text{Char}_\star(A) = \text{Char}(A).$$

(ii) Every element $a \in A$ with $a^* = a$, has real spectrum, i.e. $\text{Spec}_A(a) \subset \mathbb{R}$.

**Proof.** (i) $\Rightarrow$ (ii). Assume (i), let $a \in A$ be an element with $a^* = a$, and let $\lambda \in \text{Spec}_A(a)$. Using the properties of the Gelfand transform, there exists $\gamma \in \text{Char}(A)$, such that $\lambda = \gamma(a)$. Since every character is involutive, we have

$$\bar{\lambda} = \overline{\gamma(a)} = \gamma(a^*) = \gamma(a) = \lambda,$$

so $\lambda \in \mathbb{R}$.

(ii) $\Rightarrow$ (i). Assume (ii), and let us prove that every character $\gamma \in \text{Char}(A)$ is involutive, i.e

$$\gamma(a^*) = \overline{\gamma(a)}, \ \forall a \in A.$$  

To prove this fact, we start with some $a \in A$ and we define the elements $a_k = \frac{1}{2}(a + a^*)$ and $a^\ast = \frac{1}{2}(a - a^*)$. It is pretty obvious that $a_k = a_k^*$, $k = 1, 2$, and we also have

$a = a_1 + ia_2$ and $a^\ast = a_1 - ia_2$, so that

$$\gamma(a) = \gamma(a_1) + i\gamma(a_2) \text{ and } \gamma(a^* = \gamma(a_1) - i\gamma(a_2).$$

But now the equality (16) is clear, since $\gamma(a_k) \in \text{Spec}_A(a_k) \subset \mathbb{R}$, $k = 1, 2$. \qed
Exercise 16. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. Consider the Banach algebra $C(\mathbb{T})$, and equip it with a non-standard involution, defined by

$$f^*(\zeta) = \overline{f(\zeta)}, \quad \forall \zeta \in \mathbb{T}, \ f \in C(\mathbb{T}).$$

Prove that $(C(\mathbb{T}), \star)$ is still an involutive Banach algebra. Identify the character space $\text{Char}(C(\mathbb{T}))$ with $\mathbb{T}$, as in Exercise ???. Prove that, under this identification, one has the equality

$$\text{Char}_*(C(\mathbb{T}), \star) = \{-1, 1\}.$$

Exercise 17. Let $\Omega$ be a locally compact space. Consider the involutive Banach algebra $C_0(\Omega)$ (this means that it comes equipped with the standard involution). Prove that $C_0(\Omega)$ is “nice” in the sense defined above.

Exercise 18. Prove that the Fourier algebra $\ell^1(\mathbb{Z})$, equipped with the involution defined in Example 5.5.B, is “nice.”